Computing Exact Solutions to a Generalized Lax-Sawada-Kotera-Ito Seventh-Order KdV Equation

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The Cole-Hopf transform is used to construct exact solutions to a generalization of both the seventh-order Lax KdV equation (Lax KdV7) and the seventh-order Sawada-Kotera-Ito KdV equation (Sawada-Kotera-Ito KdV7).

1. Introduction

Many direct and computational methods have been used to handle nonlinear partial differential equations (NLPDE’s). Some methods used in a satisfactory way to obtain exact solutions to NLPDE’s are inverse scattering method [1], Hirota bilinear method [2, 3], Backlund transformations [4], Painlevé analysis [5], Lie groups [6], the tanh method [7], the generalized tanh method [8, 9], the extended tanh method [10–12], the improved tanh-coth method [13, 14], the Exp-function method [15–17], the projective Riccati equation method [18], the generalized projective Riccati equations method [19–24], the extended hyperbolic function method [25], variational iteration method [26, 27], He’s polynomials [28], homotopy perturbation method [29], and many other methods [30]. However, there is not a unified method that could be used to handle all NLPDE’s; in this sense, the implementation of new
methods or variants of the some well-known methods is relevant. The principal objective of this paper consists in obtaining exact traveling wave solutions which include periodic and soliton solutions to a particular case of the general seventh-order KdV (KdV7), which is a generalization of the seventh-order Sawada-Kotera-Ito (SKI-KdV7) equation, by using a variant of the exp-function method. The general seventh-order KdV (KdV7) equation [31] reads

\[ u_t + au^3 u_x + bu_x^3 + cu u_{xx} + du^2 u_{xxx} + eu_{2x} u_{3x} + fu_{4x} u_{5x} + gu_{5x} + h_{7x} = 0. \]  

(1.1)

The (KdV7) was introduced initially by Pomeau et al. [32] for discussing the structural stability of KdV equation under a singular perturbation. Some particular cases of (1.1) are

(i) seventh-order Lax KdV equation [1, 6] \( (a = 140, b = 70, c = 280, d = 70, e = 70, f = 42, g = 14) \):

\[ u_t + 140u^3 u_x + 70u_x^3 + 280uu_x u_{xx} + 70u^2 u_{xxx} + 70u_{2x} u_{3x} + 42u_x u_{4x} + 14uu_{5x} + u_{7x} = 0; \]

(1.2)

(ii) seventh-order Sawada-Kotera-Ito equation [1, 8–10] \( (a = 252, b = 63, c = 378, d = 126, e = 63, f = 42, g = 21) \):

\[ u_t + 252u^3 u_x + 63u_x^3 + 378uu_x u_{xx} + 126u^2 u_{xxx} + 63u_{2x} u_{3x} + 42u_x u_{4x} + 21uu_{5x} + u_{7x} = 0; \]

(1.3)

(iii) seventh-order Kaup-Kupershmidt equation [1, 7] \( (a = 2016, b = 630, c = 2268, d = 504, e = 252, f = 147, g = 42) \):

\[ u_t + 2016u^3 u_x + 630u_x^3 + 2268uu_x u_{xx} + 504u^2 u_{xxx} + 252u_{2x} u_{3x} + 147u_x u_{4x} + 42uu_{5x} + u_{7x} = 0. \]

(1.4)

2. Generalization of the Lax KdV7 and the Sawada-Kotera-Ito KdV7

Observe that (1.2) and (1.3) satisfy the relation

\[ a = \frac{d}{63} (e + f + g). \]

(2.1)

For this reason we will study equation

\[ u_t + \frac{d}{63} (e + f + g)u^3 u_x + bu_x^3 + cu u_{xx} + du^2 u_{xxx} + eu_{2x} u_{3x} + fu_{4x} u_{5x} + gu_{5x} + h_{7x} = 0. \]

(2.2)

We seek solutions to (2.2) in the Cole-Hopf form

\[ u(t, x) = A\partial_x \tanh(\xi), \]

(2.3)
Substituting (2.3) into (2.2), we obtain a polynomial equation in the variable $\zeta = \exp(\xi)$. Equating the coefficients of the different powers of $\zeta$ to zero, we obtain the following algebraic system:

\[
\begin{align*}
\lambda + 64\mu^6 &= 0, \\
64\mu^5 (A(e + f + g) - 247\mu) + 5\lambda &= 0, \\
64\mu^4 \left( A^2(b + c + d) - 3A\mu(5e + 9f + 19g) + 4293\mu^2 \right) + 9\lambda &= 0, \\
64\mu^3 \left( A^3d(e+f+g) - 63A^2\mu(3b+5c+11d) + 126A\mu^2 (28e+46f+151g) - 983997\mu^3 \right) + 315\lambda &= 0.
\end{align*}
\]

Eliminating $A, \lambda,$ and $\mu$ from system (2.5) gives

\[
\begin{align*}
b &= d + \frac{1}{126}(e + f + g)(e - 5f + 10g), \\
c &= \frac{5}{21}g(e + f + g) - 2d.
\end{align*}
\]

It is easy to verify that (1.2) and (1.3) are particular cases of general KdV7 equation (1.1) subject to (2.1) and (2.6). This motivates us to define the generalized Lax-Sawada-Kotera-Ito seventh-order equation (LSKI KdV7) as follows:

\[
u_t + \frac{1}{63}d(e + f + g)u^3u_x + \left( d + \frac{1}{126}(e + f + g)(e - 5f + 10g) \right)u_x^3 \\
+ \left( \frac{5}{21}g(e + f + g) - 2d \right) uu_xu_{xx} + du^2u_{xxx} + eu_{2x}u_{3x} + f u_xu_{4x} + guu_{5x} + u_{7x} = 0.
\]

3. Solutions to Generalized LSKI KdV7

In order to look for solutions to (2.7), we will use the exp ansatz

\[
u(\xi) = p + \frac{q}{1 + r \exp(-s \xi) + s \exp(\xi)},
\]

where $p, q, r,$ and $s$ are some constants. Substituting (3.1) into (2.7) gives an algebraic system. Solving it, we obtain

\[
\lambda = -\frac{1}{63}d(e + f + g)p^3 - \mu^2 \left( dp^2 + gp\mu^2 + \mu^4 \right), \\
q = \frac{126\mu^2}{e + f + g}, \\
s = \frac{1}{4r}, \\
r = r, \\
\mu = \mu.
\]
From (2.4), (3.1), and (3.2), we obtain following solution to (2.7) subject:

\[ u(x,t) = p + \frac{126\mu^2}{(e+f+g)(1 + r \exp(\xi) + (1/4r) \exp(-\xi))}, \]

\[ \xi = \mu(x + \lambda t + \delta), \]

\[ \lambda = -\frac{1}{63} d(e + f + g)p^3 - \mu^2 \left( dp^2 + g\mu^2 + \mu^4 \right). \]

(3.3)

In particular, if \( r = 1/2 \), equation (3.3) gives

\[ u(x,t) = p + \frac{63\mu^2}{e+f+g} \text{sech}^2\left(\frac{\mu}{2}(x + \lambda t + \delta)\right), \]

\[ \lambda = -\frac{1}{63} d(e + f + g)p^3 - \left( dp^2 + g\mu^2 + \mu^4 \right)\mu^2. \]

(3.4)

Replacing \( \mu \) with \( \mu \sqrt{-1} \) gives the following periodic solutions:

\[ u(x,t) = p - \frac{63\mu^2}{e+f+g} \text{sec}^2\left(\frac{\mu}{2}(x + \lambda t + \delta)\right), \]

\[ \lambda = -\frac{1}{63} d(e + f + g)p^3 + \left( dp^2 + g\mu^2 + \mu^4 \right)\mu^2. \]

(3.5)

On the other hand, if \( r = -1/2 \), equation (3.3) gives

\[ u(x,t) = p - \frac{63\mu^2}{e+f+g} \text{csch}^2\left(\frac{\mu}{2}(x + \lambda t + \delta)\right), \]

\[ \lambda = -\frac{1}{63} d(e + f + g)p^3 - \left( dp^2 + g\mu^2 + \mu^4 \right)\mu^2. \]

(3.6)

Replacing \( \mu \) with \( \mu \sqrt{-1} \) gives the following periodic solutions:

\[ u(x,t) = p - \frac{63\mu^2}{e+f+g} \text{csc}^2\left(\frac{\mu}{2}(x + \lambda t + \delta)\right), \]

\[ \lambda = -\frac{1}{63} d(e + f + g)p^3 + \left( dp^2 - g\mu^2 + \mu^4 \right)\mu^2. \]

(3.7)
4. Solutions to Sawada-Kotera-Ito KdV7 Equation

From (3.3)–(3.7) with \(d = 126, e = 63, f = 42,\) and \(g = 21,\) we obtain the following analytic solutions to equation (1.3):

\[
\begin{align*}
u(x, t) &= p + \frac{4r \mu^2 \exp(\mu(x + \lambda t + \delta))}{(1 + 2r \exp(\mu(x + \lambda t + \delta)))^2}, \quad \lambda = -252p^3 - 126p^2 \mu^2 - 21p \mu^4 - \mu^6, \\
u(x, t) &= p + \frac{1}{2} \mu^2 \text{sech}^2\left(\frac{1}{2} \mu(x + \lambda t + \delta)\right), \quad \lambda = -252p^3 - 126p^2 \mu^2 - 21p \mu^4 - \mu^6, \\
u(x, t) &= p - \frac{1}{2} \mu^2 \text{sec}^2\left(\frac{1}{2} \mu(x + \lambda t + \delta)\right), \quad \lambda = -252p^3 + 126p^2 \mu^2 - 21p \mu^4 + \mu^6, \\
u(x, t) &= p - \frac{1}{2} \mu^2 \text{csch}^2\left(\frac{1}{2} \mu(x + \lambda t + \delta)\right), \quad \lambda = -252p^3 - 126p^2 \mu^2 - 21p \mu^4 - \mu^6.
\end{align*}
\]

5. Conclusions

We exhibited an equation that generalizes both seventh-order Lax equation and seventh-order Sawada-Kotera-Ito equation. At the same time, we obtained exact solutions to these equations with the aid of a Cole-Hopf ansatz. These same ideas are suitable for the seventh-order Kaup-Kupershmidt equation. We think that some of the solutions in this work are new in the open literature. We may apply other methods to find exact solutions to a variety of nonlinear PDE’s. See [3, 12–52].

References


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