Research Article

Positive Solution for the Elliptic Problems with Sublinear and Superlinear Nonlinearities

Chunmei Yuan, Shujuan Guo, and Kaiyu Tong

College of Physics and Mathematics, Changzhou University, Changzhou, Jiangsu 213164, China

Correspondence should be addressed to Chunmei Yuan, chunmeiyuan@126.com

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This paper deals with the existence of positive solutions for the elliptic problems with sublinear and superlinear nonlinearities $-\Delta u = \lambda a(x)u^p + b(x)u^q$ in Ω , $u > 0$ in Ω , $u = 0$ on $\partial\Omega$, where $\lambda > 0$ is a real parameter, $0 < p < 1 < q$. Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 3$), and $a(x)$ and $b(x)$ are some given functions. By means of variational method and super-subsolution method, we obtain some results about existence of positive solutions.

1. Introduction

In this paper, we consider the elliptic problems with sublinear and superlinear nonlinearities

$$\begin{aligned} -\Delta u &= \lambda a(x)u^p + b(x)u^q \quad \text{in } \Omega, \\ u &> 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1}_\lambda$$

where $\lambda > 0$ is a real parameter, $0 < p < 1 < q$. Ω is a bounded domain in $\mathbb{R}^N$ ($N \geq 3$), and $a(x)$ and $b(x)$ are some given functions which satisfies the following assumptions:

(H₁) $a(x), b(x) \in L^\infty(\Omega)$, $a(x) \geq c_0$, $b(x) \leq -c_1$, where c_0, c_1 are positive constants,

or

(H₂) $a(x), b(x) \in L^\infty(\Omega)$, $a(x), b(x) \geq c_0$, where c_0 is a positive constant.

For convenience, we denote $((1)_\lambda)$ with hypothesis (H₁) or (H₂) by $(1)_\lambda^-$ and $(1)_\lambda^+$, respectively.

Such problems occur in various branches of mathematical physics and population dynamics, and sublinear analogues or superlinear analogues of $((1)_\lambda)$ have been considered by many authors in recent years (see [1–9] and their references). But most of such studies have been concerned with equations of the type involving sublinear nonlinearity (see [3–6, 8, 9]), with only few references dealing with the elliptic problems with sublinear and superlinear nonlinearities. In [1], Ambrosetti et al. deal with the analogue of $((1)_\lambda)$ with $a(x) = b(x) \equiv 1$. It is known from [2] that there exist $\lambda^* \in (0, \infty)$, such that problem $((1)_\lambda)$ has a solution if $\lambda \leq \lambda^*$ and has no solution if $\lambda > \lambda^*$, provided $b(x) \equiv 1$ on Ω .

Our goal in this paper is to show how variational method and super-subsolution method can be used to establish some existence results of problem $((1)_\lambda)$. We work on the Sobolev space $H_0^1(\Omega)$ equipped with the norm $\|u\| = (\int_\Omega |\nabla u|^2 dx)^{1/2}$. For $u \in H_0^1(\Omega)$ we define $I_\lambda : H_0^1(\Omega) \rightarrow \mathbb{R}$ by

$$I_\lambda(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{\lambda}{p+1} \int_\Omega a(x) |u|^{p+1} dx - \frac{1}{q+1} \int_\Omega b(x) |u|^{q+1} dx. \quad (1.1)$$

Let λ_1 be the first eigenvalue of

$$\begin{aligned} -\Delta u &= \lambda u, \quad x \in \Omega, \\ u &= 0, \quad x \in \partial\Omega. \end{aligned} \quad (1.2)$$

φ_1 denotes the corresponding eigenfunction satisfying $0 \leq \varphi_1(x) \leq 1$. $L^p(\Omega)$, $(1 \leq p \leq \infty)$, denotes Lebesgue spaces, and the norm in L^p is denoted by $\|\cdot\|_p$.

2. The Existence of Positive Solution of $(1)_\lambda^-$

It is well known that

$$\nabla \varphi_1(x) \neq 0, \quad \forall x \in \partial\Omega. \quad (2.1)$$

Define $a = \min_{\partial\Omega} |\nabla \varphi_1|^2$; from (2.1) we know $a > 0$, so we can split the domain Ω into two parts: Ω_ε and $\Omega \setminus \Omega_\varepsilon$, where $\Omega_\varepsilon = \{x \in \Omega : |\nabla \varphi_1|^2 \geq a/2\} \cap \{x \in \Omega : \varphi_1(x) \leq \varepsilon, \varepsilon \text{ is small enough}\}$. Let $b = \inf_{\Omega \setminus \Omega_\varepsilon} \varphi_1(x)$; we obtain that $b \geq \varepsilon$ by the positivity of φ_1 in Ω , and $\Omega \setminus \Omega_\varepsilon$ is nonempty when ε is small enough.

Theorem 2.1. *Let $a(x)$, $b(x)$ satisfy assumption (H_1) , and $0 < p < 1 < q < 2^* - 1$, where $2^* = 2N/(N-2)$ is the limiting exponent in the Sobolev embedding. Then there exists a constant $\tilde{\lambda} > 0$ such that $(1)_\lambda^-$ possesses at least a weak positive solution $u^*(x) \in H_0^1(\Omega)$ for $\lambda \geq \tilde{\lambda}$.*

Proof. Let $e(x)$ denote the positive solution of the following equation:

$$\begin{aligned} -\Delta e &= 1, \quad x \in \Omega, \\ e &= 0, \quad x \in \partial\Omega. \end{aligned} \quad (2.2)$$

Here and hereafter we use the following notations: $A = \|a\|_\infty$, $B = \|b\|_\infty$, $E = \|e\|_\infty$. Since $0 < p < 1$, for all $\lambda \in \mathbb{R}^+$, there exists $T = T(\lambda) > 0$ satisfying

$$T \geq \lambda A T^p E^p. \quad (2.3)$$

Observing that $b(x) \leq -c_1 < 0$, as a consequence, the function Te verifies

$$T = -\Delta(Te) \geq \lambda A (Te)^p \geq \lambda a(x) (Te)^p + b(x) (Te)^q, \quad (2.4)$$

and hence it is a supersolution of $(1)_\lambda^+$. Let $v(x) = \varphi_1^l$, $x \in \Omega$, $l > 1$. For $x \in \Omega$, we have $x \in \Omega_\varepsilon$ or $x \in \Omega \setminus \Omega_\varepsilon$. We will discuss it from two conditions.

(I) For all $x \in \Omega_\varepsilon$, observing that $l > 1$ and when ε is small enough, we have

$$al(l-1)\frac{s^{-2}}{2} - Bs^{l(q-1)} > \lambda_1 l, \quad \forall s \in (0, \varepsilon). \quad (2.5)$$

Since $x \in \Omega_\varepsilon$, then it follows that $\varphi_1(x) \leq \varepsilon$, $|\nabla \varphi_1|^2 \geq a/2$. From (2.5) we infer

$$\lambda_1 l \leq l(l-1)\varphi_1^{-2} |\nabla \varphi_1|^2 - B\varphi_1^{l(q-1)}, \quad \forall x \in \Omega_\varepsilon. \quad (2.6)$$

Multiplying (2.6) with φ_1^l , we get

$$l(1-l)\varphi_1^{l-2} |\nabla \varphi_1|^2 + \lambda_1 l \varphi_1^l \leq -B\varphi_1^{lq}. \quad (2.7)$$

It follows that

$$-\Delta(\varphi_1^l) \leq \lambda a(x) (\varphi_1^l)^p - b(x) (\varphi_1^l)^q. \quad (2.8)$$

(II) For all $x \in \Omega \setminus \Omega_\varepsilon$, there exists $\tilde{\lambda} > 0$, such that for all $\lambda \geq \tilde{\lambda}$, and we have

$$\lambda c_0 s^{pl} - Bs^{ql} \geq \lambda_1 l s^l, \quad \forall s \in R, \quad b \leq s \leq 1. \quad (2.9)$$

Since $x \in \Omega \setminus \Omega_\varepsilon$, then we have $\varphi_1(x) \geq b$ (and $\varphi_1(x) \leq 1$). From (2.9), it follows that

$$-\Delta(\varphi_1^l) \leq \lambda_1 l \varphi_1^l \leq \lambda c_0 \varphi_1^{lp} - Bs^{ql} \leq \lambda a(x) (\varphi_1^l)^p + b(x) (\varphi_1^l)^q. \quad (2.10)$$

From (2.8) and (2.10), we derive that there exists $\tilde{\lambda} > 0$ such that for all $x \in \Omega$, for all $\lambda \geq \tilde{\lambda}$,

$$-\Delta(\varphi_1^l) \leq \lambda a(x) (\varphi_1^l)^p + b(x) (\varphi_1^l)^q, \quad (2.11)$$

that is, $v(x) = \varphi_1^l(x)$ is a subsolution of $(1)_\lambda^-$. Taking T as sufficiently large, we also have

$Te > \varphi_1^l$ by minimal principle. Define $w(x) = Te(x)$, and let $K = \{u \in H_0^1(\Omega) : v(x) \leq u(x) \leq w(x), \text{ for all } x \in \Omega\}$, then K is closed and convex (and weakly closed). Let $f(s) = \lambda a(x)s^p + b(x)s^q$, for all $s \in \mathbb{R}, s > 0$. We consider the function

$$I_\lambda(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} \int_0^u f(s) ds dx. \quad (2.12)$$

Observe that $b(x) < 0$, $0 < p < 1 < q < 2^* - 1$; we infer that I_λ is coercive, bounded, since it is blow and weakly lower semicontinuous. Using this fact, we conclude that there exists $u^* \in K$, such that $I_\lambda(u^*) = \inf_K I_\lambda$ (see [10]). In the following, we will prove that u^* is a solution of problem $(1)_\lambda^-$.

For $\phi \in K$, define $h : [0, 1] \rightarrow \mathbb{R}$, such that

$$h(t) = I(t\phi + (1-t)u^*). \quad (2.13)$$

Clearly, $h(t)$ achieves its minimum at $t = 0$, and

$$h'(t)|_{t=0} = \int_{\Omega} [\nabla u^* \nabla (\phi - u^*)] dx - \int_{\Omega} f(u^*) (\phi - u^*) dx \geq 0. \quad (2.14)$$

For all $\varphi \in H_0^1(\Omega)$, $\eta > 0$, define

$$\Psi(x) = \begin{cases} v, & \text{when } u^* + \eta\varphi < v, \\ u^* + \eta\varphi, & \text{when } v \leq u^* + \eta\varphi \leq w, \\ w, & \text{when } u^* + \eta\varphi > w. \end{cases} \quad (2.15)$$

Obviously, $\Psi \in K$, and inserting (2.15) into (2.14), we find

$$\begin{aligned} 0 &\leq \int_{v \leq u^* + \eta\varphi \leq w} [\nabla u^* \cdot \nabla (\eta\varphi) - f(u^*)(\eta\varphi)] dx \\ &\quad + \int_{u^* + \eta\varphi > w} [\nabla u^* \nabla (w - u^*) - f(u^*)(w - u^*)] dx \\ &\quad + \int_{u^* + \eta\varphi < v} [\nabla u^* \nabla (v - u^*) - f(u^*)(v - u^*)] dx \end{aligned}$$

$$\begin{aligned}
&= \eta \int_{v \leq u^* + \eta\varphi \leq w} [\nabla u^* \cdot \nabla \varphi - f(u^*)\varphi] dx \\
&\quad + \int_{u^* + \eta\varphi > w} [\nabla w \cdot \nabla (w - u^*) - f(w)(w - u^*)] dx \\
&\quad + \int_{u^* + \eta\varphi < v} [\nabla v \cdot \nabla (v - u^*) - f(v)(v - u^*)] dx \\
&\quad - \int_{u^* + \eta\varphi > w} |\nabla w - \nabla u^*|^2 dx - \int_{u^* + \eta\varphi < v} |\nabla v - \nabla u^*|^2 dx \\
&\quad + \int_{u^* + \eta\varphi > w} [f(w) - f(u^*)](w - u^*) dx \\
&\quad + \int_{u^* + \eta\varphi < v} [f(v) - f(u^*)](v - u^*) dx.
\end{aligned} \tag{2.16}$$

Since $w(x)$ and $v(x)$ are supersolution and subsolution, respectively, then

$$\begin{aligned}
\int_{u^* + \eta\varphi > w} [\nabla w \cdot \nabla (w - u^*) - f(w)(w - u^*)] dx &\leq \eta \int_{u^* + \eta\varphi > w} [\nabla w \cdot \nabla \varphi - f(w)\varphi] dx, \\
\int_{u^* + \eta\varphi < v} [\nabla v \cdot \nabla (v - u^*) - f(v)(v - u^*)] dx &\leq \eta \int_{u^* + \eta\varphi < v} [\nabla v \cdot \nabla \varphi - f(v)\varphi] dx.
\end{aligned} \tag{2.17}$$

Observe that $\text{meas}[u^* + \eta\varphi > w] \rightarrow 0$, $\text{meas}[u^* + \eta\varphi < v] \rightarrow 0$, as $\eta \rightarrow 0$,

$$\begin{aligned}
\int_{u^* + \eta\varphi > w} [\nabla w \cdot \nabla \varphi - f(w)\varphi] dx &\rightarrow 0, \\
\int_{u^* + \eta\varphi < v} [\nabla v \cdot \nabla \varphi - f(v)\varphi] dx &\rightarrow 0.
\end{aligned} \tag{2.18}$$

Since $u^* \in K$, $b(x) < 0$, it follows that

$$\begin{aligned}
&\int_{u^* + \eta\varphi > w} [f(w) - f(u^*)](w - u^*) dx \\
&= \int_{u^* + \eta\varphi > w} \lambda a(x) (w^p - u^{*p}) (w - u^*) dx + \int_{u^* + \eta\varphi > w} b(x) (w^q - u^{*q}) (w - u^*) dx \\
&\leq \int_{u^* + \eta\varphi > w} \lambda a(x) (w^p - u^{*p}) (w - u^*) dx.
\end{aligned} \tag{2.19}$$

Similar to (2.19), we have

$$\begin{aligned}
& \int_{u^* + \eta\varphi < v} [f(v) - f(u^*)](v - u^*)dx \\
&= \int_{u^* + \eta\varphi < v} \lambda a(x) (v^p - u^{*p})(v - u^*)dx + \int_{u^* + \eta\varphi < v} b(x) (v^q - u^{*q})(v - u^*)dx \quad (2.20) \\
&\leq \int_{u^* + \eta\varphi < v} \lambda a(x) (v^p - u^{*p})(v - u^*)dx.
\end{aligned}$$

Similar to (2.18), as $\eta \rightarrow 0$, it follows that

$$\begin{aligned}
& \int_{u^* + \eta\varphi > w} \lambda a(x) (w^p - u^{*p})(w - u^*)dx \rightarrow 0, \\
& \int_{u^* + \eta\varphi < v} \lambda a(x) (v^p - u^{*p})(v - u^*)dx \rightarrow 0. \quad (2.21)
\end{aligned}$$

As $\eta \rightarrow 0$, we also have

$$\int_{v \leq u^* + \eta\varphi \leq w} [\nabla u^* \cdot \nabla \varphi - f(u^*)\varphi]dx \rightarrow \int_{\Omega} [\nabla u^* \cdot \nabla \varphi - f(u^*)\varphi]dx. \quad (2.22)$$

Inserting (2.17), (2.19), and (2.20) into (2.16), we find

$$\begin{aligned}
0 &\leq \eta \left\{ \int_{v \leq u^* + \eta\varphi \leq w} [\nabla u^* \cdot \nabla \varphi - f(u^*)\varphi]dx + \int_{u^* + \eta\varphi > w} [\nabla w \cdot \nabla \varphi - f(w)\varphi]dx \right. \\
&\quad + \left. \int_{u^* + \eta\varphi < v} [\nabla v \cdot \nabla \varphi - f(v)\varphi]dx \right\} \quad (2.23) \\
&\quad + \int_{u^* + \eta\varphi > w} \lambda a(x) (w^p - u^{*p})(w - u^*)dx \\
&\quad + \int_{u^* + \eta\varphi < v} \lambda a(x) (v^p - u^{*p})(v - u^*)dx.
\end{aligned}$$

Dividing by η and letting $\eta \rightarrow 0$, using (2.18), (2.21), and (2.22), we derive

$$\int_{\Omega} [\nabla u^* \cdot \nabla \varphi - f(u^*)\varphi]dx \geq 0. \quad (2.24)$$

Noting that φ is arbitrary, this holds equally for $-\varphi$, and it follows that u^* is indeed a weak solution of $(1)_{\lambda}^-$, and the strong maximum principle yields $u^* > \varphi_1^l$, in Ω . Therefore it is a weak positive solution of $(1)_{\lambda}^-$. $\square$

3. The Existence of Positive Solution of $(1)_\lambda^+$

Theorem 3.1. *Let $a(x)$, $b(x)$ satisfy assumption (H_2) , and $0 < p < 1 < q < +\infty$. Then there exists $\Lambda \in \mathbb{R}$, $\Lambda > 0$, such that*

- (i) *for all $\lambda \in (0, \Lambda)$ problem $(1)_\lambda^+$ has a minimal solution u_λ such that $I_\lambda(u_\lambda) < 0$. Moreover u_λ is increasing with respect to λ ;*
- (ii) *for $\lambda = \Lambda$ problem $(1)_\lambda^+$ has at least one weak solution $u \in H \cap L^{p+1}$;*
- (iii) *for all $\lambda > \Lambda$ problem $(1)_\lambda^+$ has no solution.*

To prove Theorem 3.1, let us define

$$\Lambda = \sup\{\lambda > 0 : (1)_\lambda^+ \text{ has a solution}\}. \quad (3.1)$$

First of all we prove a useful lemma.

Lemma 3.2. *One has $0 < \Lambda < +\infty$.*

Proof. Let $e(x)$ denote the solution of the following equation:

$$\begin{aligned} -\Delta e &= 1, & x \in \Omega, \\ e &= 0, & x \in \partial\Omega. \end{aligned} \quad (3.2)$$

Since $0 < p < 1 < q$, we can find $\lambda_0 > 0$ such that for all $0 < \lambda \leq \lambda_0$ there exists $T = T(\lambda) > 0$ satisfying

$$T \geq \lambda A T^p E^p + B T^q E^q. \quad (3.3)$$

As a consequence, the function Te verifies

$$T = -\Delta(Te) \geq \lambda A(Te)^p + B(Te)^q \geq \lambda a(x)(Te)^p + b(x)(Te)^q, \quad (3.4)$$

and hence it is a supersolution of $(1)_\lambda^+$. Moreover, let u_0 denote the solution of the following problem:

$$\begin{aligned} -\Delta u &= \lambda a(x)u_0^p, & x \in \Omega, \\ u_0 &= 0, & x \in \partial\Omega. \end{aligned} \quad (3.5)$$

(From [3] we know that u_0 exists.) Then εu_0 is a subsolution of $(1)_\lambda^+$, provided

$$-\Delta(\varepsilon u_0) = \lambda \varepsilon a(x)u_0^p \leq \lambda a(x)(\varepsilon u_0)^p + b(x)(\varepsilon u_0)^q, \quad (3.6)$$

which is satisfied for all $\varepsilon > 0$ small enough and all λ . Taking ε as possibly smaller, we also have

$$\varepsilon u_0 < Te. \quad (3.7)$$

It follows that $(1)_\lambda^+$ has a solution u , $\varepsilon u_0 \leq u \leq Te$ whenever $\lambda \leq \lambda_0$, and thus $\Lambda \geq \lambda_0$.

Next, let λ^* be such that

$$c_0(\lambda^* t^p + t^q) > \lambda_1 t, \quad \forall t > 0. \quad (3.8)$$

If λ is such that $(1)_\lambda^+$ has a solution u , multiplying $(1)_\lambda^+$ by φ_1 and integrating over Ω we find

$$\lambda_1 \int_{\Omega} u \varphi_1 dx = \lambda \int_{\Omega} a(x) u^p \varphi_1 dx + \int_{\Omega} b(x) u^q \varphi_1 dx \geq c_0 \left[\int_{\Omega} (\lambda u^p \varphi_1 + u^q \varphi_1) dx \right]. \quad (3.9)$$

This and (3.5) immediately imply that $\lambda < \lambda^*$ and show that $\Lambda \leq \lambda^*$, hence $0 < \Lambda < +\infty$. $\square$

We are now ready to give the proof of Theorem 3.1.

Proof. (i) From the proof of lemma, it follows that, for all $\lambda \in (0, \Lambda)$, problem $(1)_\lambda^+$ has a solution u_λ . Let u_0 satisfy (3.5); the iteration

$$-\Delta u_{n+1} = \lambda a(x) u_n^p + b(x) u_n^q \quad (3.10)$$

satisfies $u_n \uparrow u_\lambda$ by making use of Lemma 3.3 of [1] and maximum principle. It is easy to check that u_λ is a minimal solution of $(1)_\lambda^+$. Indeed, if u is any solution of $(1)_\lambda^+$, then $u \geq u_0$ and u is a supersolution of $(1)_\lambda^+$. Thus $u_n \leq u$, for all n , by induction, and $u_\lambda \leq u$. Next, we will prove that $I_\lambda(u_\lambda) < 0$. Indeed,

$$I_\lambda(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 dx - \frac{\lambda}{p+1} \int_{\Omega} a(x) |u|^{p+1} dx - \frac{1}{q+1} \int_{\Omega} b(x) |u|^{q+1} dx. \quad (3.11)$$

Since u_λ is a solution of $(1)_\lambda^+$ we have

$$\int_{\Omega} |\nabla u_\lambda|^2 dx = \int_{\Omega} \lambda a(x) u_\lambda^{p+1} dx + \int_{\Omega} b(x) u_\lambda^{q+1} dx. \quad (3.12)$$

From Lemma 3.5 of [1], we know

$$\int_{\Omega} \left[|\nabla \varphi|^2 - \left(\lambda p a(x) u_\lambda^{p-1} + q b(x) u_\lambda^{q-1} \right) \varphi^2 \right] dx \geq 0, \quad \forall \varphi \in H_0^1. \quad (3.13)$$

In particular with $\varphi = u_\lambda$, we infer

$$\int_{\Omega} |\nabla u_\lambda|^2 dx - \lambda p \int_{\Omega} a(x) u_\lambda^{p+1} dx - q \int_{\Omega} b(x) u_\lambda^{q+1} dx \geq 0. \quad (3.14)$$

Combining (3.12) and (3.14), we obtain

$$\begin{aligned} I_\lambda(u_\lambda) &= \lambda \left(\frac{1}{2} - \frac{1}{p+1} \right) \int_{\Omega} a(x) u_\lambda^{p+1} dx + \left(\frac{1}{2} - \frac{1}{q+1} \right) \int_{\Omega} b(x) u_\lambda^{q+1} dx \\ &\leq \frac{1-p}{2} \left(-\frac{1}{p+1} + \frac{1}{q+1} \right) \int_{\Omega} a(x) u_\lambda^{p+1} dx < 0. \end{aligned} \quad (3.15)$$

To complete the proof of (i), it remains to show that

$$u_\lambda < u_{\lambda_1} \quad \text{whenever } \lambda < \lambda_1. \quad (3.16)$$

Indeed, if $\lambda < \lambda_1$ then u_{λ_1} is a supersolution of $(1)_\lambda^+$. Since, for $\varepsilon > 0$ small, εu_0 is a subsolution of $(1)_\lambda^+$ and $\varepsilon u_0 < u_{\lambda_1}$, then $(1)_\lambda^+$ possesses a solution v , with

$$(\varepsilon u_0 \leq) v \leq u_{\lambda_1}. \quad (3.17)$$

Since u_λ is the minimal solution of $(1)_\lambda^+$, we infer that $u_\lambda \leq v \leq u_{\lambda_1}$. Moreover

$$\begin{aligned} -\Delta(u_{\lambda_1} - u_\lambda) &= \lambda_1 a(x) u_{\lambda_1}^p + b(x) u_{\lambda_1}^q - (\lambda a(x) u_\lambda^p + b(x) u_\lambda^q) \\ &\geq \lambda a(x) u_{\lambda_1}^p + b(x) u_{\lambda_1}^q - a(x) u_\lambda^p - b(x) u_\lambda^q \geq 0. \end{aligned} \quad (3.18)$$

Since $u_{\lambda_1} \neq u_\lambda$ (because $\lambda < \lambda_1$), then the Hopf Maximum principle yields $u_\lambda < u_{\lambda_1}$.

(ii) Let λ_n be a sequence such that $\lambda_n \uparrow \Lambda$; then from $I_{\lambda_n}(u_{\lambda_n}) < 0$ we deduce that there exists $C > 0$ such that

$$\begin{aligned} \|\nabla u_n\|^2 &\leq C, \\ \|u_n\|_{p+1}^{p+1} &\leq C. \end{aligned} \quad (3.19)$$

Then there exists $u^* \in H_0^1$ such that $u_n \rightarrow u^* > 0$ a.e. in Ω , strongly in L^{p+1} and weakly in H_0^1 . Such a u^* is thus a weak solution of $(1)_\Lambda^+$ for $\lambda = \Lambda$.

(iii) This follows from the definition of Λ . $\square$

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