Research Article

On the Solution n -Dimensional of the Product $\otimes^k$ Operator and Diamond Bessel Operator

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Firstly, we studied the solution of the equation $\otimes^k \diamond_B^k u(x) = f(x)$ where $u(x)$ is an unknown function for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $f(x)$ is the generalized function, k is a positive integer. Finally, we have studied the solution of the nonlinear equation $\otimes^k \diamond_B^k u(x) = f(x, \square^{k-1} L^k \Delta_B^k \square_B^k u(x))$. It was found that the existence of the solution $u(x)$ of such an equation depends on the condition of f and $\square^{k-1} L^k \Delta_B^k \square_B^k u(x)$. Moreover such solution $u(x)$ is related to the inhomogeneous wave equation depending on the conditions of p, q , and k .

1. Introduction

The operator $\diamond^k$ has been first introduced by Kananthai (see [1]), is named as the Diamond operator iterated k -times, and is defined by

$$\diamond^k = \left(\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right)^k, \quad p+q=n. \quad (1.1)$$

n is the dimension of the space $\mathbb{R}^n$, for $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and k is a nonnegative integer. The operator $\diamond^k$ can be expressed in the form $\diamond^k = \Delta^k \square^k = \square^k \Delta^k$, where Δ^k is the Laplacian operator iterated k -times defined by

$$\Delta^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \dots + \frac{\partial^2}{\partial x_n^2} \right)^k, \quad (1.2)$$

and $\square^k$ is the ultrahyperbolic operator iterated k -times defined by

$$\square^k = \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \right)^k. \quad (1.3)$$

Kananthai (see [1, Theorem 3.1, page 33]) has shown that the convolution $(-1)^k R_{2k}^e(x) * R_{2k}^H(x)$ is an elementary solution of the operator $\diamond^k$, that is,

$$\diamond^k \left((-1)^k R_{2k}^e(x) * R_{2k}^H(x) \right) = \delta(x). \quad (1.4)$$

Next, Kananthai (see [2]) has studied the linear equation

$$\diamond^k u(x) = f(x). \quad (1.5)$$

This equation is the generalization of the ultrahyperbolic equation and it can be applied to the wave equation. We obtain $u(x) = (-1)^k M_{2k,2k}(x) * f(x)$ as a solution of such an equation (1.5) where

$$M_{2k,2k} = R_{2k}^H(x) * R_{2k}^e(x). \quad (1.6)$$

The function $R_{2k}^H(x)$ is called the ultrahyperbolic kernel defined by (2.2) and $R_{2k}^e(x)$ is called the elliptic kernel defined by (2.8), with $\alpha = 2k$.

Furthermore, Yıldırım et al. (see [3]) first introduced the $\diamond_B^k$ operator that is named as Diamond Bessel operator, where $\diamond_B^k$ is defined by

$$\diamond_B^k = \left[\left(\sum_{i=1}^p B_{x_i} \right)^2 - \left(\sum_{j=p+1}^{p+q} B_{x_j} \right)^2 \right]^k, \quad (1.7)$$

and $B_{x_i} = \partial^2 / \partial x_i^2 + (2v_i / x_i)(\partial / \partial x_i)$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -1/2$, $x_i > 0$. The operator $\diamond_B^k$ can be expressed by $\diamond_B^k = \Delta_B^k \square_B^k = \square_B^k \Delta_B^k$, where

$$\Delta_B^k = \left(\sum_{i=1}^n B_{x_i} \right)^k. \quad (1.8)$$

$$\square_B^k = \left(\sum_{i=1}^p B_{x_i} - \sum_{j=p+1}^{p+q} B_{x_j} \right)^k. \quad (1.9)$$

Next, W. Satsanit has first introduced $\otimes^k$ operator and $\otimes^k$ is defined by

$$\begin{aligned}
 \otimes^k &= \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^3 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^3 \right]^k \\
 &= \left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^k \left[\left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right)^2 + \left(\sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \right) \cdot \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right) + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \right)^2 \right]^k \\
 &= \square^k \left(\Delta^2 - \frac{1}{4}(\Delta + \square)(\Delta - \square) \right)^k \\
 &= \left(\frac{3}{4} \diamond \Delta + \frac{1}{4} \square^3 \right)^k,
 \end{aligned} \tag{1.10}$$

where $\diamond$, Δ , and $\square$ are defined by (1.1), (1.2), and (1.3) with $k = 1$, respectively.

Now, firstly, the purpose of this work is to study the equation

$$\otimes^k \diamond_B^k u(x) = f(x), \tag{1.11}$$

where the operator $\otimes^k$ is defined by (1.10) and $\diamond_B^k$ defined by (1.7), $f(x)$ is a generalized function and $u(x)$ is an unknown function. Finally we study the equation

$$\otimes^k \diamond_B^k u(x) = f\left(x, \square^{k-1} L^k \Delta_B^k \square_B^k u(x)\right) \tag{1.12}$$

with f having a continuous first derivative for all $x \in \Omega \cup \partial\Omega$, where Ω is an open subset of $\mathbb{R}^n$, and $\partial\Omega$ denotes the boundary of Ω , f is bounded on Ω , that is, $|f| \leq N$, N is constant, as well as $\square^{k-1}$, L^k , Δ_B^k , and $\square_B^k$ are defined by (1.3), (2.46), (1.8) and (1.9), respectively.

We can find the solution $u(x)$ of (1.12) that is unique under the boundary condition $\square^{k-1} L^k \Delta_B^k \square_B^k u(x) = 0$ for $x \in \partial\Omega$. By [4, page 369] there exists a unique solution $W(x)$ of the equation $\square W(x) = f(x, W(x))$ for all $x \in \Omega$ with the boundary condition $W(x) = 0$ for all $x \in \partial\Omega$ where $W(x) = \square^{k-1} L^k \Delta_B^k \square_B^k u(x)$.

Moreover, if we put $p = k = 1$ in $\square^k \square_B^k M(x) = W(x)$, then we found that $M(x) = I_2^H(x) * I_2(x) * W(x)$ is a solution of the inhomogeneous equation where $I_2^H(x)$ and $I_2(x)$ are defined by (2.6) and (2.20) with $\alpha = 2, \gamma = 2$, respectively.

Before going into details, the following definitions and some important concepts are needed.

2. Preliminaries

Definition 2.1. Let $x = (x_1, x_2, \dots, x_n)$ be a point of the n -dimensional Euclidean space $\mathbb{R}^n$, denoted by

$$v = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2. \tag{2.1}$$

The nondegenerated quadratic form $p + q = n$ is the dimension of the space $\mathbb{R}^n$. Let $\Gamma_+ = \{x \in \mathbb{R}^n : x_1 > 0 \text{ and } u > 0\}$ be the interior of forward cone and let $\bar{\Gamma}_+$ denote its closure. For any complex number α , define the function

$$R_\alpha^H(v) = \begin{cases} \frac{v^{(\alpha-n)/2}}{K_n(\alpha)}, & \text{for } x \in \Gamma_+, \\ 0, & \text{for } x \notin \Gamma_+, \end{cases} \quad (2.2)$$

where the constant $K_n(\alpha)$ is given by the formula

$$K_n(\alpha) = \frac{\pi^{(n-1)/2} \Gamma((2 + \alpha - n)/2) \Gamma((1 - \alpha)/2) \Gamma(\alpha)}{\Gamma((2 + \alpha - p)/2) \Gamma((p - \alpha)/2)}. \quad (2.3)$$

The function $R_\alpha^H(v)$ is called the ultrahyperbolic kernel of Marcel Riesz and was introduced by Nozaki (see [5]).

It is well known that $R_\alpha^H(v)$ is an ordinary function if $\text{Re}(\alpha) \geq n$ and is a distribution of α if $\text{Re}(\alpha) < n$. Let $\text{supp } R_\alpha^H(v)$ denote the support of $R_\alpha^H(v)$ and suppose that $\text{supp } R_\alpha^H(v) \subset \bar{\Gamma}_+$, that is, $\text{supp } R_\alpha^H(v)$ is compact.

From Trione (see [6, page 11]), $R_{2k}^H(v)$ is an elementary solution of the operator $\square^k$, that is,

$$\square^k R_{2k}^H(v) = \delta(x). \quad (2.4)$$

By putting $p = 1$ in $R_{2k}^H(v)$ and taking into account Legendre's duplication formula for $\Gamma(z)$, that is

$$\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right), \quad (2.5)$$

we obtain

$$I_\alpha^H(v) = \frac{v^{(\alpha-n)/2}}{H_n(\alpha)}, \quad (2.6)$$

and $v = x_1^2 - x_2^2 - x_3^2 \cdots - x_n^2$ where

$$H_n(\alpha) = \pi^{(n-2)/2} 2^{\alpha-1} \Gamma\left(\frac{\alpha + 2 - n}{2}\right) \Gamma\left(\frac{\alpha}{2}\right). \quad (2.7)$$

$I_\alpha^H(v)$ is the hyperbolic kernel of Marcel Riesz.

Definition 2.2. Let $x = (x_1, x_2, \dots, x_n)$ be a point of $\mathbb{R}^n$ and $\omega = x_1^2 + x_2^2 + \cdots + x_n^2$, then the function $R_\alpha^e(\omega)$ denoted the elliptic kernel of Marcel Riesz and is defined by

$$R_\alpha^e(\omega) = \frac{\omega^{(\alpha-n)/2}}{W_n(\alpha)}, \quad (2.8)$$

where

$$\omega = x_1^2 + x_2^2 + \cdots + x_n^2, \quad (2.9)$$

$$W_n(\alpha) = \frac{\pi^{n/2} 2^\alpha \Gamma(\alpha/2)}{\Gamma((n-\alpha)/2)} \quad (2.10)$$

where α is a complex parameter and n is the dimension of $\mathbb{R}^n$.

It can be shown that $R_{-2k}^e(x) = (-1)^k \Delta^k \delta(x)$ where Δ^k is defined by (1.2). It follows that $R_0^e(x) = \delta(x)$, (see [7, page 118]).

Moreover, we obtain $(-1)^k R_{2k}^e(x)$ is an elementary solution of the operator Δ^k (see [8, Lemma 2.4, page 31]). That is

$$\Delta^k \left((-1)^k R_{2k}^e(x) \right) = \delta(x). \quad (2.11)$$

By (2.2) and (2.3) with $q = 0$, then $v^{(\alpha-n)/2}$ reduces to $\omega_p^{(\alpha-p)/2}$ where $\omega_p = x_1^2 + x_2^2 + \cdots + x_p^2$ and $K_n(\alpha)$ reduces to $K_p(\alpha) = (\pi^{(p-1)/2} \Gamma((1-\alpha)/2) \Gamma(\alpha)) / \Gamma((p-\alpha)/2)$. By using the formula

$$\begin{aligned} \Gamma(2z) &= 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right), \\ \Gamma\left(\frac{1}{2} + z\right) \Gamma\left(\frac{1}{2} - z\right) &= \pi \sec(\pi z), \end{aligned} \quad (2.12)$$

we obtain

$$K_p(\alpha) = \frac{1}{2} \sec\left(\frac{\pi\alpha}{2}\right) W_p(\alpha), \quad (2.13)$$

where $W_p(\alpha)$ is defined by (2.10) with $n = p$. Thus, for $q = 0$,

$$R_\alpha^H(v) = \frac{v^{(\alpha-p)/2}}{K_p(\alpha)} = 2 \cos\left(\frac{\pi\alpha}{2}\right) \frac{v^{(\alpha-p)/2}}{W_p(\alpha)} = 2 \cos\left(\frac{\pi\alpha}{2}\right) R_\alpha^e(\omega_p), \quad (2.14)$$

where $\omega_p = x_1^2 + x_2^2 + \cdots + x_p^2$. Thus, if $\alpha = 2k$, then

$$R_{2k}^H(\omega_p) = 2(-1)^k R_{2k}^e(\omega_p) \quad (2.15)$$

for $q = 0$ and $\omega_p = x_1^2 + x_2^2 + \cdots + x_p^2$.

Definition 2.3. Let $x = (x_1, x_2, \dots, x_n)$, $v = (v_1, v_2, \dots, v_n) \in \mathbb{R}_n^+$. For any complex number α , we define the function $S_\alpha(x)$ by

$$S_\alpha(x) = \frac{2^{n+2|v|-2\alpha} \Gamma((n+2|v|-\alpha)/2) |x|^{\alpha-n-2|v|}}{\prod_{i=1}^n 2^{v_i-1/2} \Gamma(v_i+1/2)}. \quad (2.16)$$

Definition 2.4. Let $x = (x_1, x_2, \dots, x_n)$, $v = (v_1, v_2, \dots, v_n) \in \mathbb{R}_n^+$, and $V = x_1^2 + x_2^2 + \dots + x_p^2 - x_{p+1}^2 - x_{p+2}^2 - \dots - x_{p+q}^2$ the nondegenerated quadratic form. Denote the interior of the forward cone by $\Gamma_+ = \{x \in \mathbb{R}_n^+ : x_1 > 0, x_2 > 0, \dots, x_n > 0, V > 0\}$. The function $R_\gamma(x)$ is defined by

$$R_\gamma(x) = \frac{V^{(\gamma-n-2|v|)/2}}{K_n(\gamma)}, \quad (2.17)$$

where

$$K_n(\gamma) = \frac{\pi^{(n+2|v|-1)/2} \Gamma((2+\gamma-n-2|v|)/2) \Gamma((1-\gamma)/2) \Gamma(\gamma)}{\Gamma((2+\gamma-p-2|v|)/2) \Gamma((p-2|v|-\gamma)/2)}, \quad (2.18)$$

and γ is a complex number. By putting $p = 1$ in $R_\gamma(x)$ and taking into account Legendre's duplication formula for $\Gamma(z)$, that is,

$$\Gamma(2z) = 2^{2z-1} \pi^{-1/2} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right) \quad (2.19)$$

we obtain

$$I_\gamma(x) = \frac{V^{(\gamma-n-2|v|)/2}}{N_n(\gamma)}, \quad (2.20)$$

and $V = x_1^2 - x_2^2 - x_3^2 \dots - x_n^2$ where

$$N_n(\gamma) = \pi^{(n+2|v|-1)/2} 2^{2k-1} \Gamma\left(\frac{2+\gamma-n-2|v|}{2}\right) \Gamma\left(\frac{\gamma}{2}\right). \quad (2.21)$$

Lemma 2.5. Given the equation $\Delta_B^k u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+$, where Δ_B^k is defined by (1.8), then

$$u(x) = (-1)^k S_{2k}(x), \quad (2.22)$$

where $S_{2k}(x)$ is defined by (2.16), with $\alpha = 2k$.

Proof. (See [3, page 379] and [9]). □

Lemma 2.6. Given the equation $\square_B^k u(x) = \delta(x)$ for $x \in \mathbb{R}_n^+$, where $\square_B^k$ is defined by (1.9), then

$$u(x) = R_{2k}(x), \quad (2.23)$$

where $R_{2k}(x)$ is defined by (2.17), with $\gamma = 2k$.

Proof. (See [3, page 379] and [9]). □

Lemma 2.7. *Given that P is a hyperfunction, then*

$$P\delta^k(p) + k\delta^{(k-1)}(p) = 0, \quad (2.24)$$

where $\delta^{(k)}$ is the Dirac-delta distribution with k -derivatives.

Proof. (See [8, page 233]). □

Lemma 2.8. *Given the equation*

$$\square^k u(x) = 0, \quad (2.25)$$

where $\square^k$ is defined by (1.3) and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, then $u(x) = (R_{2(k-1)}^H(v))^{(m)}$ is a solution of (2.25) with $m = (n-4)/2$, $n \geq 4$ and n is even dimension. The function $(R_{2(k-1)}^H(v))^{(m)}$ is defined by (2.2) with m -derivatives, $\alpha = 2(k-1)$, and v being defined by (2.1).

Proof. We first show the generalized function $\delta^{(m)}(r^2 - s^2)$ where $r^2 = x_1^2 + x_2^2 + \dots + x_p^2$ and $s^2 = x_{p+1}^2 + x_{p+2}^2 + \dots + x_{p+q}^2$, $p+q = n$, is a solution of the equation

$$\square u(x) = 0, \quad (2.26)$$

where $\square$ is defined by (1.3) with $k = 1$ and $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$,

$$\begin{aligned} \frac{\partial}{\partial x_i} \delta^{(m)}(r^2 - s^2) &= 2x_i \delta^{(m+1)}(r^2 - s^2), \\ \frac{\partial^2}{\partial x_i^2} \delta^{(m)}(r^2 - s^2) &= 2\delta^{(m+1)}(r^2 - s^2) + 4x_i^2 \delta^{(m+2)}(r^2 - s^2), \\ \square \delta^{(m)}(r^2 - s^2) &= \sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \delta^{(m)}(r^2 - s^2) \\ &= 2p\delta^{(m+1)}(r^2 - s^2) + 4r^2 \delta^{(m+2)}(r^2 - s^2) \\ &= 2p\delta^{(m+1)}(r^2 - s^2) + 4(r^2 - s^2)\delta^{(m+2)}(r^2 - s^2) + 4s^2 \delta^{(m+2)}(r^2 - s^2) \\ &= 2p\delta^{(m+1)}(r^2 - s^2) - 4(m+2)\delta^{(m+1)}(r^2 - s^2) + 4s^2 \delta^{(m+2)}(r^2 - s^2) \\ &= (2p - 4(m+2))\delta^{(m+1)}(r^2 - s^2) + 4s^2 \delta^{(m+2)}(r^2 - s^2). \end{aligned} \quad (2.27)$$

By Lemma 2.5 with $P = r^2 - s^2$, similarly,

$$\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2} \delta^{(m)}(r^2 - s^2) = (-2q + 4(m+2))\delta^{(m+1)}(r^2 - s^2) + 4r^2 \delta^{(m+2)}(r^2 - s^2). \quad (2.28)$$

Thus

$$\begin{aligned}
\Box \delta^{(m)}(r^2 - s^2) &= \sum_{i=1}^p \frac{\partial^2}{\partial x_i^2} \delta^{(m)}(r^2 - s^2) - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_i^2} \delta^{(m)}(r^2 - s^2) \\
&= (2(p+q) - 8(m+2)) \delta^{(m+1)}(r^2 - s^2) - 4(r^2 - s^2) \delta^{(m+2)}(r^2 - s^2) \\
&= (2n - 8(m+2)) \delta^{(m+1)}(r^2 - s^2) + 4(m+2) \delta^{(m+1)}(r^2 - s^2) \\
&= (2n - 4(m+2)) \delta^{(m+1)}(r^2 - s^2).
\end{aligned} \tag{2.29}$$

If $2n - 4(m+2) = 0$, then we have $\Box \delta^{(m)}(r^2 - s^2) = 0$. That is, $u(x) = \delta^{(m)}(r^2 - s^2)$ is a solution of (2.26) with $m = (n-4)/2$, $n \geq 4$ and n is even dimension. We write

$${}^k u(x) = \Box({}^{k-1} u(x)) = 0, \tag{2.30}$$

and from the above proof we have $\Box^{k-1} u(x) = \delta^{(m)}(r^2 - s^2)$ with $m = (n-4)/2$, $n \geq 4$ and n is even dimension. Convolving the above equation by $R_{2(k-1)}^H(v)$, we obtain

$$\begin{aligned}
R_{2(k-1)}^H(v) * \Box^{k-1} u(x) &= R_{2(k-1)}^H(v) * \delta^{(m)}(r^2 - s^2) \\
\Box^{k-1} (R_{2(k-1)}^H(v)) * u(x) &= (R_{2(k-1)}^H(v))^{(m)}, \quad \text{where } v = (r^2 - s^2) \\
\delta * u(x) = u(x) &= (R_{2(k-1)}^H(v))^{(m)}
\end{aligned} \tag{2.31}$$

by (2.2), and $v = r^2 - s^2$ is defined by Definition (2.1).

Thus $u(x) = (R_{2(k-1)}^H(v))^{(m)}$ is a solution of (2.25) with $m = (n-4)/2$, $n \geq 4$ and n is even dimension. $\square$

Lemma 2.9. *Given the equation*

$$\otimes^k G(x) = \delta(x), \tag{2.32}$$

then

$$G(x) = (R_{6k}^H(x) * (-1)^{2k} R_{4k}^e(x)) * (O^{*k}(x))^{*-1} \tag{2.33}$$

is an elementary solution for the $\otimes^k$ operator iterated k -times where $\otimes^k$ is defined by (1.10), and

$$O(x) = \frac{3}{4} R_4^H(x) + \frac{1}{4} (-1)^2 R_4^e(x) \tag{2.34}$$

where $O^{*k}(x)$ denotes the convolution of $O(x)$ itself k -times and $(O^{*k}(x))^{*-1}$ denotes the inverse of $O^{*k}(x)$ in the convolution algebra. Moreover $G(x)$ is a tempered distribution.

Proof. From (3.1), we have

$$\otimes^k G(x) = \left(\frac{3}{4} \diamond \Delta + \frac{1}{4} \square^3 \right)^k G(x) = \delta(x), \quad (2.35)$$

or we can write

$$\left(\frac{3}{4} \diamond \Delta + \frac{1}{4} \square^3 \right) \left(\frac{3}{4} \diamond \Delta + \frac{1}{4} \square^3 \right)^{k-1} G(x) = \delta(x). \quad (2.36)$$

Convolving both sides of the above equation by $R_6^H(x) * (-1)^2 R_4^e(x)$,

$$\left(\frac{3}{4} \diamond \Delta + \frac{1}{4} \square^3 \right) * \left(R_6^H(x) * (-1)^2 R_4^e(x) \right) \left(\frac{3}{4} \diamond \Delta + \frac{1}{4} \square^3 \right)^{k-1} G(x) = \delta(x) * R_6^H(x) * (-1)^2 R_4^e(x), \quad (2.37)$$

or

$$\begin{aligned} & \left(\frac{3}{4} \square \left(R_2^H(x) \right) * \Delta^2 (-1)^2 R_4^e(x) * R_4^H(x) + \frac{1}{4} \square^3 R_6^H(x) * (-1)^2 R_4^e(x) \right) \\ & * \left(\frac{3}{4} \diamond \Delta + \frac{1}{4} \square^3 \right)^{k-1} G(x) = \delta(x) * R_6^H(x) * (-1)^2 R_4^e(x). \end{aligned} \quad (2.38)$$

By (2.4) and (2.8), we obtain

$$\left(\frac{3}{4} \delta * \delta * R_4^H(x) + \frac{1}{4} \delta * (-1)^2 R_4^e(x) \right) * \left(\frac{3}{4} \diamond \Delta + \frac{1}{4} \square^3 \right)^{k-1} G(x) = \delta(x) * R_6^H(x) * (-1)^2 R_4^e(x). \quad (2.39)$$

Thus

$$\left(\frac{3}{4} R_4^H(x) + \frac{1}{4} (-1)^2 R_4^e(x) \right) * \left(\frac{3}{4} \diamond \Delta + \frac{1}{4} \square^3 \right)^{k-1} G(x) = R_6^H(x) * (-1)^2 R_4^e(x). \quad (2.40)$$

Keeping on convolving both sides of the above equation by $R_6^H(x) * (-1)^2 R_4^e(x)$ up to $k-1$ times, we obtain

$$O^{*k}(x) * G(x) = \left(R_6^H(x) * (-1)^2 R_4^e(x) \right)^{*k} \quad (2.41)$$

where the symbol $*k$ denotes the convolution of itself k -times. By properties of $R_\alpha(x)$, we have

$$\left(R_6^H(x) * (-1)^2 R_4^e(x) \right)^{*k} = R_{6k}^H(x) * (-1)^{2k} R_{4k}^e(x). \quad (2.42)$$

Thus,

$$O^{*k}(x) * G(x) = R_{6k}^H(x) * (-1)^{2k} R_{4k}^e(x). \quad (2.43)$$

Now, consider the function $O^{*k}(x)$, since $R_6^H(x) * (-1)^2 R_4^e(x)$ is a tempered distribution. Thus $O(x)$ defined by (2.34) is a tempered distribution, and we obtain that $O^{*k}(x)$ is a tempered distribution and $R_{6k}^H(x) * (-1)^{2k} R_{4k}^e(x) \in \mathcal{S}'$ is the space of tempered distribution. Choose $\mathcal{S}' \subset \mathfrak{D}'_{\mathcal{R}}$ where $\mathfrak{D}'_{\mathcal{R}}$ is the right-side distribution which is a subspace of $\mathfrak{D}'$ of distribution.

Thus $R_{6k}^H(x) * (-1)^{2k} R_{4k}^e(x) \in \mathfrak{D}'_{\mathcal{R}}$. It follows that $R_{6k}^H(x) * (-1)^{2k} R_{4k}^e(x)$ is an element of convolution algebra, since $\mathfrak{D}'_{\mathcal{R}}$ is a convolution algebra. Hence by the method of Zemanian (see [10]), (2.33) has a unique solution

$$G(x) = \left(R_{6k}^H(x) * (-1)^{2k} R_{4k}^e(x) \right) * \left(O^{*k}(x) \right)^{*^{-1}}, \quad (2.44)$$

where $(O^{*k}(x))^{*^{-1}}$ is an inverse of $O^{*k}(x)$ in the convolution algebra and $G(x)$ is called the Green function of the $\otimes^k$ operator. $\square$

Lemma 2.10. *Given the equation*

$$L^k K(x) = \delta(x), \quad (2.45)$$

where L^k is the operator defined by

$$L^k = \left(\frac{3}{4} \Delta^2 + \frac{1}{4} \square^2 \right)^k \quad (2.46)$$

and Δ and $\square$ are defined by (1.2) and (1.3) with $k = 1$, respectively, one obtains that $K(x)$ is an elementary solution of the L^k operator where

$$\begin{aligned} K(x) &= \left(R_{4k}^H(x) * (-1)^{2k} R_{4k}^e(x) \right) * \left(O^{*k}(x) \right)^{*^{-1}}, \\ O(x) &= \frac{3}{4} R_4^H(x) + \frac{1}{4} (-1)^2 R_4^e(x), \end{aligned} \quad (2.47)$$

where $O^{*k}(x)$ denotes the convolution of $O(x)$ itself k -times and $(O^{*k}(x))^{*^{-1}}$ denotes the inverse of $O^{*k}(x)$ in the convolution algebra. Moreover $K(x)$ is a tempered distribution.

Proof. The proof of Lemma 2.10 is similar to the proof of Lemma 2.9. $\square$

Lemma 2.11. *Given the equation*

$$\square u(x) = f(x, u(x)), \quad (2.48)$$

where f is defined and has continuous first derivatives for all $x \in \Omega \cup \partial\Omega$, where Ω is an open subset of $\mathbb{R}^n$ and $\partial\Omega$ is the boundary of Ω , assume that f is bounded, that is, $|f(x, u(x))| \leq N$ for all $x \in \Omega$.

Then one obtains a continuous function $u(x)$ as unique solution of (2.48) with the boundary condition $u(x) = 0$ for $x \in \partial\Omega$.

Proof. We can prove the existence of the solution $u(x)$ of (2.48) by the method of iterations and Schuder's estimates. The details of the proof are given by Courant and Hilbert; (see [4, pages 369–372]). $\square$

Lemma 2.12. *The function $R_{-2k}^H(x)$ and $S_{-2k}(x)$ are the inverse of the convolution algebra of R_{2k}^H and S_{2k} , respectively, that is,*

$$\begin{aligned} R_{-2k}^H(x) * R_{2k}^H(x) &= R_{-2k+2k}^H(x) = R_0^H(x) = \delta, \\ S_{-2k}(x) * S_{2k}(x) &= S_{-2k+2k}(x) = S_0(x) = \delta. \end{aligned} \quad (2.49)$$

Proof. (See [7, page 158] and [11]). $\square$

3. Main Results

Theorem 3.1. *Given the equation*

$$\otimes^k \diamond_B^k u(x) = 0, \quad (3.1)$$

where $\otimes^k$ is the Otimes operator iterated k -times and $\diamond_B^k$ is Diamond Bessel operator iterated k -times defined by (1.10) and (1.7), respectively, and $u(x)$ is an unknown function, one obtains that $u(x)$ is a solution of (3.1) where

$$u(x) = K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * (-1)^{k-1} (R_{2(k-1)}^H(v))^{(m)} \quad (3.2)$$

where $K(x)$ is defined by (2.47), as well as $S_{2k}(x)$, $R_{2k}(x)$, and $(R_{2(k-1)}^H(v))^{(m)}$ are defined by (2.16), (2.17), and (2.2) with $\alpha = 2k$, $\gamma = 2k$ and $\alpha = 2(k-1)$, respectively.

Proof. Since

$$\otimes^k = \left(\frac{3}{4} \diamond \Delta + \frac{1}{4} \square^3 \right)^k, \quad \diamond_B^k = \Delta_B^k \square_B^k. \quad (3.3)$$

Consider the homogeneous equation

$$\otimes^k \diamond_B^k u(x) = 0. \quad (3.4)$$

The above equation can be written as

$$\left(\frac{3}{4} \diamond \Delta + \frac{1}{4} \square^3 \right)^k \Delta_B^k \square_B^k u(x) = 0, \quad (3.5)$$

or

$$\square^k \left(\frac{3}{4} \Delta^2 + \frac{1}{4} \square^2 \right)^k \Delta_B^k \square_B^k u(x) = 0. \quad (3.6)$$

That is,

$$\square^k L^k \Delta_B^k \square_B^k u(x) = 0, \quad (3.7)$$

where $\square^k, L^k, \Delta_B^k$, and $\square_B^k$ are defined by (1.3), (2.46), (1.8), and (1.9), respectively. By Lemma 2.8, we obtain

$$L^k \Delta_B^k \square_B^k u(x) = (R_{2(k-1)}^H(v))^{(m)}. \quad (3.8)$$

Since $(-1)^k S_{2k}(x), R_{2k}(x)$ are the elementary solution of the operators Δ_B^k and $\square_B^k$, respectively, and by Lemma 2.10, we have that $K(x)$ is an elementary of the operator L^k defined by (2.46), that is,

$$\begin{aligned} \Delta_B^k (-1)^k S_{2k}(x) &= \delta(x), & \square_B^k R_{2k}(x) &= \delta(x), \\ L^k K(x) &= \delta(x). \end{aligned} \quad (3.9)$$

Convolving both sides of (3.8) by $K(x) * (-1)^k S_{2k}(x) * R_{2k}(x)$, we obtain

$$K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * L^k \Delta_B^k \square_B^k u(x) = K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * (R_{2(k-1)}^H(v))^{(m)}. \quad (3.10)$$

By properties of convolution

$$L^k K(x) * \Delta_B^k (-1)^k S_{2k}(x) * \square_B^k R_{2k}(x) * u(x) = K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * (R_{2(k-1)}^H(v))^{(m)}. \quad (3.11)$$

By Lemmas 2.10, 2.5, and 2.6, we obtain

$$\delta(x) * \delta(x) * \delta(x) * u(x) = K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * (R_{2(k-1)}^H(v))^{(m)}. \quad (3.12)$$

Thus

$$u(x) = K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * (R_{2(k-1)}^H(v))^{(m)} \quad (3.13)$$

is the solution of (3.1). □

Theorem 3.2. *Given the equation*

$$\otimes^k \diamond_B^k u(x) = f(x), \quad (3.14)$$

where $\otimes^k$ is the Otimes operator iterated k -times defined by (1.10), and $\diamond_B^k$ is the Diamond Bessel operator iterated k -times defined by (1.7), $f(x)$ is the generalized function, $u(x)$ is an unknown function, $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and n is even,

One obtains that

$$u(x) = K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * \left(R_{2(k-1)}^H(v) \right)^{(m)} + G(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * f(x) \quad (3.15)$$

is a general solution of (3.14) and $G(x)$ is defined by (2.33), $K(x)$ is defined by (2.47), as well as $S_{2k}(x)$ and $R_{2k}(x)$ are defined by (2.16) and (2.17) with $\alpha = 2k$ and $\gamma = 2k$, respectively.

Proof. Consider the equation

$$\otimes^k \diamond_B^k u(x) = f(x) \quad (3.16)$$

or

$$\otimes^k \Delta_B^k \square_B^k u(x) = f(x). \quad (3.17)$$

Convolving both sides of (3.14) by $G(x) * (-1)^k S_{2k}(x) * R_{2k}(x)$, we obtain

$$G(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * \otimes^k \Delta_B^k \square_B^k u(x) = G(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * f(x). \quad (3.18)$$

By properties of convolution,

$$\otimes^k G(x) * \Delta_B^k (-1)^k S_{2k}(x) * \square_B^k R_{2k}(x) * u(x) = G(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * f(x). \quad (3.19)$$

By Lemmas 2.9, 2.5, and 2.6, we obtain

$$\delta(x) * \delta(x) * \delta(x) * u(x) = G(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * f(x). \quad (3.20)$$

Thus

$$u(x) = G(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * f(x). \quad (3.21)$$

Consider the homogeneous equation

$$\otimes^k \diamond_B^k u(x) = 0. \quad (3.22)$$

By Theorem 3.1, we have a homogeneous solution

$$u(x) = K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * \left(R_{2(k-1)}^H(v) \right)^{(m)}. \quad (3.23)$$

Thus, the general solution of (3.14) is

$$u(x) = K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * \left(R_{2(k-1)}^H(v) \right)^{(m)} + G(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * f(x), \quad (3.24)$$

as required. $\square$

Theorem 3.3. Consider the nonlinear equation

$$\otimes^k \diamond_B^k u(x) = f\left(x, \square^{k-1} L^k \Delta_B^k \square_B^k u(x)\right) \quad (3.25)$$

where $\otimes^k, \diamond_B^k, \square^{k-1}, L^k, \Delta_B^k$, and $\square_B^k$ are defined by (1.10), (1.7), (1.3), (2.44), and (1.9), respectively. Let f be defined, and having continuous first derivative for all $x \in \Omega \cup \partial\Omega$, Ω is an open subset of R^n and $\partial\Omega$ denotes the boundary function, that is,

$$\left| f\left(x, \square^{k-1} L^k \Delta_B^k \square_B^k u(x)\right) \right| \leq N \quad (3.26)$$

for all $x \in \Omega$ and the boundary condition

$$\square^{k-1} L^k \Delta_B^k \square_B^k u(x) = 0 \quad (3.27)$$

for all $x \in \partial\Omega$. Then one obtains

$$u(x) = R_{2(k-1)}^H(x) * G(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * W(x) \quad (3.28)$$

as a solution of (3.25) with the boundary condition

$$u(x) = (R_{2(k-2)}^H(v))^{(m)} * G(x) * (-1)^k S_{2k}(x) * R_{2k}(x) \quad (3.29)$$

for all $x \in \partial\Omega, m = (n-4)/2$, and $W(x)$ is a continuous function for $x \in \Omega \cup \partial\Omega$, while $R_{2(k-2)}^H(v), S_{2k}(x)$, and $R_{2k}(x)$ are given by (2.2), (2.16), and (2.17) with $\alpha = 2(k-2), \alpha = 2k$, and $\gamma = 2k$, respectively. Moreover, for $k = 1$ one obtains

$$M(x) = \left(R_{-4}^H(x) * (-1)^2 R_{-4}^e(x) \right) * \left(O^{*1}(x) \right) * (-1)^k S_{-2}(x) * u(x) \quad (3.30)$$

as a solution of the inhomogeneous equation

$$\square \square_B M(x) = W(x), \quad (3.31)$$

where $\square$ and $\square_B$ are defined by (1.3) and (1.9) with $k = 1$, respectively, and $u(x)$ is obtained from (3.28). Furthermore, If one puts $p = k = 1$, then the operators $\square^k$ and $\square_B^k$ reduce to

$$\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2}, \quad B_{x_1} - B_{x_2} - B_{x_3} - \cdots - B_{x_n}, \quad (3.32)$$

respectively, and the solution $M(x) = I_2^H(x) * I_2(x) * W(x)$ is the inhomogeneous wave equation

$$\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2} \right) \cdot (B_{x_1} - B_{x_2} - B_{x_3} - \cdots - B_{x_n})M(x) = W(x), \quad (3.33)$$

where $I_2^H(x)$ is defined by (2.6) with $\alpha = 2$ and $I_2(x)$ is defined by (2.20) with $\gamma = 2$.

Proof. Since

$$\otimes^k \diamond_B^k u(x) = \square \square^{k-1} L^k \Delta_B^k \square_B^k u(x) = f(x, \square^{k-1} L^k \Delta_B^k \square_B^k u(x)), \quad (3.34)$$

$u(x)$ has continuous derivative up to order $6k$ for $k = 1, 2, 3, \dots$, and $\square^{k-1} L^k \Delta_B^k \square_B^k u(x)$ exists as the generalized function. Thus we can assume that

$$\square^{k-1} L^k \Delta_B^k \square_B^k u(x) = W(x), \quad \forall x \in \Omega. \quad (3.35)$$

Then (3.34) can be written in the form

$$\otimes^k \diamond_B^k u(x) = \square W(x) = f(x, W(x)). \quad (3.36)$$

By (3.26)

$$|f(x, W(x))| \leq N, \quad x \in \Omega, \quad (3.37)$$

and by (3.27) $W(x) = 0, x \in \partial\Omega$, or

$$\square^{k-1} L^k \Delta_B^k \square_B^k u(x) = 0, \quad \forall x \in \partial\Omega. \quad (3.38)$$

We obtain a unique solution of (3.28) which satisfies (3.27) by Lemma 2.8.

Since $R_{2(k-1)}^H(x)$, $(-1)^k S_{2k}(x)$, and $R_{2k}(x)$ are the elementary solution of the operators $\square^{k-1}$, Δ_B^k , and $\square_B^k$, respectively, and by Lemma 2.10, we have that $K(x)$ is an elementary of the operator L^k where $L^k = ((3/4)\Delta^2 + (1/4)\square^2)^k$, that is,

$$\begin{aligned} \square^{k-1} R_{2(k-1)}^H(x) &= \delta, & \Delta_B^k (-1)^k S_{2k}(x) &= \delta, \\ {}_B^k R_{2k}(x) &= \delta, & L^k K(x) &= \delta. \end{aligned} \quad (3.39)$$

From (3.35), we have

$$\square^{k-1} L^k \Delta_B^k \square_B^k u(x) = W(x). \quad (3.40)$$

Convolving the above equation by

$$R_{2(k-1)}^H(x) * K(x) * (-1)^k S_{2k}(x) * R_{2k}(x), \quad (3.41)$$

we obtain

$$\begin{aligned} & \left(R_{2(k-1)}^H(x) * K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) \right) * \left(\square^{k-1} L^k \Delta_B^k \square_B^k u(x) \right) \\ &= \left(R_{2(k-1)}^H(x) * K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) \right) * W(x). \end{aligned} \quad (3.42)$$

By properties of convolution, we obtain

$$\begin{aligned} & \left(\square^{k-1} R_{2(k-1)}^H(x) \right) * \left(L^k K(x) \right) * \left(\Delta_B^k * (-1)^k S_{2k} \right) * \left(\square_B^k R_{2k} \right) * u(x) \\ &= \left(R_{2(k-1)}^H(x) * K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) \right) * W(x). \end{aligned} \quad (3.43)$$

By (3.39) we obtain

$$\delta * \delta * \delta * \delta * u(x) = \left(R_{2(k-1)}^H(x) * K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) \right) * W(x). \quad (3.44)$$

Thus

$$u(x) = \left(R_{2(k-1)}^H(x) * K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) \right) * W(x), \quad (3.45)$$

as a solution of (3.25).

Next, consider the boundary condition (3.38). From

$$\square^{k-1} L^k \Delta_B^k \square_B^k u(x) = 0, \quad (3.46)$$

by Lemma 2.8, we have

$$L^k \Delta_B^k \square_B^k u(x) = (R_{2(k-2)}^H(v))^{(m)}, \quad (3.47)$$

where $m = (n-4)/2$, $n \geq 4$ and n is even. Convolving both sides of (3.47) by

$$K(x) * (-1)^k S_{2k}(x) * R_{2k}(x), \quad (3.48)$$

we obtain

$$\begin{aligned} & \left(K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) \right) * \left(L^k \Delta_B^k \square_B^k \right) * u(x) \\ &= K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * \left(R_{2(k-2)}^H(v) \right)^{(m)}. \end{aligned} \quad (3.49)$$

By the properties of convolution, we obtain

$$\begin{aligned} & \left(L^k K(x) \right) * \left(\Delta_B^k (-1)^k S_{2k} \right) * \left(\square_B^k R_{2k} \right) * u(x) \\ &= K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * \left(R_{2(k-2)}^H(v) \right)^{(m)}. \end{aligned} \quad (3.50)$$

By (3.39), we obtain

$$\delta * \delta * \delta * u(x) = \left(K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) \right) * \left(R_{2(k-2)}^H(v) \right)^{(m)}. \quad (3.51)$$

Thus, for $x \in \partial\Omega$ and $k = 2, 3, 4, 5, \dots$,

$$u(x) = K(x) * (-1)^k S_{2k}(x) * R_{2k}(x) * \left(R_{2(k-2)}^H(v) \right)^{(m)}, \quad (3.52)$$

as required.

Now, for $k = 1$ in (3.28), we have

$$u(x) = \delta(x) * G(x) * (-1) S_2(x) * R_2(x) * W(x). \quad (3.53)$$

By (2.47), we have

$$G(x) = \left(R_6^H(x) * (-1)^2 R_4^e(x) \right) * \left(O^{*1}(x) \right)^{* -1}. \quad (3.54)$$

Taking into account (3.53), we obtain

$$u(x) = \left(R_6^H(x) * (-1)^2 R_4^e(x) \right) * \left(O^{*1}(x) \right)^{* -1} * (-1) S_2(x) * R_2(x) * W(x) \quad (3.55)$$

as a solution of (3.25) for $k = 1$.

Convolving both sides of (3.55) by

$$\left(R_{-4}^H(x) * (-1)^2 R_{-4}^e(x) \right) * \left(O^{*1}(x) \right) * (-1) S_{-2}(x), \quad (3.56)$$

by Lemma 2.12, we obtain

$$\left(R_{-4}^H(x) * (-1)^2 R_{-4}^e(x) \right) * \left(O^{*1}(x) \right) * (-1) S_{-2}(x) * u(x) = R_2^H(x) * R_2(x) * W(x). \quad (3.57)$$

By Lemma 2.6, we obtain

$$M(x) = \left(R_{-4}^H(x) * (-1)^2 R_{-4}^e(x) \right) * \left(O^{*1}(x) \right) * (-1) S_{-2}(x) * u(x) \quad (3.58)$$

as a solution of the inhomogeneous equation

$$\square \square_B M(x) = W(x). \quad (3.59)$$

Now, consider the boundary condition for $k = 1$ in (3.27); we have

$$L \Delta_B \square_B u(x) = 0, \quad \text{or} \quad \square_B L \Delta_B u(x) = 0 \quad (3.60)$$

for $x \in \partial\Omega$. Thus by Lemma 2.8, for $k = 1$, we have

$$L \Delta_B u(x) = \delta^{(m)}(v) \quad \text{for } x \in \partial\Omega, \quad (3.61)$$

where $\delta^{(m)}(x) = R_0^H(x)$. Convolving the above equation by $K(x) * (-1) S_2(x)$ where $K(x)$ is defined by (2.47) with $k = 1$ and $S_2(x)$ is defined by (2.16) with $\alpha = 2$, we obtain

$$K(x) * (-1) S_2(x) * (L \Delta_B u(x)) = \delta^{(m)}(v) * K(x) * (-1) S_2(x). \quad (3.62)$$

By properties of convolution,

$$L K(x) * \Delta_B (-1) S_2(x) * u(x) = \delta^{(m)}(v) * K(x) * (-1) S_2(x). \quad (3.63)$$

By Lemmas 2.10 and 2.5, we obtain

$$\delta(x) * \delta(x) * u(x) = \delta^{(m)}(v) * K(x) * (-1) S_2(x). \quad (3.64)$$

It follows that

$$u(x) = \delta^{(m)}(v) * K(x) * (-1) S_2(x). \quad (3.65)$$

By (2.47) with $k = 1$, we have

$$K(x) = \left(R_4^H(x) * (-1)^2 R_4^e(x) \right) * \left(O^{*1}(x) \right)^{* -1}. \quad (3.66)$$

Taking into account (3.65), we obtain

$$u(x) = \delta^{(m)}(v) * \left(R_4^H(x) * (-1)^2 R_4^e(x) \right) * \left(O^{*1}(x) \right)^{* -1} * (-1) S_2(x) \quad \text{for } x \in \partial\Omega. \quad (3.67)$$

Now consider the case $k = 1, p = 1$, and $q = n - 1$, that is, from (3.59), $R_2^H(x)$ reduced to $I_2^H(x)$ where $I_2^H(x)$ is defined by (2.2) with $\alpha = 2$ and $R_2(x)$ reduced to $I_2(x)$ where $I_2(x)$ is defined by (2.17) with $\gamma = 2$, and then the operator $\square$ defined by (1.3) reduces to the wave operator

$$\square^* = \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2}, \quad (3.68)$$

$\square_B$ defined by (1.9) reduces to the Bessel wave operator

$$\square_B^* = B_{x_1} - B_{x_2} - B_{x_3} - \cdots - B_{x_n}, \quad (3.69)$$

and then the solution $M(x)$ reduced to

$$M(x) = I_2^H(x) * I_2(x) * W(x), \quad (3.70)$$

which is the solution of inhomogeneous wave equation

$$\square^* \square_B^* M(x) = W(x), \quad (3.71)$$

or

$$\left(\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2} - \cdots - \frac{\partial^2}{\partial x_n^2} \right) \cdot (B_{x_1} - B_{x_2} - B_{x_3} - \cdots - B_{x_n}) M(x) = W(x). \quad (3.72)$$

With the boundary condition for $x \in \partial\Omega$,

$$L^* \square_B^* \Delta_B u(x) = 0, \quad (3.73)$$

where $L^* = (3/4)\Delta^2 + (1/4)(\square^*)^2$ and $\square^*$ is defined by (3.68), or for $x \in \partial\Omega$ and by (3.65), we obtain

$$u(x) = \delta^{(m)}(s) * \left(I_4^H(x) * (-1)^2 R_4^e(x) \right) * \left(D^{*1}(x) \right)^{*-1} * (-1) S_2(x), \quad (3.74)$$

where $I_4(x)$ is defined by (2.20) with $\gamma = 4$, $s = x_1^2 - x_2^2 - x_3^2 - \cdots - x_n^2$, and $D(x)$ reduced from $O(x)$ where it is defined by (2.34), that is, $D(x) = (3/4)I_4^H(x) + (1/2)(-1)^2 R_4^e(x)$. $\square$

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