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Research Article

The Stability of Some Differential Equations

M. M. Pourpasha,[1] Th. M. Rassias,[2] R. Saadati,[3] and S. M. Vaezpour[4]

[1] *Department of Mathematics, Chalous Branch, Islamic Azad University (IAU), Chalous 46615, Iran*
[2] *Department of Mathematics, National Technical University of Athens, Zografou Campus, 157 80 Athens, Greece*
[3] *Department of Mathematics, Science and Research Branch, Islamic Azad University (IAU), Tehran 14778, Iran*
[4] *Department of Mathematics and Computer Science, Amirkabir University of Technology, 424 Hafez Ave., Tehran, Iran*

Correspondence should be addressed to R. Saadati, rsaadati@eml.cc

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We generalize the results obtained by Jun and Min (2009) and use fixed point method to obtain the stability of the functional equation $f(x+\sigma(y)) = F[f(x), f(y)]$, for a class of functions of a vector space into a Banach space where σ is an involution. Then we obtain the stability of the differential equations of the form $y' = F[q(x), P(x)y(x)]$.

1. Introduction and Preliminary

The stability problem of functional equations originated from a question of Ulam [1] in 1940, concerning the stability of group homomorphisms.

The stability concept that was introduced by Rassias' theorem [2] in 1978 provided a large influence to a number of mathematicians to develop the notion of what is known today by the term Hyers-Ulam-Rassias stability of the linear mapping. Since then, the stability of several functional equations has been extensively investigated by several mathematicians, see [3–5]. They have many applications in Information Theory, Physics, Economic Theory, and Social and Behavior Sciences.

In 1996, Isac and Rassias [6] were the first to use the fixed point methods to investigate the Hyers-Ulam-Rassias stability.

Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a generalized metric on X if and only if d satisfies

(1) $d(x,y) = 0$, if and only if $x = y$,
(2) $d(x,y) = d(y,x)$, for all $x, y \in X$,
(3) $d(x,z) \leq d(x,y) + d(y,z)$, for all $x, y, z \in X$.

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

We now introduce one of fundamental results of fixed point theory. For the proof, refer to [7]. For an extensive theory of fixed point theorems and other nonlinear methods, the reader is referred to the book of Hyers et al. [8].

Theorem 1.1. *Let (X,d) be a generalized complete metric space. Assume that $J : X \to X$ is a strictly contractive operator with the Lipschitz constant $0 < L < 1$. If there exists a nonnegative integer k such that $d(J^{k+1}f, J^k f) < \infty$ for some $f \in X$, then the followings are true:*

(a) *the sequence $\{J^n f\}$ converges to a fixed point f^* of J,*

(b) *f^* is the unique fixed point of J in*

$$X^* = \left\{ g \in X : d\left(J^k f, g\right) < \infty \right\}, \tag{1.1}$$

(c) *if $g \in X^*$, then*

$$d(g, f^*) \leq \frac{1}{1-L} d(Jg, g). \tag{1.2}$$

2. Stability of the Generalized Functional Equations

The stability problem for a general equation of the form

$$f(G(x,y)) = H[f(x), f(y)] \tag{2.1}$$

was investigated by Cholewa [9] in 1984. Indeed, Cholewa proved the superstability of the above equation under some additional assumptions on the functions and spaces involved.

Recently, Jung and Min [10] applied the fixed point method to the investigate the stability of functional equation

$$f(x+y) = F[f(x), f(y)]. \tag{2.2}$$

In this section, we generalized the Jun and Min's results and use fixed point approach to obtain the stability of the functional equation

$$f(x+\sigma(y)) = F[f(x), f(y)] \tag{2.3}$$

for a class of functions of a vector space into a Banach space where σ is an involution.

Theorem 2.1. *Let X and $(Y, \|\cdot\|)$ be a vector space over K and a Banach space over K, respectively. Let $(X \times X, \|\cdot\|_2)$ be a Banach space over K. Assume that $F : X \times X \to Y$ is a bounded linear transformation, whose norm is denoted by $\|F\|$, satisfying*

$$F(F(u,u), F(v,v)) = F(F(u,v), F(u,v)) \tag{2.4}$$

for all $u, v : X \to X$ and there exists a real number $\kappa > 0$ with

$$\|(u(x), u(\sigma(x))) - (v(x), v(\sigma(x)))\|_2 \leq \kappa \|u(x) - v(x)\| \tag{2.5}$$

for all $u, v : X \to X$. Moreover, assume that $\varphi : X \times X \to [0, \infty)$ is a given function satisfying

$$\varphi(x, \sigma(y)) \leq \varphi(2x, 2y) \tag{2.6}$$

for all $x, y \in X$. If $\kappa\|F\| < 1$ and a function $f : X \to Y$ satisfies the inequality

$$\|f(x + \sigma(y)) - F[f(x), f(y)]\| \leq \varphi(x, y) \tag{2.7}$$

for any $x, y \in X$, then there exists a unique solution $f^* : X \to Y$ of (2.3) *such that*

$$\|f(x) - f^*(x)\| \leq \frac{1}{1 - \kappa\|F\|}\varphi(x, x). \tag{2.8}$$

Proof. First, we denote by X the set of all functions $h : X \to Y$ and by d the generalized metric on X defined as

$$d(g, h) = \inf\{C \in [0, \infty) : \|g(x) - h(x)\| \leq CM'(x, y),\ \forall x \in E_1\}. \tag{2.9}$$

By a similar method used at the proof of [4, Theorem 3.1], we can show that (X, d) is a generalized complete metric space. Now, let us define an operator $J : X \to X$ by

$$(Jh)(x) = F\left[h\left(\frac{x}{2}\right), h\left(\sigma\left(\frac{x}{2}\right)\right)\right] \tag{2.10}$$

for every $x \in X$. We assert that J is strictly contractive on X. Given $g, h \in X$, let $C \in [0, \infty]$ be an arbitrary constant with $d(g, h) \leq C$, that is,

$$\|g(x) - h(x)\| \leq C\varphi(x, y) \tag{2.11}$$

for each $x \in X$. By (2.5), (2.6), (2.10), and (2.11), we have

$$\begin{aligned}
\|Jg(x) - Jh(x)\| &\le \left\| F\left[g\left(\frac{x}{2}\right), g\left(\sigma\left(\frac{x}{2}\right)\right)\right] - F\left[h\left(\frac{x}{2}\right), h\left(\sigma\left(\frac{x}{2}\right)\right)\right]\right\| \\
&\le \|F\| \left\| \left[g\left(\frac{x}{2}\right), g\left(\sigma\left(\frac{x}{2}\right)\right)\right] - \left[h\left(\frac{x}{2}\right), h\left(\sigma\left(\frac{x}{2}\right)\right)\right]\right\| \\
&\le \|F\|\kappa \left\| g\left(\frac{x}{2}\right) - h\left(\frac{x}{2}\right)\right\| \\
&\le \|F\|\kappa C\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \\
&\le \|F\|\kappa C\varphi(x, y)
\end{aligned} \tag{2.12}$$

for every $x \in X$. Then, from (2.9) we have $d(Jg, Jh) \le \kappa\|F\|d(g, h)$ for any $g, h \in X$, where $\kappa\|F\|$ is the Lipschitz constant with $0 < \kappa\|F\| < 1$. Thus, J is strictly contractive.

Now, we claim that $d(Jf, f) \le \infty$. Replacing $x/2$ by x and $\sigma(x/2)$ by y in (2.7), then it follows from (2.6) and (2.10) that

$$\begin{aligned}
\left\| f\left(\frac{x}{2} + \sigma\left(\sigma\left(\frac{x}{2}\right)\right)\right) - F\left[f\left(\frac{x}{2}\right), f\left(\sigma\left(\frac{x}{2}\right)\right)\right]\right\| &\le \varphi\left(\frac{x}{2}, \sigma\left(\frac{x}{2}\right)\right) \\
\|f(x) - (Jf)(x)\| &\le \varphi\left(\frac{x}{2}, \sigma\left(\frac{x}{2}\right)\right) \\
&\le \varphi(x, x)
\end{aligned} \tag{2.13}$$

for every $x \in X$. Then,

$$d(Jf, f) \le 1 \le \infty. \tag{2.14}$$

Now, it follows from Theorem 1.1(a) that there exists a function $f^* : E_1 \to E_2$ which is a fixed point of J, such that

$$\lim_{n\to\infty} d(J^n f, f^*) = 0. \tag{2.15}$$

From Theorem 1.1(c), we get

$$d(J^n f, f^*) \le \frac{1}{1 - \kappa\|F\|} d(Jf, f) \le \frac{1}{1 - \kappa\|F\|}, \tag{2.16}$$

which implies the validity of (2.8). According to Theorem 1.1(b), f^* is the unique fixed point of J with $d(f, f^*) < \infty$.

We now assert that

$$\|(J^n f)(x + \sigma(y)) - F[(J^n f)(x), (J^n f)(y)]\| \le (\kappa\|F\|)^n \varphi(x, x) \tag{2.17}$$

for all $n \in N$ and $x, y \in X$. Indeed, it follows from (2.4), (2.5), (2.6), (2.7), and (2.10) that

$$\begin{aligned}
&\|(Jf)(x+\sigma(y)) - F[(Jf)(x),(Jf)(y)]\| \\
&\quad = \left\| F\left[f\left(\frac{x+\sigma(y)}{2}\right), f\left(\sigma\left(\frac{x+\sigma(y)}{2}\right)\right)\right] \right. \\
&\qquad \left. -F\left[F\left[f\left(\frac{x}{2}\right), f\left(\sigma\left(\frac{x}{2}\right)\right)\right], F\left[f\left(\frac{y}{2}\right), f\left(\sigma\left(\frac{y}{2}\right)\right)\right]\right]\right\| \\
&\quad \le \|F\| \left\| \left[f\left(\frac{x+\sigma(y)}{2}\right), f\left(\sigma\left(\frac{x+\sigma(y)}{2}\right)\right)\right] \right. \\
&\qquad \left. -\left[F\left[f\left(\frac{x}{2}\right), f\left(\sigma\left(\frac{x}{2}\right)\right)\right], F\left[f\left(\frac{y}{2}\right), f\left(\sigma\left(\frac{y}{2}\right)\right)\right]\right]\right\| \\
&\quad \le \|F\|\kappa \left\| f\left(\frac{x+\sigma(y)}{2}\right) - \left[F\left[f\left(\frac{x}{2}\right), f\left(\frac{y}{2}\right)\right]\right]\right\| \\
&\quad \le \|F\|\kappa\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \\
&\quad \le \|F\|\kappa\varphi(x,x)
\end{aligned} \tag{2.18}$$

for any $x, y \in X$. Then, it follows from (2.4), (2.5), (2.6), (2.10), and (2.17) that

$$\begin{aligned}
&\left\|\left(J^{n+1}f\right)(x+\sigma(y)) - F\left[\left(J^{n+1}f\right)(x), \left(J^{n+1}f\right)(y)\right]\right\| \\
&\quad = \left\| F\left[J^n f\left(\frac{x+\sigma(y)}{2}\right), J^n f\left(\sigma\left(\frac{x+\sigma(y)}{2}\right)\right)\right] \right. \\
&\qquad \left. -F\left[F\left[J^n f\left(\frac{x}{2}\right), J^n f\left(\sigma\left(\frac{x}{2}\right)\right)\right], F\left[J^n f\left(\frac{y}{2}\right), J^n f\left(\sigma\left(\frac{y}{2}\right)\right)\right]\right]\right\| \\
&\quad \le \|F\|\kappa \left\| J^n f\left(\frac{x+\sigma(y)}{2}\right) - \left[F\left[J^n f\left(\frac{x}{2}\right), J^n f\left(\frac{y}{2}\right)\right]\right]\right\| \\
&\quad \le (\|F\|\kappa)^{n+1}\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \\
&\quad \le (\|F\|\kappa)^{n+1}\varphi(x,x)
\end{aligned} \tag{2.19}$$

for all $n \in N$, which proves the validity of (2.17).

Finally, we prove that $f^*(x+\sigma(y)) = F[f^*(x), f^*(y)]$ for any $x, y \in X$. Since F is continuous as a bounded linear transformation, it follows from (2.15) and (2.17) that

$$\begin{aligned}
&\left\| f^*(x+\sigma(y)) - F\left[f^*(x), f^*(y)\right] \right\| \\
&\quad = \lim_{n\to\infty} \left\| J^n f\left(\frac{x+\sigma(y)}{2}\right) - \left[F\left[J^n f\left(\frac{x}{2}\right), J^n f\left(\frac{y}{2}\right)\right]\right] \right\| \\
&\quad \le \lim_{n\to\infty} (\|F\|\kappa)^n \varphi(x,x) = 0
\end{aligned} \tag{2.20}$$

for all $x, y \in X$, which implies that f^* is a solution of (2.7). □

Corollary 2.2. *Let X and $(Y, \|\cdot\|)$ be a vector space over K and a Banach space over K, respectively, and let $(Y \times Y, \|\cdot\|_2)$ be a Banach space over K. Assume that $F : Y \times Y \to Y$ is a bounded linear transformation, whose norm is denoted by F, satisfying condition* (2.4) *and that there exists a real number $\kappa > 0$ satisfying condition* (2.5). *If $\kappa\|F\| < 1$ and a function $f : X \to Y$ satisfies the inequality*

$$\left\| f(x+y) - F\left[f(x), f(y)\right] \right\| \le \epsilon\left(\|x\|^p + \|y\|^p\right) \tag{2.21}$$

for all $x, y \in X$ and for some nonnegative real constants θ and p, then there exists a unique solution $f^* : X \to Y$ of 1.2 such that

$$\left\| f(x) - f^*(x) \right\| \le \frac{2\theta}{1-\kappa\|F\|}\|x\|^p \tag{2.22}$$

for all $x \in X$.

Example 2.3. Assume that $X = Y = \mathbb{C}$, and consider the Banach spaces $(\mathbb{C}, |\cdot|)$ and $(\mathbb{C}\times\mathbb{C}, |\cdot|_2)$, where we define $|(u(t), v(t))|_2 = \sqrt{|u|^2 + |v|^2}$ for all $u, v : \mathbb{C} \to \mathbb{C}$. Let A and B be fixed complex numbers with $|A| + |B| < 1/\sqrt{2}$, and let $F : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ be a linear transformation defined by

$$F(u(t), v(t)) = Au(t) + Bv(t). \tag{2.23}$$

Then it is easy to show that F satisfies condition (2.13).

If u and v are complex numbers satisfying $|(u(t), v(t))|_2 \le 1$ for all t, then

$$|F(u,v)| \le |A|\,|u| + |B|\,|v| \le |A| + |B|. \tag{2.24}$$

Thus, we get

$$\|F\| \le \sup\{|F(u,v)| : u, v \in \mathbb{C} \text{ with } |(u,v)|_2 \le 1\} \le |A| + |B|, \tag{2.25}$$

which implies the boundedness of the linear transformation F.

On the other hand, we obtain

$$\|(u(x), u(\sigma(x))) - (v(x), v(\sigma(x)))\|_2 \leq \sqrt{2}\|u(x) - v(x)\| \tag{2.26}$$

for any $u, v \in \mathbb{C}$, then we have

$$\|F\|\kappa \leq \sqrt{2}(|A| + |B|) \leq 1. \tag{2.27}$$

If the function $f : \mathbb{C} \to \mathbb{C}$ satisfies the inequality

$$\left\| f^*(x + \sigma(y)) - F\left[f^*(x), f^*(y)\right] \right\| \tag{2.28}$$

for all $x, y \in \mathbb{C}$ and for some $\varepsilon > 0$, then Corollary 2.2 (with $\theta = \varepsilon/2$ and $p = 0$) implies that there exists a unique function $f^* : \mathbb{C} \to \mathbb{C}$ such that

$$\left\| f^*(x + \sigma(y)) = F\left[f^*(x), f^*(y)\right] \right\| \tag{2.29}$$

for all $x, y \in \mathbb{C}$ and

$$\left| f^*(x) - f(x) \right| \leq \frac{\varepsilon}{1 - \sqrt{2}(|A| + |B|)} \tag{2.30}$$

for any $x \in \mathbb{C}$.

3. Stability of the Generalized Differential Equations

Let Y be a normed space, and let I be an open interval. Assume that for any function $y : I \to Y$ satisfying the differential inequality

$$\left\| a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x) \right\| \leq \varepsilon \tag{3.1}$$

for all $x \in I$ and for some $\varepsilon \geq 0$, there exists a solution $y_0 : I \to Y$ of the differential equation

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0(x)y(x) + h(x) = 0 \tag{3.2}$$

such that $\|y(x) - y_0(x)\| \leq K(\varepsilon)$ for any $x \in I$, where $K(\varepsilon)$ is an expression of ε only. Then, we say that the above differential equation has the Hyers-Ulam stability.

If the above statement is also true when we replace ε and $K(\varepsilon)$ by $\varphi(x)$ and $\Phi(x)$, where $\varphi, \Phi : I \to [0, \infty)$ are functions not depending on y and y_0 explicitly, then we say that the corresponding differential equation has the Hyers-Ulam-Rassias stability (or the generalized Hyers-Ulam stability).

We may apply these terminologies for other differential equations. For more detailed definitions of the Hyers-Ulam stability and the Hyers-Ulam-Rassias stability, refer to [11, 12].

In 1998, Alsina and Ger investigated the Hyers-Ulam stability of differential equations. They proved in [13] that if a differentiable function $y : I \to \mathbb{R}$ satisfies the differential inequality $|y'(t) - y(t)| \leq \varepsilon$, where I is an open subinterval of $\mathbb{R}$, then there exists a differentiable function $y_0 : I \to \mathbb{R}$ satisfying $y_0'(t) = y_0(t)$ and $|y_0(t) - y(t)| \leq 3\varepsilon$ for any $t \in I$.

Alsina and Ger's results have been generalized by Takahasi et al. [14]. They proved that the Hyers-Ulam stability holds for the Banach space-valued differential equation $y'(x) = \lambda y(x)$ (see also [15]).

Recently, Takahasi et al. also proved the Hyers-Ulam stability of linear differential equations of first order, $y'(x) + g(x)y(x) = 0$, where $g(x)$ is a continuous function, and they also proved the Hyers-Ulam stability of linear differential equations of other types (see [16–18]).

In this section, for a bounded and continuous function $F(x, y)$, we will adopt the idea of Cădariu and Radu [19, 20] and prove the Hyers-Ulam-Rassias stability as well as the Hyers-Ulam stability of the differential equations of the form

$$y'(x) = F\big(q(x), p(x)y(x)\big). \tag{3.3}$$

Theorem 3.1. *For given real numbers a and b with $a < b$, let $I = [a, b]$ be a closed interval and choose $c \in I$. Let K and L be positive constants with $0 < KL < 1$. Assume that $F : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies a Lipschitz condition*

$$\big|F(x, y) - F(x, z)\big| \leq L\big|y - z\big| \tag{3.4}$$

for any $x \in I$ and $y, z \in \mathbb{R}$. If a continuously differentiable function $y : I \to \mathbb{R}$ satisfies

$$\big|y'(x) - F\big(q(x), p(x)y(x)\big)\big| \leq \varphi(x) \tag{3.5}$$

for all $x \in I$, where $p(x), q(x)$ are continuous functions in which $|p(x)| \leq c$ and $\varphi : I \to (0, \infty)$ is a continuous function with

$$\left|\int_c^x \varphi(t)dt\right| \leq K\varphi(x) \tag{3.6}$$

for each $x \in I$, then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ such that

$$y_0(x) = y(c) + \int_c^x F\big(q(x), p(x)y(x)\big)dt \tag{3.7}$$

(consequently, y_0 is a solution to (2.15)*) and*

$$\big|y(x) - y_0(x)\big| \leq \frac{K}{1 - KL}\varphi(x) \tag{3.8}$$

for all $x \in I$.

Proof. Let us define a set X of all continuous functions $f : I \to \mathbb{R}$ by

$$X = \{f : I \to \mathbb{R} \mid f \text{ is continuous}\} \tag{3.9}$$

and introduce a generalized metric on X as follows:

$$d(f,g) = \inf\{C \in [0,\infty) : |f(x) - g(x)| \le C\varphi(x),\ \forall x \in I\}. \tag{3.10}$$

By a similar method used at the proof of [4, Theorem 3.1], we assert that (X,d) is complete. Let $\{h_n\}$ be a Cauchy sequence in (X,d).

Then, for any $\varepsilon > 0$, there exists an integer $N_\varepsilon > 0$ such that $d(h_m,h_n) \le \varepsilon$ for all $m,n \in N_\varepsilon$. It further follows from (3.10) that

$$\forall\varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N}\ \forall m,n \in N_\varepsilon\ \forall x \in I : |h_m(x) - h_n(x)| \le \varepsilon\varphi(x). \tag{3.11}$$

Equation (3.11) implies that $\{h_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, $\{h_n(x)\}$ converges for each $x \in I$. Thus, we can define a function $h : I \to \mathbb{R}$ by

$$h(x) = \lim_{n\to\infty} h_n(x). \tag{3.12}$$

Let m increase to infinity, then by (3.11) we have

$$\forall\varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N}\ \forall n \in N_\varepsilon\ \forall x \in I : |h(x) - h_n(x)| \le \varepsilon\varphi(x). \tag{3.13}$$

Since φ is bounded on I, $\{h_n\}$ converges uniformly to h. Hence, h is continuous and $h \in X$.

Further, considering (3.10) and (3.13), then

$$\forall\varepsilon > 0\ \exists N_\varepsilon \in \mathbb{N}\ \forall n \in N_\varepsilon : d(h,h_n) \le \varepsilon. \tag{3.14}$$

Then, the Cauchy sequence $\{h_n\}$ converges to h in (X,d). Hence, (X,d) is complete.

Now, define the operator $J : X \to X$ by

$$(Jf)(x) = y(c) + \int_c^x F(q(x), p(x)f(x))\,dt \qquad x \in I \tag{3.15}$$

for all $f \in X$. (Indeed, according to the Fundamental Theorem of Calculus, Jf is continuously differentiable on I, since F and f are continuous functions. Hence, we may conclude that $Jf \in X$.) We prove that J is strictly contractive on X. For any $f,g \in X$, let $C_{fg} \in [0,\infty]$ be an arbitrary constant with $d(f,g) \le C_{fg}$, then, by (2.15), we have

$$|f(x) - g(x)| \le C_{fg}\varphi(x) \tag{3.16}$$

for any $x \in I$. It then follows from (3.4), (3.6), (3.10), (3.15), and (3.16) that

$$\begin{aligned}
\left|(Jf)(x) - (Jg)(x)\right| &\leq \left|\int_c^x \{F(q(x), p(x)f(x)) - F(q(x), p(x)g(x))\}dt\right| \\
&\leq \left|\int_c^x \left|F(q(x), p(x)f(x)) - F(q(x), p(x)g(x))\right|dt\right| \\
&\leq L\left|\int_c^x |f(t) - g(t)|dt\right| \\
&\leq LC_{fg}\left|\int_c^x \varphi(t)dt\right| \\
&\leq KLC_{fg}\varphi(x)
\end{aligned} \tag{3.17}$$

for all $x \in I$. Then, $d(Jf, Jg) \leq KLC_{fg}$. Hence, we can conclude that $d(Jf, Jg) \leq KLd(f, g)$ for any $f, g \in X$ (note that $0 < KL < 1$). It follows from (3.9) and (3.15) that for an arbitrary $g_0 \in X$, there exists a constant $0 < C < 1$ with

$$\left|(Jg_0)(x) - g_0(x)\right| = \left|y(c) + \int_c^x F(p(t), g_0(t))dt - g_0(x)\right| \leq C\varphi(x) \tag{3.18}$$

for all $x \in I$, since $F(x, g_0(x))$ and $g_0(x)$ are bounded on I and $\min_{x\in I}\varphi(x) > 0$. Thus, (3.10) implies that

$$d(Jg_0, g_0) < \infty. \tag{3.19}$$

Therefore, according to Theorem 1.1(a), there exists a continuous function $y_0 : I \to \mathbb{R}$ such that $J^n g_0 \to y_0$ in (X, d) and $Jy_0 = y_0$, that is, y_0 satisfies (3.7) for every $x \in I$. For any $g \in X$, since g and g_0 are bounded on I and $\min_{x\in I}\varphi(x) > 0$, there exists a constant $0 < C_g < 1$ such that

$$\left|g(x) - g_0(x)\right| \leq C_g\varphi(x) \tag{3.20}$$

for any $x \in I$. Hence, we have $d(g_0, g) < \infty$ for all $g \in X$, that is, $\{g \in X \mid d(g_0, g) < \infty\} = X$. Hence, in view of Theorem 1.1(b), we conclude that y_0 is the unique continuous function with the property (3.7).

On the other hand, it follows from (3.5) that

$$-\varphi(x) \leq y'(x) - F(q(x), p(x)y(x)) \leq \varphi(x) \tag{3.21}$$

for all $x \in I$. If we integrate each term in the above inequality from c to x, then we obtain

$$\left|y(x) - y(c) - \int_c^x F(q(x), p(x)y(x))dt\right| \leq \left|\int_c^x \varphi(t)dt\right| \tag{3.22}$$

for any $x \in I$. Thus, by (3.6) and (3.15), we get

$$\left|y(x) - Jy(x)\right| \leq K\varphi(x) \tag{3.23}$$

for each $x \in I$, which implies that

$$d(Jy, y) \leq K. \tag{3.24}$$

Finally, Theorem 1.1(c) and (3.24) implys that

$$d(Jy, y_0) \leq \frac{1}{1-KL} d(Jy, y) \leq \frac{K}{1-KL}, \tag{3.25}$$

which means that inequality (3.24) holds true for all $x \in I$. □

Now, we prove the last theorem for unbounded intervals. Also we show that Theorem 3.1 is also true if I is replaced by an unbounded interval such as $(-\infty, b]$, $\mathbb{R}$, or $[a, \infty)$.

Theorem 3.2. *For given real numbers a and b, let I denote either $(-\infty, b]$, $\mathbb{R}$, or $[a, \infty)$. Set either $c = a$ for $I = [a, \infty)$ or $c = b$ for $I = (-\infty, b]$ or c is a fixed real number if $I = \mathbb{R}$. Let K and L be positive constants with $0 < KL < 1$. Assume that $F : I \times \mathbb{R} \to \mathbb{R}$ is a continuous function which satisfies Lipschitz condition* (3.4) *for any $x \in I$ and $y, z \in \mathbb{R}$. If a continuously differentiable function $y : I \to \mathbb{R}$ satisfies*

$$\left|y'(x) - F(q(x), p(x)y(x))\right| \leq \varphi(x) \tag{3.26}$$

for all $x \in I$, where $p(x)$ is a continuous function and $\varphi : I \to (0, \infty)$ is a continuous function satisfying condition (3.6) *for each $x \in I$, then there exists a unique continuous function $y_0 : I \to \mathbb{R}$ such that*

$$\left|y(x) - y_0(x)\right| \leq \frac{K}{1-KL}\varphi(x) \tag{3.27}$$

for all $x \in I$.

Proof. We prove for $I = \mathbb{R}$. The other cases can be proved similarly. For any $n \in \mathbb{N}$, we define $I_n = [c-n, c+n]$. (We set $I_n = [b-n, b]$ for $I = (-\infty, b]$ and $I_n = [a, a+n]$ for $I = [a, \infty)$). By Theorem 3.1, there exists a unique continuous function $y_n : I_n \to \mathbb{R}$ such that

$$y_n(x) = y(c) + \int_c^x F(q(x), p(x)y_n(t))dt \tag{3.28}$$

$$\left|y_n(x) - y(x)\right| \leq \frac{K}{1-KL}\varphi(x) \tag{3.29}$$

for all $x \in I_n$. The uniqueness of y_n implies that, if $x \in I_n$, then

$$y_n(x) = y_{n+1}(x) = y_{n+2}(x) = \cdots . \tag{3.30}$$

For any $x \in \mathbb{R}$, define $n(x) \in \mathbb{N}$ as

$$n(x) = \min\{n \in N \mid x \in I_n\}. \tag{3.31}$$

Moreover, define a function $y_0 : \mathbb{R} \to \mathbb{R}$ by

$$y_0(x) = y_{n(x)}(x), \tag{3.32}$$

and we assert that y_0 is continuous. For an arbitrary $x_1 \in \mathbb{R}$, we choose the integer $n_1 = n(x_1)$. Then, x_1 belongs to the interior of I_{n_1+1} and there exists an $\varepsilon > 0$ such that $y_0(x) = y_{n_1+1}(x)$ for all x with $x_1 - \varepsilon < x < x_1 + \varepsilon$. Since y_{n_1+1} is continuous at x_1, so is y_0. That is, y_0 is continuous at x_1 for any $x_1 \in \mathbb{R}$.

We will now show that y_0 satisfies (3.8) for all $x \in \mathbb{R}$. For an arbitrary $x \in \mathbb{R}$, we choose the integer $n(x)$. Then, it holds that $x \in I_{n(x)}$ and it follows from (3.28) and (3.32) that

$$y_0(x) = y_{n(x)}(x) = y(c) + \int_c^x F\big(q(x), p(x)y_{n(x)}(t)\big)dt = y(c) + \int_c^x F\big(q(x), p(x)y_0(t)\big)dt \tag{3.33}$$

since $n(t) \le n(x)$ for any $t \in I_n(x)$. Then, from (3.30) and (3.32) we have

$$y_{n(x)}(t) = y_{n(t)}(t) = y_0(t). \tag{3.34}$$

Since $x \in I_n(x)$ for every $x \in \mathbb{R}$, by (3.29) and (3.32), we have

$$\big|y_0(x) - y(x)\big| = \big|y_{n(x)}(x) - y(x)\big| \le \frac{K}{1 - KL}\varphi(x) \tag{3.35}$$

for any $x \in \mathbb{R}$.

Finally, we show that y_0 is unique. Let $z_0 : \mathbb{R} \to \mathbb{R}$ be another continuous function which satisfies (3.8), with z_0 in place of y_0, for all $x \in \mathbb{R}$. Suppose x is an arbitrary real number. Since the restrictions $y_0|_{In(x)} (= y_n(x))$ and $z_0|_{In}(x)$ both satisfy (3.7) and (3.8) for all $x \in I_n(x)$, the uniqueness of $y_n(x) = y_0|_{In(x)}$ implies that

$$y_0(x) = y_0|_{In(x)}(x) = z_0|_{In(x)}(x) = z_0(x) \tag{3.36}$$

as required. □

Corollary 3.3. *Given $c \in \mathbb{R}$ and $r > 0$, let I denote a closed ball of radius r and centered at c, that is, $I = \{x \in \mathbb{R} \mid c - r \le x \le c + r\}$, and let $F : I \times \mathbb{R} \to \mathbb{R}$ be a continuous function which satisfies*

a Lipschitz condition (3.4) *for all* $x \in I$ *and* $y, z \in R$, *where* L *is a constant with* $0 < Lr < 1$. *If a continuously differentiable function* $y : I \to \mathbb{R}$ *satisfies the differential inequality*

$$\left|y'(x) - F(x, y(x))\right| \leq \varepsilon \tag{3.37}$$

for all $x \in I$ *and for some* $\varepsilon \geq 0$, *then there exists a unique continuous function* $y_0 : I \to \mathbb{R}$ *satisfying* (3.7) *and*

$$\left|y(x) - y_0(x)\right| \leq \frac{r}{1 - rL}\varepsilon \tag{3.38}$$

for any $x \in I$.

Example 3.4. We choose positive constants K and L with $KL < 1$. For a positive number $\varepsilon < 2K$, let $I = [0, 2K - \varepsilon]$ be a closed interval. Given a polynomial $p(x)$, we assume that a continuously differentiable function $y : I \to \mathbb{R}$ satisfies

$$\left|y'(x) - Ly(x) - p(x)\right| \leq x + \varepsilon \tag{3.39}$$

for all $x \in I$. If we set $F(x, y) = Ly + p(x)$ and $\varphi(x) = x + \varepsilon$, then the above inequality has the identical form. Moreover, we obtain

$$\left|\int_c^x \varphi(t)dt\right| = \frac{x^2}{2} + \varepsilon x \leq K\varphi(x) \tag{3.40}$$

for each $x \in I$, since $K\varphi(x) - x^2/2 - \varepsilon x \geq 0$ for all $x \in I$. By Theorem 3.1, there exists a unique continuous function $y_0 : I \to \mathbb{R}$ such that

$$\begin{gathered} y_0(x) = y(0) + \int_0^x \{Ly(t) - p(t)\}dt \\ \left|y_0(x) - y(x)\right| \leq \frac{K}{1 - KL}(x + \varepsilon) \end{gathered} \tag{3.41}$$

for any $x \in I$.

Example 3.5. Let a be a constant greater than 1 and choose a constant L with $0 < L < \ln a$. Given an interval $I = [0, 1)$ and a polynomial $p(x)$, suppose $y : I \to \mathbb{R}$ is a continuously differentiable function satisfying

$$\left|y'(x) - Ly(x) - p(x)\right| \leq a^x \tag{3.42}$$

for all $x \in I$. If we set $\varphi(x) = a^x$, then we have

$$\left|\int_c^x \varphi(t)dt\right| \leq \frac{1}{\ln a}\varphi(x) \tag{3.43}$$

for any $x \in I$. By Theorem 3.2, there exists a unique continuous function $y_0 : I \to \mathbb{R}$ with

$$y_0(x) = y(0) + \int_0^x \{Ly(t) - p(t)\}dt$$
$$|y_0(x) - y(x)| \leq \frac{1}{\ln a - L} a^x \tag{3.44}$$

for any $x \in I$.

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