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Research Article

A Note on Improved Homotopy Analysis Method for Solving the Jeffery-Hamel Flow

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This paper presents the solution of the nonlinear equation that governs the flow of a viscous, incompressible fluid between two converging-diverging rigid walls using an improved homotopy analysis method. The results obtained by this new technique show that the improved homotopy analysis method converges much faster than both the homotopy analysis method and the optimal homotopy asymptotic method. This improved technique is observed to be much more accurate than these traditional homotopy methods.

1. Introduction

The mathematical study of the flow of a viscous incompressible two-dimensional fluid in a wedge-shaped channel with a sink or source at the vertex was pioneered by Jeffery [1] and Hamel [2]. The problem has since been studied extensively by, among others, Axford [3] who included the effects of an externally applied magnetic field and Rosenhead [4] who obtained a general solution containing elliptic functions.

Instability and bifurcation are other aspects of the Jeffery-Hamel problem that have attracted widespread interest; see, for example, Akulenko and Kumakshev [5, 6]. Three-dimensional extensions to and bifurcations of the Jeffery-Hamel flow have been made by Stow et al. [7] while McAlpine and Drazin [8] presented a normal mode analysis of two-dimensional perturbations of a viscous incompressible fluid driven between inclined plane walls by a line source at the intersection of the walls. Banks et al.

[9] investigated various perturbations and the linear temporal stability of such flows and found evidence of a strong interaction between the disturbances up- and downstream if the angle between the planes exceeds a certain Reynolds-number-dependent critical value. Makinde and Mhone [10] investigated the temporal stability of MHD Jeffery-Hamel flows. They showed that an increase in the magnetic field intensity has a strong stabilizing effect on both diverging and converging channel geometries. A review of the theory of instabilities and bifurcations in channels is given by Drazin [11].

As with most problems in science and engineering, the equations governing the Jeffery-Hamel problem are highly nonlinear and so generally do not have closed form analytical solutions. Nonlinear equations can, in principle, be solved by any one of a wide variety of numerical methods. However, numerical solutions rarely give intuitive insights into the effects of various parameters associated with a problem. Consequently, most recent studies of flows in diverging and converging channels have centred on the use of the Jeffery-Hamel flow equations as a testing and proving tool for the accuracy, reliability, and robustness of new techniques for solving nonlinear equations. Examples of such techniques include the summation series technique [12], the homotopy analysis method [13, 14], the decomposition method [15], the homotopy perturbation method [16], Hermite-Padé approximation [17], and the spectral-homotopy analysis method [18–21]. The study by Joneidi et al. [14] used three methods: the differential transform method (DTM), the homotopy analysis method (HAM), and the homotopy perturbation method (HPM) to solve the Jeffery-Hamel problem. The study confirmed that although both the DTM and the HPM give acceptable accuracy, the HAM is by far the superior method delivering faster convergence and better accuracy. Nonetheless, important improvements have been made to the HAM by Motsa et al. [18, 19]. The spectral modification of the HAM or SHAM proposed by Motsa et al. [18, 19] removes some of the prescriptive assumptions associated with the HAM and further accelerates the convergence rate of the method.

A recent study by Esmaeilpour and Ganji [22] reported on the solution of the Jeffery-Hamel problem using an optimal homotopy asymptotic method (OHAM). This method was developed by Marinca and Herişanu [23] for the approximate solution of nonlinear problems of thin film flow of a fourth-grade fluid. The OHAM was used by these authors and others to solve the equations for the steady flow of a fourth-grade fluid in a porous medium [24, 25] and has since been applied to many other nonlinear problems including the squeezing flow problem by Idrees et al. [26]. Studies by Islam et al. [27, 28] further suggest that the OHAM is more general than both the HAM and the HPM with the latter methods being special cases of the OHAM.

In this paper, we report on a new and improved method known as improved homotopy analysis method (IHAM) for solving general boundary value problems. The IHAM is an algorithm that seeks to improve the initial approximation that is later used in the HAM to solve the governing nonlinear equation resulting in significant improvement in the accuracy and convergence rate of the solutions. We seek to demonstrate the application of this method by solving the Jeffery-Hamel problem and to show its accuracy and rapid convergence by comparing the present solutions with those in the literature, including the HAM, SHAM, and the OHAM. The IHAM solutions are further compared with numerical solutions. Finally, we believe that this work will motivate further improvements to the homotopy-based semi-analytical methods.

2. Governing Equations

We consider the steady two-dimensional flow of a viscous incompressible fluid between two nonparallel rigid planes with angle 2α. Assuming radial flow with velocity $\mathbf{v} = u(r,\theta)$, the motion of such a fluid is described by the equations (see [14, 19, 22])

$$\frac{\rho}{r}\frac{\partial}{\partial r}(ru) = 0, \tag{2.1}$$

$$u\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} - \frac{u}{r^2}\right], \tag{2.2}$$

$$-\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \frac{2\nu}{r^2}\frac{\partial u}{\partial \theta} = 0, \tag{2.3}$$

where ρ is the fluid density, p the pressure, and ν the kinematic viscosity. The continuity equation implies that

$$h(\theta) = ru(r,\theta), \tag{2.4}$$

and by using the nondimensional parameters

$$f(y) = \frac{f(\theta)}{f_{\max}}, \qquad y = \frac{\theta}{\alpha}, \tag{2.5}$$

Equations (2.2)-(2.3) reduce to

$$f''' + 2\alpha \operatorname{Re} f f' + 4\alpha^2 f' = 0, \tag{2.6}$$

where

$$\operatorname{Re} = \frac{\alpha f_{\max}}{\nu} = \frac{U_{\max} r\alpha}{\nu} \tag{2.7}$$

is the Reynolds number and $U_{\max}$ is the maximum velocity at the centre of the channel. The appropriate boundary conditions are

$$f(0) = 1, \quad f'(0) = 0, \qquad f(1) = 0. \tag{2.8}$$

3. Improved Homotopy Analysis Method (IHAM) Solution

In this section, we describe the use of the improved homotopy analysis method (IHAM) in the governing equation (2.6). To apply the IHAM, we assume that the solution $f(y)$ can be expanded as

$$f(y) = f_i(y) + \sum_{n=0}^{i-1} f_n(y), \quad i = 1, 2, 3, \ldots, \tag{3.1}$$

where f_i are unknown functions whose solutions are obtained using the HAM approach at the ith iteration and f_n, $(1 \leq n \leq i-1)$ are known from previous iterations. The algorithm starts with the initial approximation $f_0(y)$ which is chosen to satisfy the boundary conditions (2.8). An appropriate initial guess is

$$f_0(y) = 1 - y^2. \tag{3.2}$$

Substituting (3.1) in the governing equation (2.6)-(2.8) gives

$$f_i''' + a_{1,i-1} f_i' + a_{2,i-1} f_i + 2\alpha \operatorname{Re} f_i f_i' = r_{i-1}, \tag{3.3}$$

subject to the boundary conditions

$$f_i(0) = 0, \quad f_i'(0) = 0, \qquad f_i(1) = 0, \tag{3.4}$$

where the coefficient parameters $a_{k,i-1}$, $(k = 1, 2)$ and r_{i-1} are defined as

$$\begin{gathered} a_{1,i-1} = 2\alpha \operatorname{Re} \sum_{n=0}^{i-1} f_n + 4\alpha^2, \qquad a_{2,i-1} = 2\alpha \operatorname{Re} \sum_{n=0}^{i-1} f_n', \\ r_{i-1} = -\left[\sum_{n=0}^{i-1} f_n''' + 2\alpha \operatorname{Re} \sum_{n=0}^{i-1} f_n' \sum_{n=0}^{i-1} f_n + 4\alpha^2 \sum_{n=0}^{i-1} f_n' \right]. \end{gathered} \tag{3.5}$$

Starting from the initial approximation (3.2), the subsequent solutions f_i $(i \geq 1)$ are obtained by recursively solving equation (3.3) using the HAM approach [29, 30]. To find the HAM solutions of (3.3), we begin by rewriting (3.3) as

$$\mathcal{N}[f_i(y)] = r_{i-1}, \tag{3.6}$$

where $\mathcal{N}$ is a nonlinear operator defined by

$$\mathcal{N}[f_i(y)] = f_i''' + a_{1,i-1} f_i' + a_{2,i-1} f_i + 2\alpha \operatorname{Re} f_i f_i'. \tag{3.7}$$

Let $f_{i,0}(y)$ denote the initial guess for the unknown function $f_i(y)$, and let $\hbar \neq 0$ be an auxiliary parameter. Using an embedding parameter $q \in [0,1]$ we construct a homotopy

$$\mathscr{H}[F_i(y;q); f_{i,0}(y), \hbar, q] = (1-q)\mathcal{L}[F_i(y;q) - f_{i,0}(y)] - q\hbar\{\mathcal{N}[F_i(y;q)] - r_{i-1}\}, \tag{3.8}$$

where L is an auxiliary linear operator with the property that $\mathcal{L}[0] = 0$ and $F_i(y;q)$ is an unknown function. Upon equating $\mathscr{H}$ to 0, we obtain the zero-order deformation equation

$$(1-q)\mathcal{L}[F_i(y;q) - f_{i,0}(y)] = q\hbar\{\mathcal{N}[F_i(y;q)] - r_{i-1}\}. \tag{3.9}$$

When, $q = 0$ (3.9) becomes

$$\mathcal{L}[F_i(y;0) - f_{i,0}(y)] = 0. \tag{3.10}$$

This equation holds provided

$$F_i(y;0) = f_{i,0}(y) \tag{3.11}$$

as $\mathcal{L}[0] = 0$. When $q = 1$, (3.9) is simplified to

$$\mathcal{N}[F_i(y;1)] = r_{i-1} \tag{3.12}$$

as $\hbar \neq 0$. Equation (3.12) is the same as equation (3.6) provided

$$F_i(y;1) = f_i(y). \tag{3.13}$$

It follows from (3.11) and (3.13) that as q increases from 0 to 1, the unknown function $F(y;q)$ varies continuously from initial guess $f_{i,0}(y)$ to exact solution $f_i(y)$ of (3.6).

Differentiating (3.9) m times with respect to q and then setting $q = 0$ and finally dividing the resulting equations by $m!$ yields the mth-order deformation equations

$$\begin{aligned}&\mathcal{L}[f_{i,m}(\eta) - \chi_m f_{i,m-1}]\\&\quad = \hbar\left(f'''_{i,m-1} + a_{1,i-1}f'_{i,m-1} + a_{2,i-1}f_{i,m-1} + 2\alpha\,\mathrm{Re}\sum_{j=0}^{m-1} f_{i,j}f''_{i,m-1-j} - (1-\chi_m)r_{i-1}\right),\end{aligned} \tag{3.14}$$

subject to the boundary conditions

$$f_{i,m}(0) = f'_{i,m}(0) = f_{i,m}(1) = 0, \tag{3.15}$$

where

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{3.16}$$

If we define the linear operator $\mathcal{L}$ by

$$\mathcal{L}[F_i(y;q)] = \frac{\partial^3 F_i}{\partial y^3}, \tag{3.17}$$

then the initial approximation $f_{i,0}$ that is used in the higher-order equations (3.14) is obtained by solving the differential equation

$$\mathcal{L}[f_{i,0}] = f_{i,0}''' = r_{i-1}, \tag{3.18}$$

subject to the boundary conditions

$$f_{i,0}(0) = f_{i,0}'(0) = f_{i,0}(1) = 0. \tag{3.19}$$

Thus, starting from the initial approximation, which is obtained-from (3.18), higher order approximations $f_{i,m}(y)$ for $m \geq 1$ can be obtained through the recursive formula (3.14). We note that (3.14) forms a set of linear ordinary differential equations and can be easily solved analytically, especially by means of symbolic computation software such as Maple, Mathematica, Matlab, and others.

Expanding $F_i(y;q)$ in Taylor series about $q = 0$ gives

$$F_i(y;q) = F_i(y;0) + \sum_{k=1}^{+\infty} \frac{q^k}{k!} \frac{\partial^k F_i(y;q)}{\partial y^k}\bigg|_{q=0}. \tag{3.20}$$

If we set $q = 1$ in (3.20) and define

$$f_{i,k}(y) := \frac{1}{k!} \frac{\partial^k F_i(y;q)}{\partial y^k}\bigg|_{q=0} \tag{3.21}$$

for each $k = 1, 2, \ldots$, then making use of (3.11) and (3.13) transforms (3.20) to

$$f_i(y) = f_{i,0}(y) + \sum_{k=1}^{+\infty} f_{i,k}(y). \tag{3.22}$$

Upon truncating the infinite series in (3.22), the solutions for f_i are then generated using the solutions for $f_{i,m}$ as follows:

$$f_i = f_{i,0} + f_{i,1} + f_{i,2} + f_{i,3} + f_{i,4} + \cdots + f_{i,m}. \tag{3.23}$$

The $[i, m]$ approximate solution for $f(y)$ is then obtained by substituting f_i (obtained from (3.23)) in (3.1).

Table 1: Comparison between the numerical results and the order $[m, n]$ IHAM approximate results (using $\hbar = -1$) for $f(y)$ against the OHAM results reported in [22] when Re = 50, $\alpha = 5$.

y	OHAM [22]	$[2,2]$	$[2,3]$	$[2,4]$	Numerical
0	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000
0.1	0.98251809	0.98243121	0.98243124	0.98243124	0.98243124
0.2	0.93156589	0.93122583	0.93122596	0.93122596	0.93122596
0.3	0.85138155	0.85061033	0.85061061	0.85061062	0.85061062
0.4	0.74826040	0.74679035	0.74679080	0.74679080	0.74679080
0.5	0.62953865	0.62694761	0.62694817	0.62694817	0.62694817
0.6	0.50242894	0.49823389	0.49823445	0.49823445	0.49823445
0.7	0.37293383	0.36696589	0.36696634	0.36696634	0.36696634
0.8	0.24508198	0.23812347	0.23812375	0.23812375	0.23812375
0.9	0.12071562	0.11515181	0.11515193	0.11515193	0.11515193
1	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000

4. Results and Discussion

The accuracy and reliability of the IHAM is determined by comparing the current results with the optimal homotopy analysis method (OHAM) of Esmaeilpour and Ganji [22], the homotopy analysis results (HAM) of Joneidi et al. [14], and the numerical results obtained using the MATLAB `bvp4c` routine which is a boundary value solver based on the adaptive Lobatto quadrature scheme [31, 32]. In Table 1 we show a comparison of the current velocity results at various dimensionless angles y for different orders $[m, n]$ of the IHAM. The Reynolds number is fixed at Re = 50, and we consider a diverging channel with angle $\alpha = 5°$. Convergence of the current results to the numerical results to six decimal places is achieved at order $[2, 2]$ while convergence to ten decimal places is achieved at order $[2, 3]$. On the other hand, convergence of the OHAM is evidently slow, and up to the twentieth order of approximation, the method mostly fails to converge to the numerical results.

The slow convergence of the OHAM is confirmed in Table 2 where the absolute errors of the two methods in relation to the numerical solution are given. At any angle y, the absolute error using the OHAM is much larger than that obtained using the IHAM. The largest absolute error at any angle y using the IHAM is 5.6×10^{-7} while, by comparison, the largest absolute error when using the OHAM is 6.958×10^{-3}.

Table 3 gives a comparison of the convergence rate of the current method with the HAM [14]. Convergence of the IHAM to the numerical results is found to be rapid, with agreement to ten decimal places being archived at order $[3, 3]$ of the IHAM algorithm. The sixteenth-order HAM on the other hand fails to achieve the accuracy of both the IHAM and the numerical results. The poor performance of the HAM is further confirmed in Table 4 where the largest absolute error incurred by the IHAM at order $[3, 2]$ is 7.2×10^{-9} while the largest absolute error incurred by using the sixteenth-order HAM at any angle y is approximately a hundred times larger at 4.7×10^{-7}.

Figure 1 shows the effect of the Reynolds number on the fluid velocity for different values of α. It can be seen from the figure that the fluid velocity increases with Reynolds numbers in the case of convergent channels ($\alpha < 0$) but decreases with Re in the case of divergent channels ($\alpha > 0$). This observation is consistent with the observations made in [13, 14, 19].

Table 2: Comparison between the errors of the order $[m, n]$ IHAM results against the OHAM results reported in [22] for Re = 50 and $\alpha = 5$.

OHAM	[2, 2]	[2, 3]	[2, 4]
0.00000000	0.00000000	0.00000000	0.00000000
0.00008685	0.00000003	0.00000000	0.00000000
0.00033993	0.00000013	0.00000000	0.00000000
0.00077093	0.00000029	0.00000001	0.00000000
0.00146960	0.00000045	0.00000000	0.00000000
0.00259048	0.00000056	0.00000000	0.00000000
0.00419449	0.00000056	0.00000000	0.00000000
0.00596749	0.00000045	0.00000000	0.00000000
0.00695823	0.00000028	0.00000000	0.00000000
0.00556369	0.00000012	0.00000000	0.00000000
0.00000000	0.00000000	0.00000000	0.00000000

Table 3: Comparison between the numerical results and the order $[m, n]$ IHAM approximate results (using $\hbar = -1$) for $f(y)$ against the HAM results reported in [14] when Re = 110, $\alpha = 3$.

y	HAM [14]	[3, 2]	[3, 3]	[3, 4]	Numerical
0	1.0000000000	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0.1	0.9792357062	0.9792357059	0.9792357065	0.9792357065	0.9792357065
0.2	0.9192658842	0.9192658839	0.9192658856	0.9192658856	0.9192658856
0.3	0.8265336102	0.8265336105	0.8265336123	0.8265336123	0.8265336123
0.4	0.7102211838	0.7102211831	0.7102211833	0.7102211832	0.7102211832
0.5	0.5804994700	0.5804994617	0.5804994588	0.5804994588	0.5804994588
0.6	0.4469350941	0.4469350728	0.4469350671	0.4469350670	0.4469350670
0.7	0.3174084545	0.3174084348	0.3174084276	0.3174084276	0.3174084276
0.8	0.1976410661	0.1976411012	0.1976410945	0.1976410945	0.1976410945
0.9	0.0912302288	0.0912304252	0.0912304211	0.0912304211	0.0912304211
1	0.0000004700	0.0000000000	0.0000000000	0.0000000000	0.0000000000

Table 4: Comparison between the errors of the order $[m, n]$ IHAM results against the HAM results reported in [14] for Re = 110 and $\alpha = 3$.

y	HAM [14]	[3, 2]	[3, 3]	[3, 4]
0	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.1	0.0000000003	0.0000000006	0.0000000000	0.0000000000
0.2	0.0000000014	0.0000000017	0.0000000000	0.0000000000
0.3	0.0000000021	0.0000000018	0.0000000000	0.0000000000
0.4	0.0000000006	0.0000000001	0.0000000001	0.0000000000
0.5	0.0000000112	0.0000000029	0.0000000000	0.0000000000
0.6	0.0000000271	0.0000000058	0.0000000001	0.0000000000
0.7	0.0000000269	0.0000000072	0.0000000000	0.0000000000
0.8	0.0000000284	0.0000000067	0.0000000000	0.0000000000
0.9	0.0000001923	0.0000000041	0.0000000000	0.0000000000
1	0.0000004700	0.0000000000	0.0000000000	0.0000000000

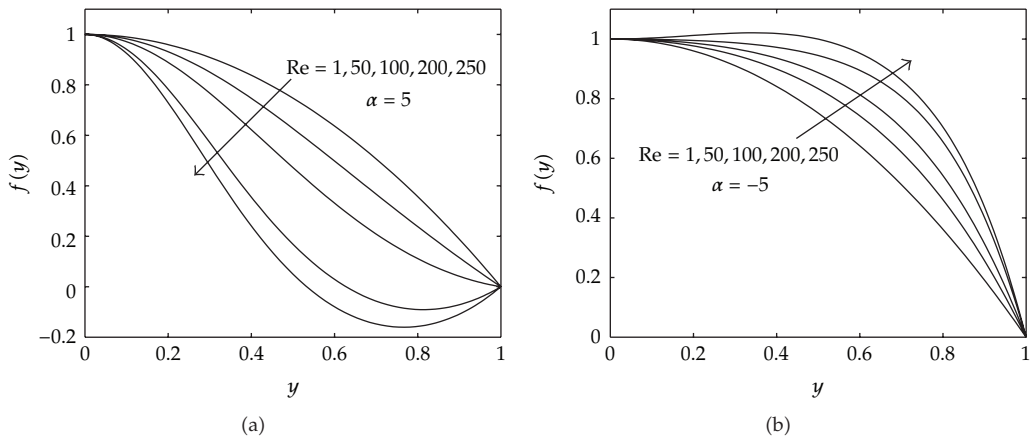

Figure 1: Velocity profile $f(y)$ for different values of Re when $\alpha = 5$ and $\alpha = -5$, respectively.

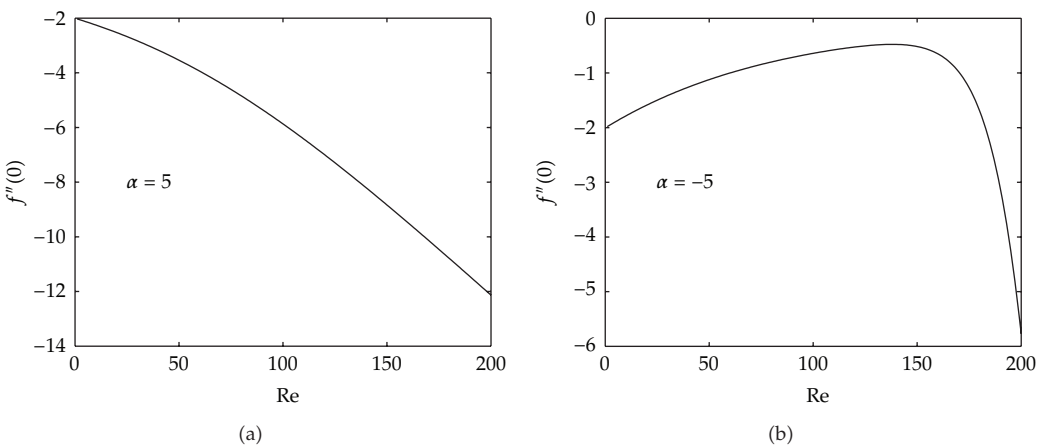

Figure 2: Variation of $f''(0)$ against the Reynolds number when $\alpha = 5$ and $\alpha = -5$, respectively.

Figure 2 shows the effect of the Reynolds number on $f''(0)$, which is related to wall shear stress, for different values of α. It can be seen from the figure that $f''(0)$ decreases monotonically for $\alpha > 0$. In the case of $\alpha < 0$, $f''(0)$ increases for a range of Re until it reaches a peak then decreases.

5. Conclusion

In this brief note we have proposed an improved homotopy analysis method (IHAM) for the solution of general nonlinear differential equations. We have compared the performance of the new algorithm against the optimal homotopy analysis method (OHAM), the standard homotopy analysis method (HAM), and numerical approximations by solving the Jeffery-Hamel problem for Newtonian flow in converging/diverging channels. Numerical computations show that the IHAM is accurate and converges to the numerical approximations at lower orders compared to both the HAM and the OHAM. However, at this juncture, we

cannot say with certainty that this method is better than these other existing methods. We recommend that we need to use this improved homotopy analysis method (IHAM) to solve more nonlinear differential equations.

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References

[1] G. B. Jeffery, "The two-dimensional steady motion of a viscous fluid," *Philosophical Magazine*, vol. 6, no. 29, pp. 455–465, 1915.

[2] G. Hamel, "Spiralförmige Bewgungen Zher Flssigkeiten," *Jahresbericht der Deutschen Mathematiker-Vereinigung*, vol. 25, pp. 34–60, 1916.

[3] W. I. Axford, "The magnetohydrodynamic Jeffrey-Hamel problem for a weakly conducting fluid," *The Quarterly Journal of Mechanics and Applied Mathematics*, vol. 14, pp. 335–351, 1961.

[4] L. Rosenhead, "The steady two-dimensional radial flow of viscous fluid between two inclinedplane walls," *Proceedings of the Royal Society A*, vol. 175, pp. 436–467, 1940.

[5] L. D. Akulenko and S. A. Kumakshev, "Bifurcation of a main steady-state viscous fluid flow in a plane divergent channel," *Izvestiya. Mekhanika Zhidkosti i Gaza*, no. 3, pp. 25–36, 2005.

[6] L. D. Akulenko and S. A. Kumakshev, "Bifurcation of multimode flows of a viscous fluid in a plane diffuser," *Journal of Applied Mathematics and Mechanics*, vol. 72, pp. 296–302, 2008.

[7] S. R. Stow, P. W. Duck, and R. E. Hewitt, "Three-dimensional extensions to Jeffery-Hamel flow," *Fluid Dynamics Research*, vol. 29, no. 1, pp. 25–46, 2001.

[8] A. McAlpine and P. G. Drazin, "On the spatio-temporal development of small perturbations of Jeffery-Hamel flows," *Fluid Dynamics Research*, vol. 22, no. 3, pp. 123–138, 1998.

[9] W. H. H. Banks, P. G. Drazin, and M. B. Zaturska, "On perturbations of Jeffery-Hamel flow," *Journal of Fluid Mechanics*, vol. 186, pp. 559–581, 1988.

[10] O. D. Makinde and P. Y. Mhone, "Temporal stability of small disturbances in MHD Jeffery-Hamel flows," *Computers & Mathematics with Applications*, vol. 53, no. 1, pp. 128–136, 2007.

[11] P. G. Drazin, "Flow through a diverging channel: instability and bifurcation," *Fluid Dynamics Research*, vol. 24, no. 6, pp. 321–327, 1999.

[12] M. A. H. Khan, P. G. Drazin, and Y. Tourigny, "The summation of series in several variables and its applications in fluid dynamics," *Fluid Dynamics Research*, vol. 33, no. 1-2, pp. 191–205, 2003.

[13] G. Domairry, A. Mohsenzadeh, and M. Famouri, "The application of homotopy analysis method to solve nonlinear differential equation governing Jeffery-Hamel flow," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 1, pp. 85–95, 2009.

[14] A. A. Joneidi, G. Domairry, and M. Babaelahi, "Three analytical methods applied to Jeffery-Hamel flow," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 11, pp. 3423–3434, 2010.

[15] Q. Esmaili, A. Ramiar, E. Alizadeh, and D. D. Ganji, "An approximation of the analytical solution of the Jeffery-Hamel flow by decomposition method," *Physics Letters A*, vol. 372, no. 19, pp. 3434–3439, 2008.

[16] Z. Z. Ganji, D. D. Ganji, and M. Esmaeilpour, "Study on nonlinear Jeffery-Hamel flow by He's semi-analytical methods and comparison with numerical results," *Computers & Mathematics with Applications*, vol. 58, no. 11-12, pp. 2107–2116, 2009.

[17] O. D. Makinde and P. Y. Mhone, "Hermite-Padé approximation approach to MHD Jeffery-Hamel flows," *Applied Mathematics and Computation*, vol. 181, no. 2, pp. 966–972, 2006.

[18] S. S. Motsa, P. Sibanda, and S. Shateyi, "A new spectral-homotopy analysis method for solving a nonlinear second order BVP," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 9, pp. 2293–2302, 2010.

[19] S. S. Motsa, P. Sibanda, F. G. Awad, and S. Shateyi, "A new spectral-homotopy analysis method for the MHD Jeffery-Hamel problem," *Computers & Fluids*, vol. 39, no. 7, pp. 1219–1225, 2010.

[20] Z. G. Makukula, P. Sibanda, and S. S. Motsa, "A note on the solution of the von Kármán equations using series and Chebyshev spectral methods," *Boundary Value Problems*, vol. 2010, Article ID 471793, 17 pages, 2010.

[21] P. Sibanda, Z. G. Makukula, and S. S. Motsa, "A novel numerical technique for two-dimensional laminar flow between two moving porous walls," *Mathematical Problems in Engineering*, vol. 2010, Article ID 528956, 15 pages, 2010.

[22] M. Esmaeilpour and D. D. Ganji, "Solution of the Jeffery-Hamel flow problem by optimal homotopy asymptotic method," *Computers & Mathematics with Applications*, vol. 59, no. 11, pp. 3405–3411, 2010.

[23] V. Marinca and N. Herişanu, "Application of optimal homotopy asymptotic method for solving nonlinear equations arising in heat transfer," *International Communications in Heat and Mass Transfer*, vol. 35, no. 6, pp. 710–715, 2008.

[24] V. Marinca, N. Herişanu, C. Bota, and B. Marinca, "An optimal homotopy asymptotic method applied to the steady flow of a fourth-grade fluid past a porous plate," *Applied Mathematics Letters*, vol. 22, no. 2, pp. 245–251, 2009.

[25] V. Marinca and N. Herisanu, "Optimal homotopy perturbation method for strongly nonlinear differentialequations," *Nonlinear Science Letters A*, vol. 1, no. 3, pp. 273–280, 2010.

[26] M. Idrees, S. Islam, S. Haq, and S. Islam, "Application of the optimal homotopy asymptotic method to squeezing flow," *Computers & Mathematics with Applications*, vol. 59, no. 12, pp. 3858–3866, 2010.

[27] S. Islam, R. A. Shah, and I. Ali, "Couette and Poiseuille flows for fourth grade fluids using optimal homotopy asymptotic method," *World Applied Sciences Journal*, vol. 9, no. 11, pp. 1228–1236, 2010.

[28] J. Ali, S. Islam, S. Islam, and G. Zaman, "The solution of multipoint boundary value problems by the optimal homotopy asymptotic method," *Computers & Mathematics with Applications*, vol. 59, no. 6, pp. 2000–2006, 2010.

[29] S. Liao, *Beyond Perturbation: Introduction to Homotopy Analysis Method*, vol. 2 of *CRC Series: Modern Mechanics and Mathematics*, Chapman & Hall/CRC, Boca Raton, Fla, USA, 2004.

[30] S. Liao, "Notes on the homotopy analysis method: some definitions and theorems," *Communications in Nonlinear Science and Numerical Simulation*, vol. 14, no. 4, pp. 983–997, 2009.

[31] J. Kierzenka and L. F. Shampine, "A BVP solver based on residual control and the MATLAB PSE," *ACM Transactions on Mathematical Software*, vol. 27, no. 3, pp. 299–316, 2001.

[32] L. F. Shampine, I. Gladwell, and S. Thompson, *Solving ODEs with MATLAB*, Cambridge University Press, Cambridge, UK, 2003.