Letter to the Editor

A Note on the Paper “Multipoint BVPs for Second-Order Differential Equations with Impulses” by Xuxin Yang, Zhimin He, and Jianhua Shen

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We give a counter example to the comparison principle for the multipoint BVPs (by Xuxin Yang, Zhimin He, and Jianhua Shen, in Mathematical Problems in Engineering, Volume 2009, Article ID 258090, doi:10.1155/2009/258090). Then we suggest and prove a corrected version of the comparison principle.

1. Introduction and Preliminaries

Consider the following multipoint BVPs [1]:

\[-u''(t) = f(t,u(t),u(\theta(t))), \quad t \neq t_k, \; t \in J = [0,1],\]
\[\Delta u'(t_k) = I_k(u(t_k)), \quad k = 1,2,\ldots,m,\]
\[u(0) - au'(0) = cu(\eta), \quad u(1) + bu'(1) = du(\xi),\]

where \(0 \leq \theta(t) \leq t, \theta \in C(J), 0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = 1, f\) is continuous everywhere except at \(\{t_k\} \times R^2; f(t_k,\cdot,\cdot)\) and \(f(t_{k-1,\cdot,\cdot})\) exist with \(f(t_{k,\cdot,\cdot}) = f(t_{k-1,\cdot,\cdot}); I_k \in C(R,R),\) and \(\Delta u'(t_k) = u'(t_k^+) - u'(t_k^-), a \geq 0, b \geq 0, 0 \leq c \leq 1, 0 \leq d \leq 1, a + c > 0, b + d > 0, 0 < \eta, \xi < 1.\)

Let \(PC(f) = \{x : J \rightarrow R; x(t)\) be continuous everywhere except for some \(t_k\) at which \(x(t_k^+)\) and \(x(t_k^-)\) exist and \(x(t_k) = x(t_k^+), k = 1,2,\ldots,m\}; PC^1(f) = \{x \in PC(f) : x'(t)\) is continuous everywhere except for some \(t_k\) at which \(x'(t_k^+)\) and \(x'(t_k^-)\) exist and \(x'(t_k) = \)
Let \( J^- = J \setminus \{ t_k, k = 1, 2, \ldots, m \} \), and \( E = PC^1(J, R) \cap C^2(J^-, R) \). A function \( x \in E \) is called a solution of BVPS (1.1) if it satisfies (1.1).

The purpose of this note is to point out that the results basing on the comparison principle [1, Theorem 2.1] are not true. Then we give a new comparison principle.

2. Problem and Statement

The authors [1] proved some existence results for multipoint BVPs (1.1) by use of the following comparison principle [1, Theorem 2.1].

Assume that \( u \in E \) satisfies

\[
-u''(t) + Mu(t) + Nu(\theta(t)) \leq 0, \quad t \neq t_k, \quad t \in J = [0, 1],
\]

\[
\Delta u'(t_k) \geq L_k u(t_k), \quad k = 1, 2, \ldots, m,
\]

\[
u(0) - au'(0) \leq cu(\eta), \quad u(1) + bu'(1) \leq du(\xi),
\]

where \( a \geq 0, b \geq 0, 0 \leq c \leq 1, 0 \leq d \leq 1, a + c > 0, b + d > 0, 0 < \eta, \xi < 1, L_k \geq 0 \), and constants \( M, N \) satisfy

\[
M > 0, N \geq 0, \quad \frac{M + N}{2} + \sum_{k=1}^{m} L_k \leq 1.
\]

Then \( u(t) \leq 0 \) for \( t \in J \).

However, the comparison principle above is not true.

A Counter Example

Let

\[
u(t) = \begin{cases} 
\frac{3}{2} t^2 + 20, & t \in \left[ 0, \frac{1}{2} \right], \\
\frac{5}{2} t^2 + 3, & t \in \left( \frac{1}{2}, 1 \right].
\end{cases}
\]

Then \( u(t) \leq 0 \) for \( t \in J \).
Then

\[ u'(t) = \begin{cases} 
3t, & t \in [0, \frac{1}{2}], \\
5t, & t \in \left(\frac{1}{2}, 1\right]. 
\end{cases} \]  

(2.4)

\[ u''(t) = \begin{cases} 
3, & t \in [0, \frac{1}{2}], \\
5, & t \in \left(\frac{1}{2}, 1\right]. 
\end{cases} \]

And let \( M = N = 1/1000, \ a = b = c = d = 1, m = 1, t_1 = 1/2, L_1 = 1/1000, \theta(t) = (1/2)t, \ \eta = 1/3, \) and \( \xi = 1/6. \) When \( t \in [0, 1/2], \) then

\[ \frac{1}{1000} \left( \frac{3}{2} t^2 + 20 \right) + \frac{1}{1000} \left( \frac{3}{2} \times \frac{t^2}{4} + 20 \right) \leq 3. \]  

(2.5)

When \( t \in (1/2, 1], \) then

\[ \frac{1}{1000} \left( \frac{5}{2} t^2 + 3 \right) + \frac{1}{1000} \left( \frac{5}{2} \times \frac{t^2}{4} + 3 \right) \leq 5. \]  

(2.6)

Hence \( -u''(t) + Mu(t) + Nu(\theta(t)) \leq 0. \)

\[ \Delta u' \left( \frac{1}{2} \right) = u' \left( \frac{1}{2}^+ \right) - u' \left( \frac{1}{2} \right) = 5 \times \frac{1}{2} - \left( 3 \times \frac{1}{2} \right) = 1, \]  

(2.7)

\[ \frac{1}{1000} u' \left( \frac{1}{2} \right) = \frac{1}{1000} \left( \frac{3}{2} \times \frac{1}{4} + 20 \right) = \frac{1}{1000} \times \frac{163}{8}. \]  

(2.8)

Hence \( \Delta u'(t_1) \geq L_1 u(t_1). \)

\[ u(0) - u'(0) = 20, \quad u \left( \frac{1}{3} \right) = \frac{3}{2} \times \frac{1}{9} + 20. \]  

(2.9)

Hence \( u(0) - au'(0) \leq cu(1/3). \)

\[ u(1) + u'(1) = \frac{5}{2} + 3 + 5 = \frac{21}{2}, \quad u \left( \frac{1}{6} \right) = \frac{3}{2} \times \frac{1}{36} + 20. \]  

(2.10)
Proof. Suppose to contrary that there exist some $u \in E \cap C(J)$ such that

\begin{equation}
\frac{M + N}{2} + \sum_{k=1}^{m} L_k = \frac{2}{100} < 1.
\end{equation}

But we easily show that $u(t) > 0$, for all $t \in [0, 1]$, which is a contradiction with (Theorem 2.1) in [1]. In fact, we can correct Theorem 2.1 in [1] as follows.

**Theorem 2.1.** Suppose $u \in E \cap C(J)$ such that

\begin{align*}
-u''(t) + Mu(t) + Nu(\theta(t)) &\leq 0 \quad t \neq t_k, t \in J = [0, 1], \\
\Delta u'(t_k) &\geq L_k u(t_k), \quad k = 1, 2, \ldots, m, \\
0 &\leq u(t) \leq \Delta u(t), \\
0 &< u'(t), \\
M &> 0, N > 0, \quad M + N \geq 0, \quad \frac{M + N}{2} + \sum_{k=1}^{m} L_k \leq 1.
\end{align*}

Then $u(t) \leq 0$ for $t \in J$.

**Remark 2.2.** In this Theorem, we have to add $u \in C(J)$.

**Proof.** Suppose to contrary that there exist some $t \in J$, such that $u(t) > 0$.

If $u(1) = \max_{t \in J} u(t) > 0$, we have $u'(1) \geq 0$, $u(1) \geq u(\xi)$, and

\begin{equation}
du(\xi) \leq u(1) \leq u(1) + bu'(1) \leq du(\xi).
\end{equation}

Therefore, $d = 1$ and $u(\xi)$ is maximum value.

If $u(0) = \max_{t \in J} u(t) > 0$, we have $u'(0) \geq 0$, $u(0) \geq u(\eta)$, and

\begin{equation}
cu(\eta) \leq u(0) \leq u(0) - au'(0) \leq cu(\eta).
\end{equation}

Therefore, $c = 1$ and $u(\eta)$ is maximum value.
Mathematical Problems in Engineering

So there is a $\delta \in (0, 1)$ such that

$$u(\delta) = \max_{t \in J} u(t) > 0, \quad \text{by } \Delta u = 0, \quad \text{then } u'(\delta^+) \leq 0, \quad u'(\delta^-) \geq 0. \quad (2.16)$$

It is obvious to see that $\delta \notin \{t_k, k = 1, 2, \ldots, m\}$ by

$$\Delta u'(\delta) = u'(\delta^+) - u'(\delta) \geq L_k u(\delta) > 0 \quad (2.17)$$

which is a contradiction because of (2.16).

(i) Suppose that $u(t) \geq 0$ for $t \in [0, \delta]$.

By $u(\delta) = \max_{t \in J} u(t) > 0$, we get $\delta \in J^-, u''(\delta) \leq 0$. On the other hand, by (2.12), we have

$$0 < M u(\delta) + Nu(\theta(t)) \leq u''(\delta) \quad (2.18)$$

which is a contradiction.

(ii) Suppose there exists $t_* \in [0, \delta]$ such that $u(t_*) = \min_{t \in (0, \delta)} u(t) < 0$. By (2.12), we get

$$u''(t) \geq (M + N) u(t_*), \quad t \in [0, \delta), t \neq t_k,$$

$$\Delta u(t_k) = 0, \quad \Delta u'(t_k) \geq L_k u(t_k), \quad k = 1, 2, \ldots, m. \quad (2.19)$$

Integrating from $s(t_* \leq s \leq \delta)$ to $\delta$, we get

$$u'(\delta) - u'(s) \geq \int_s^\delta (M + N) u(t_*) ds + \sum_{s \leq t \leq \delta} L_k u(t_k)$$

$$= (\delta - s)(M + N) u(t_*) + \sum_{s \leq t_k \leq \delta} L_k u(t_k) \quad (2.20)$$

$$\geq (\delta - s)(M + N) u(t_*) + \sum_{k=1}^m L_k u(t_*).$$

Hence

$$-u'(s) \geq (\delta - s)(M + N) u(t_*) + \sum_{k=1}^m L_k u(t_*), \quad t_* \leq s \leq \delta. \quad (2.21)$$
Then integrate from $t_*$ to $\delta$ to obtain

$$-u(t_*) < u(\delta) - u(t_*)$$

$$\leq \int_{t_*}^{\delta} (M + N)u(t_*)(s - \delta)ds - \sum_{k=1}^{m} L_k u(t_*)(\delta - t_*)$$

$$= (M + N)u(t_*) \left[ -\frac{(t_* - \delta)^2}{2} \right] - \sum_{k=1}^{m} L_k u(t_*)(\delta - t_*)$$

$$\leq -\left[ \frac{M + N}{2} (\delta - t_*)^2 + \sum_{k=1}^{m} L_k \right] u(t_*)$$

$$\leq -\left( \frac{M + N}{2} + \sum_{k=1}^{m} L_k \right) u(t_*) .$$

By (2.13), we get $u(t_*) > 0$ which is a contradiction. We complete the proof.

This implies that in order to get the existence results of the multipoint BVPs [1], we have to require an additional continuity hypotheses on the function space.

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**References**

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