Research Article

Weingarten and Linear Weingarten Type Tubular Surfaces in E^3

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We study tubular surfaces in Euclidean 3-space satisfying some equations in terms of the Gaussian curvature, the mean curvature, the second Gaussian curvature, and the second mean curvature. This paper is a completion of Weingarten and linear Weingarten tubular surfaces in Euclidean 3-space.

1. Introduction

Let f and g be smooth functions on a surface M in Euclidean 3-space E^3 . The Jacobi function $\Phi(f, g)$ formed with f, g is defined by

$$\Phi(f, g) = \det \begin{pmatrix} f_s & f_t \\ g_s & g_t \end{pmatrix}, \quad (1.1)$$

where $f_s = \partial f / \partial s$ and $f_t = \partial f / \partial t$. In particular, a surface satisfying the Jacobi equation $\Phi(K, H) = 0$ with respect to the Gaussian curvature K and the mean curvature H on a surface M is called a Weingarten surface or a W -surface. Also, if a surface satisfies a linear equation with respect to K and H , that is, $aK + bH = c$, $(a, b, c) \neq (0, 0, 0)$, $a, b, c \in \mathbb{R}$, then it is said to be a linear Weingarten surface or a LW -surface [1].

When the constant $b = 0$, a linear Weingarten surface M reduces to a surface with constant Gaussian curvature. When the constant $a = 0$, a linear Weingarten surface M reduces to a surface with constant mean curvature. In such a sense, the linear Weingarten surfaces can be regarded as a natural generalization of surfaces with constant Gaussian curvature or with constant mean curvature [1].

If the second fundamental form $\mathbf{II}$ of a surface M in E^3 is nondegenerate, then it is regarded as a new pseudo-Riemannian metric. Therefore, the Gaussian curvature $K_{\mathbf{II}}$ is the second Gaussian curvature on M [1].

For a pair (X, Y) , $X \neq Y$, of the curvatures K , H , $K_{\mathbf{II}}$ and $H_{\mathbf{II}}$ of M in E^3 , if M satisfies $\Phi(X, Y) = 0$ by $aX + bY = c$, then it said to be a (X, Y) -Weingarten surface or (X, Y) -linear Weingarten surface, respectively [1].

Several geometers have studied W -surfaces and LW -surfaces and obtained many interesting results [1–9]. For the study of these surfaces, Kühnel and Stamou investigated ruled (X, Y) -Weingarten surfaces in Euclidean 3-space E^3 [7, 9]. Also, Baikoussis and Koufogiorgos studied helicoidal $(H, K_{\mathbf{II}})$ -Weingarten surfaces [10]. Dillen, and Sodsiri, and Kühnel, gave a classification of ruled (X, Y) -Weingarten surfaces in Minkowski 3-space E_1^3 , where $(X, Y) \in \{K, H, K_{\mathbf{II}}\}$ [2–4]. Koufogiorgos, Hasanis, and Koutroufiotis investigated closed ovaloid (X, Y) -linear Weingarten surfaces in E^3 [11, 12]. Yoon, Blair and Koufogiorgos classified ruled (X, Y) -linear Weingarten surfaces in E^3 [8, 13, 14]. Ro and Yoon studied tubes in Euclidean 3-space which are (K, H) , $(K, K_{\mathbf{II}})$, $(H, K_{\mathbf{II}})$ -Weingarten, and linear Weingarten tubes, satisfying some equations in terms of the Gaussian curvature, the mean curvature, and the second Gaussian curvature [1].

Following the Jacobi equation and the linear equation with respect to the Gaussian curvature K , the mean curvature H , the second Gaussian curvature $K_{\mathbf{II}}$, and the second mean curvature $H_{\mathbf{II}}$, an interesting geometric question is raised: classify all surfaces in Euclidean 3-space satisfying the conditions

$$\begin{aligned}\Phi(X, Y) &= 0, \\ aX + bY &= c,\end{aligned}\tag{1.2}$$

where $X, Y \in \{K, H, K_{\mathbf{II}}, H_{\mathbf{II}}\}$, $X \neq Y$ and $(a, b, c) \neq (0, 0, 0)$.

In this paper, we would like to contribute the solution of the above question by studying this question for tubes or tubular surfaces in Euclidean 3-space E^3 .

2. Preliminaries

We denote a surface M in E^3 by

$$M(s, t) = (m_1(s, t), m_2(s, t), m_3(s, t)).\tag{2.1}$$

Let U be the standard unit normal vector field on a surface M defined by

$$U = \frac{M_s \wedge M_t}{\|M_s \wedge M_t\|},\tag{2.2}$$

where $M_s = \partial M(s, t) / \partial s$. Then, the first fundamental form I and the second fundamental form $\mathbf{II}$ of a surface M are defined by, respectively,

$$\begin{aligned}I &= Eds^2 + 2Fds\,dt + Gdt^2, \\ \mathbf{II} &= eds^2 + 2fds\,dt + gdt^2,\end{aligned}\tag{2.3}$$

where

$$\begin{aligned} E &= \langle M_s, M_s \rangle, \quad F = \langle M_s, M_t \rangle, \quad G = \langle M_t, M_t \rangle, \\ e &= -\langle M_s, U_s \rangle = \langle M_{ss}, U \rangle, \quad f = -\langle M_s, U_t \rangle = \langle M_{st}, U \rangle, \quad g = -\langle M_t, U_t \rangle = \langle M_{tt}, U \rangle, \end{aligned} \quad (2.4)$$

[14]. On the other hand, the Gaussian curvature K and the mean curvature H are

$$\begin{aligned} K &= \frac{eg - f^2}{EG - F^2}, \\ H &= \frac{Eg - 2Ff + Ge}{2(EG - F^2)}, \end{aligned} \quad (2.5)$$

respectively. From Brioschi's formula in a Euclidean 3-space, we are able to compute K_{II} and H_{II} of a surface by replacing the components of the first fundamental form E, F , and G by the components of the second fundamental form e, f , and g , respectively [14]. Consequently, the second Gaussian curvature K_{II} of a surface is defined by

$$K_{\text{II}} = \frac{1}{(|eg| - f^2)^2} \left\{ \begin{vmatrix} -\frac{1}{2}e_{tt} + f_{st} - \frac{1}{2}g_{ss} & \frac{1}{2}e_s & f_s - \frac{1}{2}e_t \\ f_t - \frac{1}{2}g_s & e & f \\ \frac{1}{2}g_t & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_t & \frac{1}{2}g_s \\ \frac{1}{2}e_t & e & f \\ \frac{1}{2}g_s & f & g \end{vmatrix} \right\}, \quad (2.6)$$

and the second mean curvature H_{II} of a surface is defined by

$$H_{\text{II}} = H - \frac{1}{2\sqrt{|\det \Pi|}} \sum_{i,j} \frac{\partial}{\partial u^i} \left(\sqrt{|\det \Pi|} L^{ij} \frac{\partial}{\partial u^j} \left(\ln \sqrt{|K|} \right) \right), \quad (2.7)$$

where u^i and u^j stand for " s " and " $\theta = t$ ", respectively, and $L^{ij} = (L_{ij})^{-1}$, where L_{ij} are the coefficients of the second fundamental form [3, 4].

Remark 2.1. It is well known that a minimal surface has a vanishing second Gaussian curvature, but that a surface with the vanishing second Gaussian curvature need not to be minimal [14].

3. Weingarten Tubular Surfaces

Definition 3.1. Let $\alpha : [a, b] \rightarrow \mathbb{E}^3$ be a unit-speed curve. A tubular surface of radius $\lambda > 0$ about α is the surface with parametrization

$$M(s, \theta) = \alpha(s) + \lambda[N(s) \cos \theta + B(s) \sin \theta], \quad (3.1)$$

$a \leq s \leq b$, where $N(s)$, $B(s)$ are the principal normal and the binormal vectors of α , respectively [1].

The curvature and the torsion of the curve α are denoted by κ , τ . Then, Frenet formula of $\alpha(s)$ is defined by

$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}, \quad (3.2)$$

[1]. Furthermore, we have the natural frame $\{M_s, M_\theta\}$ given by

$$\begin{aligned} M_s &= (1 - \lambda\kappa \cos \theta)T - \lambda\tau \sin \theta N + \lambda\tau \cos \theta B, \\ M_\theta &= -\lambda \sin \theta N + \lambda \cos \theta B. \end{aligned} \quad (3.3)$$

The components of the first fundamental form are

$$E = \lambda^2 \tau^2 + \sigma^2, \quad F = \lambda^2 \tau, \quad G = \lambda^2, \quad (3.4)$$

where $\sigma = 1 - \lambda\kappa \cos \theta$.

On the other hand, the unit normal vector field U is obtained by

$$U = \frac{M_s \wedge M_\theta}{\|M_s \wedge M_\theta\|} = -\varepsilon \cos \theta N - \varepsilon \sin \theta B. \quad (3.5)$$

As $\lambda > 0$, ε is the sign of σ such that if $\sigma < 0$, then $\varepsilon = -1$ and if $\sigma > 0$, then $\varepsilon = 1$. From this, the components of the second fundamental form of M are given by

$$e = \varepsilon \lambda \tau^2 - \varepsilon \kappa \cos \theta \sigma, \quad f = \varepsilon \lambda \tau, \quad g = \varepsilon \lambda. \quad (3.6)$$

If the second fundamental form is nondegenerate, $eg - f^2 \neq 0$, that is, κ , σ and $\cos \theta$ are nowhere vanishing. In this case, we can define formally the second Gaussian curvature K_{II} and the second mean curvature H_{II} on M . On the other hand, the Gauss curvature K , the

mean curvature H , the second Gaussian curvature K_{II} and the second mean curvature H_{II} are obtained by using (2.5), (2.6) and (2.7) as follows:

$$\begin{aligned} K &= -\frac{\kappa \cos \theta}{\lambda \sigma}, \\ H &= \frac{\varepsilon(1 - 2\lambda\kappa \cos \theta)}{2\lambda\sigma}, \\ K_{II} &= -\frac{\varepsilon\kappa(\cos^2\theta - 6\lambda\kappa\cos^3\theta + 4\kappa^2\lambda^2\cos^4\theta + 1)}{4\cos\theta\sigma}, \\ H_{II} &= \frac{1}{-8\varepsilon\lambda\kappa^3\cos^3\theta\sigma^3} \left(\sum_{i=0}^6 g_i \cos^i\theta \right), \end{aligned} \quad (3.7)$$

and where the coefficients g_i are

$$\begin{aligned} g_0 &= 3\lambda^2\kappa^2\tau^2, \\ g_1 &= 2\lambda\kappa(\kappa_s\tau - \kappa\tau_s)\sin\theta - (1 + 6\lambda^2\tau^2)\kappa^3, \\ g_2 &= 2\lambda^2\kappa^2(\kappa\tau_s - 4\kappa_s\tau)\sin\theta + \lambda(3(\kappa_s)^2 + 3\kappa^4 - 2\kappa\kappa_{ss} - \kappa^2\tau^2), \\ g_3 &= 2\lambda^2\kappa(2\kappa^2\tau^2 - \kappa^3 + \kappa\kappa_{ss} - 3(\kappa_s)^2) - \kappa^3, \\ g_4 &= 16\lambda\kappa^4, \\ g_5 &= -20\lambda^2\kappa^5, \\ g_6 &= 8\lambda^3\kappa^6. \end{aligned} \quad (3.8)$$

Differentiating K , K_{II} , H , and H_{II} with respect to s and θ , after straightforward calculations, we get,

$$K_s = -\frac{\kappa_s \cos \theta}{\lambda \sigma^2}, \quad K_\theta = \frac{\kappa \sin \theta}{\lambda \sigma^2}, \quad (3.9)$$

$$H_s = -\frac{\varepsilon\kappa_s \cos \theta}{2\sigma^2}, \quad H_\theta = \frac{\varepsilon\kappa \sin \theta}{2\sigma^2}, \quad (3.10)$$

$$(K_{II})_s = \frac{\varepsilon\kappa_s(8\lambda^3\kappa^3\cos^5\theta - 18\lambda^2\kappa^2\cos^4\theta + 12\lambda\kappa\cos^3\theta - \cos^2\theta - 1)}{4\cos\theta\sigma^2}, \quad (3.11)$$

$$(K_{II})_\theta = -\frac{\varepsilon\kappa \sin \theta(8\lambda^3\kappa^3\cos^5\theta - 18\lambda^2\kappa^2\cos^4\theta + 12\lambda\kappa\cos^3\theta + \sin^2\theta - 2\lambda\kappa \cos \theta)}{4\cos^2\theta\sigma^2}, \quad (3.12)$$

$$(H_{II})_s = \frac{1}{8\varepsilon\kappa^4\cos^3\theta\sigma^4} \left(\sum_{i=0}^6 f_i \cos^i\theta \right), \quad (3.13)$$

and where f_i are

$$\begin{aligned}
f_0 &= 3\kappa^2\tau(\kappa_s\tau - 2\kappa\tau_s), \\
f_1 &= 2\kappa(2\kappa_s(\kappa_s\tau - \kappa\tau_s) - \kappa\kappa_{ss}\tau) \sin \theta + (3\kappa\tau_s - 2\kappa_s\tau)6\lambda\kappa^3\tau, \\
f_2 &= 2\lambda\kappa^2(9\kappa_s(\kappa\tau_s - \kappa_s\tau) + 2\kappa\kappa_{ss}\tau) \sin \theta + 6\lambda^2\kappa^4\tau(3\kappa_s\tau - 2\kappa\tau_s) + \kappa_s(9(\kappa_s)^2 - 10\kappa\kappa_{ss}) \\
&\quad + \kappa^2\tau(2\kappa\tau_s - \kappa_s\tau), \\
f_3 &= 2\lambda^2\kappa^3(\kappa_s(16\kappa_s\tau - 7\kappa\tau_s) - 4\kappa\tau\kappa_{ss}) \sin \theta \\
&\quad + 2\lambda\kappa(15\kappa\kappa_s\kappa_{ss} - (15(\kappa_s)^2 + \kappa^4)\kappa_s + \kappa^2\tau(2\tau\kappa_s - 5\kappa\tau_s)), \\
f_4 &= 2\lambda^2\kappa^2(5\kappa_s(3(\kappa_s)^2 - 2\kappa\kappa_{ss}) + 2\kappa^2\tau(2\kappa\tau_s - 3\tau\kappa_s) + \kappa^4\kappa_s) - 2\kappa^4\kappa_s, \\
f_5 &= 6\lambda\kappa^5\kappa_s, \\
f_6 &= -4\lambda^2\kappa^6\kappa_s,
\end{aligned} \tag{3.14}$$

$$(H_{\Pi})_{\theta} = \frac{1}{8\epsilon\lambda\kappa^3\cos^4\theta\sigma^4} \left(\sum_{i=0}^6 h_i \cos^i \theta \right), \tag{3.15}$$

and where the coefficients h_i are

$$\begin{aligned}
h_0 &= -9\lambda\kappa^2\tau^2 \sin \theta, \\
h_1 &= 2\kappa^3(1 + 15\lambda^2\tau^2) \sin \theta + 4\lambda\kappa(\kappa\tau_s - \kappa_s\tau), \\
h_2 &= \lambda(2\kappa\kappa_{ss} - 8\kappa^4 + \kappa^2\tau^2(1 - 30\lambda^2\kappa^2) - 3(\kappa_s)^2) \sin \theta + 6\lambda^2\kappa^2(3\kappa_s\tau - 2\kappa\tau_s), \\
h_3 &= 4\lambda^2\kappa(2\kappa^4 - \kappa^2\tau^2 - 2\kappa\kappa_{ss} + 3(\kappa_s)^2) \sin \theta + 2\lambda\kappa(\kappa_s\tau - \kappa\tau_s + 4\lambda^2\kappa^2(\kappa\tau_s - 4\kappa_s\tau)), \\
h_4 &= 2\lambda\kappa^3(3\lambda^2(2\kappa\tau^2 - \kappa^3 + \kappa_{ss}) + \kappa) \sin \theta + 2\lambda^2\kappa^2(4(\kappa\tau_s - \tau\kappa_s) - 9\lambda(\kappa_s)^3), \\
h_5 &= 6\lambda^2\kappa^3(\lambda(4\kappa_s\tau - \kappa\tau_s) - \kappa^2 \sin \theta), \\
h_6 &= 4\lambda^3\kappa^6 \sin \theta.
\end{aligned} \tag{3.16}$$

Now, we consider a tubular surface M in E^3 satisfying the Jacobi equation $\Phi(K, H_{\Pi}) = 0$. By using (3.9), (3.13), and (3.15), we obtain $\Phi(K, H_{\Pi})$ in the following form:

$$K_s(H_{\Pi})_{\theta} - K_{\theta}(H_{\Pi})_s = \frac{-\epsilon}{4\lambda^2\kappa^3\sigma^5\cos^3\theta} \sum_{i=0}^4 u_i \cos^i \theta, \tag{3.17}$$

with respect to the Gaussian curvature K and the second mean curvature H_{II} , where

$$\begin{aligned}
u_0 &= -3\lambda\tau\kappa^2(\kappa_s\tau + \kappa\tau_s)\sin\theta, \\
u_1 &= \kappa^3\left((6\lambda^2\tau^2 + 1)\kappa_s + 6\lambda^2\kappa\tau\tau_s\right)\sin\theta - \lambda\kappa^2\kappa_{ss}\tau + \lambda\kappa^3\tau_{ss}, \\
u_2 &= \lambda\left(\kappa^2\kappa_{sss} - 4\kappa\kappa_s\kappa_{ss} - 3\kappa^4\kappa_s + 3(\kappa_s)^3 + \kappa^3\tau\tau_s\right)\sin\theta + \lambda^2\kappa^3(3\kappa_s\tau_s + 4\kappa_{ss}\tau - \kappa\tau_{ss}), \\
u_3 &= \lambda\kappa\left\{\left(7\lambda\kappa\kappa_s\kappa_{ss} - \lambda\kappa^2\kappa_{sss} - 6\lambda(\kappa_s)^3 + 2\lambda\kappa^4\kappa_s - 4\lambda\kappa^3\tau\tau_s\right)\sin\theta\right. \\
&\quad \left.+ \left(\kappa\kappa_{ss}\tau + \kappa\kappa_s\tau_s - (\kappa_s)^2\tau - \kappa^2\tau_{ss}\right)\right\}, \\
u_4 &= -\lambda^2\kappa^2\left\{4\kappa\kappa_s\tau_s - 4\tau(\kappa_s)^2 - \kappa^2\tau_{ss} + \kappa\kappa_{ss}\tau\right\}.
\end{aligned} \tag{3.18}$$

Then, by $\Phi(K, H_{II}) = 0$, (3.17) becomes

$$\sum_{i=0}^4 u_i \cos^i \theta = 0. \tag{3.19}$$

Hence, we have the following theorem.

Theorem 3.2. *Let M be a tubular surface defined by (3.1) with nondegenerate second fundamental form. M is a (K, H_{II}) -Weingarten surface if and only if M is a tubular surface around a circle or a helix.*

Proof. Let us assume that M is a (K, H_{II}) -Weingarten surface, then the Jacobi equation (3.19) is satisfied. Since polynomial in (3.19) is equal to zero for every θ , all its coefficients must be zero. Therefore, the solutions of $u_0 = u_1 = u_2 = u_3 = u_4 = 0$ are $\kappa_s = 0$, $\tau = 0$ and $\kappa_s = 0$, $\tau_s = 0$ that is, M is a tubular surface around a circle or a helix, respectively.

Conversely, suppose that M is a tubular surface around a circle or a helix, then it is easily to see that $\Phi(K, H_{II}) = 0$ is satisfied for the cases both $\kappa_s = 0$, $\tau = 0$ and $\kappa_s = 0$, $\tau_s = 0$. Thus M is a (K, H_{II}) -Weingarten surface.

We suppose that a tubular surface M with nondegenerate second fundamental form in E^3 is (H, H_{II}) -Weingarten surface. From (3.10), (3.13), and (3.15), $\Phi(H, H_{II})$ is

$$H_s(H_{II})_\theta - H_\theta(H_{II})_s = \frac{1}{8\lambda\kappa^3\sigma^5\cos^3\theta} \sum_{i=0}^4 v_i \cos^i \theta, \tag{3.20}$$

with respect to the variable $\cos\theta$, where

$$\begin{aligned}
v_0 &= 3\lambda\tau\kappa^2(\kappa\tau_s + \kappa_s\tau)\sin\theta, \\
v_1 &= -\kappa^3\left(\kappa_s + 6\lambda^2\tau(\kappa_s\tau + \kappa\tau_s)\right)\sin\theta + \lambda\kappa^2(\kappa_{ss}\tau - \kappa\tau_{ss}),
\end{aligned}$$

$$\begin{aligned}
v_2 &= \lambda \left(3\kappa^4 \kappa_s - 3(\kappa_s)^3 + 4\kappa \kappa_s \kappa_{ss} - \kappa^3 \tau \tau_s - \kappa^2 \kappa_{sss} \right) \sin \theta \\
&\quad + \lambda^2 \kappa^3 (\kappa \tau_{ss} - 3\kappa_s \tau_s - 4\kappa_{ss} \tau), \\
v_3 &= \lambda^2 \kappa \left(6(\kappa_s)^3 + \kappa^2 \kappa_{sss} - 7\kappa \kappa_s \kappa_{ss} - 2\kappa^4 \kappa_s + 4\kappa^3 \tau \tau_s \right) \sin \theta \\
&\quad + \lambda \kappa \left(\kappa^2 \tau_{ss} + (\kappa_s)^2 \tau - \kappa \kappa_{ss} \tau - \kappa \kappa_s \tau_s \right), \\
v_4 &= -\lambda^2 \kappa^2 \left(\kappa^2 \tau_{ss} - 4\kappa \kappa_{ss} \tau - 4\kappa \kappa_s \tau_s + 4(\kappa_s)^2 \tau \right).
\end{aligned} \tag{3.21}$$

Then, by $\Phi(H, H_{II}) = 0$, (3.22) becomes in following form:

$$\sum_{i=0}^4 v_i \cos^i \theta = 0. \tag{3.22}$$

Thus, we state the following theorem. □

Theorem 3.3. *Let M be a tubular surface defined by (3.1) with nondegenerate second fundamental form. M is a (H, H_{II}) -Weingarten surface if and only if M is a tubular surface around a circle or a helix.*

Proof. Considering $\Phi(H, H_{II}) = 0$ and by using (3.13), one can obtain the solutions $\kappa_s = 0$, $\tau = 0$, and $\kappa_s = 0$, $\tau_s = 0$ of the equations $v_0 = v_1 = v_2 = v_3 = v_4 = 0$ for all θ . Thus, it is easily proved that M is a (H, H_{II}) -Weingarten surface if and only if M is a tubular surface around a circle or a helix. □

We consider a tubular surface M is (K_{II}, H_{II}) -Weingarten surface with nondegenerate second fundamental form in E^3 . By using (3.11), (3.12), (3.13), and (3.15), $\Phi(K_{II}, H_{II})$ is

$$(K_{II})_s (H_{II})_\theta - (K_{II})_\theta (H_{II})_s = \frac{-1}{16\lambda \kappa^3 \sigma^5 \cos^5 \theta} \sum_{i=0}^9 w_i \cos^i \theta, \tag{3.23}$$

where

$$\begin{aligned}
w_0 &= 3\lambda \tau \kappa^2 (\kappa \tau_s - 2\kappa_s \tau) \sin \theta, \\
w_1 &= \kappa^3 \left(\kappa_s + 18\lambda^2 \tau (\kappa_s \tau - 2\kappa \tau_s) \right) \sin \theta + \lambda \kappa \left(4\kappa_s (\kappa \tau_s - \kappa_s \tau) + \kappa \kappa_{ss} \tau - \kappa^2 \tau_{ss} \right), \\
w_2 &= \left\{ \left(6\kappa \kappa_{ss} - 18\lambda^2 \kappa^4 \tau^2 - 3\kappa^4 - 6(\kappa_s)^2 - 2\kappa^2 \tau^2 \right) \lambda \kappa_s + 4 \left(3\lambda^2 \kappa^2 - 1 \right) \lambda \kappa^3 \tau \tau_s - \lambda \kappa^2 \kappa_{sss} \right\} \sin \theta \\
&\quad + 3\lambda^2 \kappa^2 \left(\kappa_s (6\kappa_s \tau - 5\kappa \tau_s) - 2\kappa \kappa_{ss} \tau + \kappa^2 \tau_{ss} \right),
\end{aligned}$$

$$\begin{aligned}
w_3 &= \left\{ \left(\kappa^2 + 38\lambda^2\kappa^2\tau^2 + 4\lambda^2\kappa^4 - 23\lambda^2\kappa\kappa_{ss} + 24\lambda^2(\kappa_s)^2 \right) \kappa\kappa_s + 48\lambda^2\kappa^4\tau\tau_s + 3\lambda^2\kappa^3\kappa_{sss} \right\} \sin \theta \\
&\quad - \lambda\kappa \left\{ 2\left(\lambda^2\kappa^2 - 1 \right) \kappa^2\tau_{ss} + \left(32\lambda^2\kappa^2 - 3 \right) (\kappa_s)^2\tau_s - \left(14\lambda^2\kappa^2 - 3 \right) \kappa\kappa_s\tau_s \right. \\
&\quad \left. + 2\left(1 - 4\lambda^2\kappa^2 \right) \kappa\kappa_{ss}\tau \right\}, \\
w_4 &= -\lambda \left\{ \left(2\lambda^2\kappa^2 - 1 \right) \kappa^2\kappa_{ss} + 4\left(1 - 5\lambda^2\kappa^2 \right) \kappa\kappa_s\kappa_{ss} + \left(134\lambda^2\kappa^2 - 1 \right) \kappa^3\tau\tau_s \right. \\
&\quad \left. + \left(3\left(10\lambda^2\kappa^2 - 1 \right) (\kappa_s)^2 + \left(2\left(57\tau^2 + \kappa^2 \right) \lambda^2 + 13 \right) \kappa^4 \right) \kappa_s \right\} \sin \theta \\
&\quad + \lambda^2\kappa^2 \left(22\kappa\kappa_{ss}\tau + 17\kappa\kappa_s\tau_s - 14(\kappa_s)^2\tau - 16\kappa^2\tau_{ss} \right), \\
w_5 &= \lambda^2\kappa \left\{ 55\kappa\kappa_s\kappa_{ss} + 4\left(33\lambda^2\kappa^2 - 4 \right) \kappa^3\tau\tau_s + 2\left(66\lambda^2\tau^2 + 25 \right) \kappa^4\kappa_s - \left(13\kappa^2 + 42 \right) \kappa_{sss} \right\} \sin \theta \\
&\quad - \lambda\kappa \left\{ \left(50\lambda^2\kappa^2 - 1 \right) \kappa\kappa_s\tau_s + \left(1 - 32\lambda^2\kappa^2 \right) (\kappa_s)^2\tau + \left(1 - 32\lambda^2\kappa^2 \right) \kappa^2\tau_{ss} \right. \\
&\quad \left. + \left(74\lambda^2\kappa^2 - 1 \right) \kappa\kappa_{ss}\tau \right\}, \\
w_6 &= 2\lambda^3\kappa^2 \left(63(\kappa_s)^3 - 24\lambda^2\kappa^4\kappa_s\tau^2 - 41\kappa^4\kappa_s + 33\kappa^3\tau\tau_s - 78\kappa\kappa_s\kappa_{ss} - 24\lambda^2\kappa^5\tau\tau_s + 15\kappa^2\kappa_{sss} \right) \sin \theta \\
&\quad + \lambda^2\kappa^2 \left(16(\kappa_s)^2\tau - 26\lambda^2\kappa^4\tau_{ss} - 16\kappa\kappa_{ss}\tau + 13\kappa^2\tau_{ss} - 16\kappa\kappa_s\tau_s + 54\lambda^2\kappa^3\kappa_s\tau_s + 80\lambda^2\kappa^3\kappa_{ss}\tau \right), \\
w_7 &= 2\lambda^4\kappa^3 \left(30\kappa^4\kappa_s - 13\kappa^2\kappa_{sss} - 40\kappa^3\tau\tau_s + 79\kappa\kappa_s\kappa_{ss} - 60(\kappa_s)^3 \right) \sin \theta + \\
&\quad - 2\lambda^3\kappa^3 \left(33(\kappa_s)^2\tau_s - 33\kappa\kappa_s\tau_s - 4\lambda^2\kappa^4\tau_{ss} + 12\lambda^2\kappa^3\kappa_s\tau_s + 15\kappa^2\tau_{ss} + 16\lambda^2\kappa^3\kappa_{ss}\tau - 33\kappa\kappa_{ss}\tau \right), \\
w_8 &= -8\lambda^5\kappa^4 \left(-6(\kappa_s)^3 + 2\kappa^4\kappa_s - 4\kappa^3\tau\tau_s + 7\kappa\kappa_s\kappa_{ss} - \kappa^2\kappa_{sss} \right) \sin \theta \\
&\quad + 2\lambda^4\kappa^4 \left(13\kappa^2\tau_{ss} + 4(\kappa_s)^2\tau - 40\kappa(\kappa_{ss}\tau + \kappa_s\tau_s) \right), \\
w_9 &= 8\lambda^5\kappa^5 \left\{ 4\kappa\kappa_s\tau_s - 4(\kappa_s)^2\tau - \kappa^2\tau_{ss} + 4\kappa\kappa_{ss}\tau \right\}.
\end{aligned} \tag{3.24}$$

Since $\Phi(K_{II}, H_{II}) = 0$, then (3.23) becomes in following form:

$$\sum_{i=0}^9 w_i \cos^i \theta = 0. \tag{3.25}$$

Hence, we have the following theorem.

Theorem 3.4. *Let M be a tubular surface defined by (3.1) with nondegenerate second fundamental form. M is a (K_{II}, H_{II}) -Weingarten surface if and only if M is a tubular surface around a circle or a helix.*

Proof. It can be easily proved similar to Theorems 3.2 and 3.3. □

Consequently, we can give the following main theorem for the end of this part.

Theorem 3.5. *Let $(X, Y) \in \{(K, H_{II}), (H, K_{II}), (H_{II}, K_{II})\}$, and let M be a tubular surface defined by (3.1) with nondegenerate second fundamental form. M is a (X, Y) -Weingarten surface if and only if M is a tubular surface around a circle or a helix.*

Thus, the study of Weingarten tubular surfaces in 3-dimensional Euclidean space is completed with [1].

4. Linear Weingarten Tubular Surfaces

In last part of this paper, we study on $(K, H_{II}), (H, H_{II}), (H_{II}, K_{II}), (K, H, H_{II}), (K, H, K_{II}), (H, K_{II}, H_{II}), (K, K_{II}, H_{II})$, and (K, H, K_{II}, H_{II}) linear Weingarten tubular surfaces in E^3 . $(K, H), (K, K_{II})$, and (H, K_{II}) linear Weingarten tubes are studied in [1].

Let a_1, a_2, a_3, a_4 , and b be constants. In general, a linear combination of K, H, K_{II} and H_{II} can be constructed as

$$a_1 K + a_2 H + a_3 K_{II} + a_4 H_{II} = b. \quad (4.1)$$

By the straightforward calculations, we obtained the reduced form of (4.1)

$$8b\kappa^3 \varepsilon \sigma^3 \cos^3 \theta + \sum_{i=0}^8 p_i \cos^i \theta = 0, \quad (4.2)$$

where the coefficients are

$$\begin{aligned} p_0 &= 3a_4 \lambda \kappa^2 \tau^2, \\ p_1 &= a_4 \kappa \left(2\lambda (\kappa_s \tau - \kappa \tau_s) \sin \theta - \kappa^2 (6\lambda^2 \tau^2 + 1) \right), \\ p_2 &= a_4 \lambda \left(2\lambda \kappa^2 (\kappa \tau_s - 4\kappa_s \tau) \sin \theta + \kappa^2 (3\kappa^2 - \tau^2) - 2\kappa \kappa_{ss} + 3\kappa_s^2 \right) + 2a_3 \lambda \kappa^4, \\ p_3 &= a_4 \kappa \left(2\lambda^2 (\kappa \kappa_{ss} - \kappa^4 + 2\kappa^2 \tau^2 - 3\kappa_s^2) - 5\kappa^2 \right) - 4a_2 \kappa^3 - 4a_3 \lambda^2 \kappa^5, \\ p_4 &= 8a_1 \varepsilon \kappa^4 + 16a_2 \lambda \kappa^4 + 2a_3 \lambda \kappa^4 (1 + \lambda^2 \kappa^2) + 17a_4 \lambda \kappa^4, \\ p_5 &= -16a_1 \varepsilon \lambda \kappa^5 - 20a_2 \lambda^2 \kappa^5 - 16a_3 \lambda^2 \kappa^5 - 20a_4 \lambda^2 \kappa^5, \\ p_6 &= 8a_1 \varepsilon \lambda^2 \kappa^6 + 8a_2 \lambda^3 \kappa^6 + 34a_3 \lambda^3 \kappa^6, \\ p_7 &= -28a_3 \lambda^4 \kappa^7, \\ p_8 &= 8a_3 \lambda^5 \kappa^8. \end{aligned} \quad (4.3)$$

Then, p_0, p_1, p_2, p_7 , and p_8 are zero for any $b \in IR$. If $a_4 \neq 0$ or $a_3 \neq 0$, from $p_0 = p_1 = p_7 = p_8 = 0$, one has $\kappa = 0$. Hence, we can give the following theorems.

Theorem 4.1. Let $(X, Y) \in \{(K, H_{\text{II}}), (H, H_{\text{II}}), (K_{\text{II}}, H_{\text{II}})\}$. Then, there are no (X, Y) -linear Weingarten tubular surfaces M in Euclidean 3-space defined by (3.1) with nondegenerate second fundamental form.

Theorem 4.2. Let $(X, Y, Z) \in \{(H, K_{\text{II}}, H_{\text{II}}), (K, K_{\text{II}}, H_{\text{II}}), (K, H, H_{\text{II}}), (K, H, K_{\text{II}})\}$. Then, there are no (X, Y, Z) -linear Weingarten tubular surfaces M in Euclidean 3-space defined by (3.1) with nondegenerate second fundamental form.

Theorem 4.3. Let M be a tubular surface defined by (3.1) with nondegenerate second fundamental form. Then, there are no $(K, H, K_{\text{II}}, H_{\text{II}})$ -linear Weingarten surface in Euclidean 3-space.

Consequently, the study of linear Weingarten tubular surfaces in 3-dimensional Euclidean space is completed with [1].

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References

- [1] J. S. Ro and D. W. Yoon, "Tubes of weingarten types in a euclidean 3-space," *Journal of the Chungcheong Mathematical Society*, vol. 22, no. 3, pp. 359–366, 2009.
- [2] F. Dillen and W. Kühnel, "Ruled Weingarten surfaces in Minkowski 3-space," *Manuscripta Mathematica*, vol. 98, no. 3, pp. 307–320, 1999.
- [3] F. Dillen and W. Sodsiri, "Ruled surfaces of Weingarten type in Minkowski 3-space," *Journal of Geometry*, vol. 83, no. 1-2, pp. 10–21, 2005.
- [4] F. Dillen and W. Sodsiri, "Ruled surfaces of Weingarten type in Minkowski 3-space. II," *Journal of Geometry*, vol. 84, no. 1-2, pp. 37–44, 2005.
- [5] W. Kühnel, "Ruled W-surfaces," *Archiv der Mathematik*, vol. 62, no. 5, pp. 475–480, 1994.
- [6] R. López, "Special Weingarten surfaces foliated by circles," *Monatshefte für Mathematik*, vol. 154, no. 4, pp. 289–302, 2008.
- [7] G. Stamou, "Regelflächen vom Weingarten-Typ," *Colloquium Mathematicum*, vol. 79, no. 1, pp. 77–84, 1999.
- [8] D. W. Yoon, "Some properties of the helicoid as ruled surfaces," *JP Journal of Geometry and Topology*, vol. 2, no. 2, pp. 141–147, 2002.
- [9] W. Kühnel and M. Steller, "On closed Weingarten surfaces," *Monatshefte für Mathematik*, vol. 146, no. 2, pp. 113–126, 2005.
- [10] C. Baikoussis and T. Koufogiorgos, "On the inner curvature of the second fundamental form of helicoidal surfaces," *Archiv der Mathematik*, vol. 68, no. 2, pp. 169–176, 1997.
- [11] T. Koufogiorgos and T. Hasanis, "A characteristic property of the sphere," *Proceedings of the American Mathematical Society*, vol. 67, no. 2, pp. 303–305, 1977.
- [12] D. Koutroufiotis, "Two characteristic properties of the sphere," *Proceedings of the American Mathematical Society*, vol. 44, pp. 176–178, 1974.
- [13] D. E. Blair and Th. Koufogiorgos, "Ruled surfaces with vanishing second Gaussian curvature," *Monatshefte für Mathematik*, vol. 113, no. 3, pp. 177–181, 1992.
- [14] D. W. Yoon, "On non-developable ruled surfaces in Euclidean 3-spaces," *Indian Journal of Pure and Applied Mathematics*, vol. 38, no. 4, pp. 281–290, 2007.