Research Article

Warped Product Semi-Invariant Submanifolds of Nearly Cosymplectic Manifolds

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We study warped product semi-invariant submanifolds of nearly cosymplectic manifolds. We prove that the warped product of the type \( M_\perp \times f M_T \) is a usual Riemannian product of \( M_\perp \) and \( M_T \), where \( M_\perp \) and \( M_T \) are anti-invariant and invariant submanifolds of a nearly cosymplectic manifold \( M \), respectively. Thus we consider the warped product of the type \( M_T \times f M_\perp \) and obtain a characterization for such type of warped product.

1. Introduction

The notion of warped product manifolds was introduced by Bishop and O’Neill in 1969 as a natural generalization of the Riemannian product manifolds. Later on, the geometrical aspect of these manifolds has been studied by many researchers (cf., [1–3]). Recently, Chen [1] (see also [4]) studied warped product CR-submanifolds and showed that there exists no warped product CR-submanifolds of the form \( M = M_\perp \times f M_T \) such that \( M_\perp \) is a totally real submanifold and \( M_T \) is a holomorphic submanifold of a Kaehler manifold \( \overline{M} \). Therefore he considered warped product CR-submanifold in the form \( M = M_T \times f M_\perp \) which is called CR-warped product, where \( M_T \) and \( M_\perp \) are holomorphic and totally real submanifolds of a Kaehler manifold \( \overline{M} \). Motivated by Chen’s papers, many geometers studied CR-warped product submanifolds in almost complex as well as contact setting (see [3, 5, 6]).

Almost contact manifolds with Killing structure tensors were defined in [7] as nearly cosymplectic manifolds, and it was shown that normal nearly cosymplectic manifolds are
cosymplectic (see also [8]). Later on, Blair and Showers [9] studied nearly cosymplectic structure \((\phi, \xi, \eta, g)\) on a manifold \(\overline{M}\) with \(\eta\) closed from the topological viewpoint.

In this paper, we have generalized the results of Chen’ [1] in this more general setting of nearly cosymplectic manifolds and have shown that the warped product in the form \(M = M_L \times f(M_T)\) is simply Riemannian product of \(M_L\) and \(M_T\) where \(M_L\) is an anti-invariant submanifold and \(M_T\) is an invariant submanifold of a nearly cosymplectic manifold \(\overline{M}\). Thus we consider the warped product submanifold of the type \(M = M_T \times f(M_L)\) by reversing the two factors \(M_L\) and \(M_T\) and simply will be called warped product semi-invariant submanifold. Thus, we derive the integrability of the involved distributions in the warped product and obtain a characterization result.

2. Preliminaries

A \((2n + 1)\)-dimensional \(C^\infty\) manifold \(\overline{M}\) is said to have an almost contact structure if there exist on \(\overline{M}\) a tensor field \(\phi\) of type \((1,1)\), a vector field \(\xi\), and a 1-form \(\eta\) satisfying [9]

\[
\phi^2 = -I + \eta \otimes \xi, \quad \phi \xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1.
\] (2.1)

There always exists a Riemannian metric \(g\) on an almost contact manifold \(\overline{M}\) satisfying the following compatibility condition:

\[
\eta(X) = g(X, \xi), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),
\] (2.2)

where \(X\) and \(Y\) are vector fields on \(\overline{M}\) [9].

An almost contact structure \((\phi, \xi, \eta)\) is said to be normal if the almost complex structure \(J\) on the product manifold \(\overline{M} \times \mathbb{R}\) given by

\[
J\left(X, f \frac{d}{dt}\right) = \left(\phi X - f \xi, \eta(X) \frac{d}{dt}\right),
\] (2.3)

where \(f\) is a \(C^\infty\)-function on \(\overline{M} \times \mathbb{R}\), has no torsion, that is, \(J\) is integrable, and the condition for normality in terms of \(\phi, \xi\) and \(\eta\) is \([\phi, \phi] + 2d\eta \otimes \xi = 0\) on \(\overline{M}\), where \([\phi, \phi]\) is the Nijenhuis tensor of \(\phi\). Finally the fundamental 2-form \(\Phi\) is defined by \(\Phi(X, Y) = g(X, \phi Y)\).

An almost contact metric structure \((\phi, \xi, \eta, g)\) is said to be cosymplectic, if it is normal and both \(\Phi\) and \(\eta\) are closed [9]. The structure is said to be nearly cosymplectic if \(\phi\) is Killing, that is, if

\[
(\nabla_X \phi)Y + (\nabla_Y \phi)X = 0,
\] (2.4)

for any \(X, Y \in T\overline{M}\), where \(T\overline{M}\) is the tangent bundle of \(\overline{M}\) and \(\nabla\) denotes the Riemannian connection of the metric \(g\). Equation (2.4) is equivalent to \((\nabla_X \phi)X = 0\), for each \(X \in T\overline{M}\). The structure is said to be closely cosymplectic if \(\phi\) is Killing and \(\eta\) is closed. It is well known that an almost contact metric manifold is cosymplectic if and only if \(\nabla \phi\) vanishes identically, that is, \((\nabla_X \phi)Y = 0\) and \(\nabla_X \xi = 0\).
Proposition 2.1 (see [9]). On a nearly cosymplectic manifold, the vector field $\xi$ is Killing.

From the above proposition we have $\nabla_X \xi = 0$, for any vector field $X$ tangent to $\overline{M}$, where $\overline{M}$ is a nearly cosymplectic manifold.

Let $M$ be submanifold of an almost contact metric manifold $\overline{M}$ with induced metric $g$, and if $\nabla$ and $\nabla^\perp$ are the induced connections on the tangent bundle $TM$ and the normal bundle $T^\perp M$ of $M$, respectively, then, Gauss and Weingarten formulae are given by

$$\nabla_X Y = \nabla_X Y + h(X,Y), \quad (2.5)$$
$$\nabla_X N = -A_N X + \nabla^\perp_X N, \quad (2.6)$$

for each $X, Y \in TM$ and $N \in T^\perp M$, where $h$ and $A_N$ are the second fundamental form and the shape operator (corresponding to the normal vector field $N$), respectively, for the immersion of $M$ into $\overline{M}$. They are related as

$$g(h(X,Y), N) = g(A_N X, Y), \quad (2.7)$$

where $g$ denotes the Riemannian metric on $\overline{M}$ as well as being induced on $M$.

For any $X \in TM$, we write

$$\phi X = TX + FX, \quad (2.8)$$

where $TX$ is the tangential component and $FX$ is the normal component of $\phi X$.

Similarly for any $N \in T^\perp M$, we write

$$\phi N = BN + CN, \quad (2.9)$$

where $BN$ is the tangential component and $CN$ is the normal component of $\phi N$. The covariant derivatives of the tensor fields $P$ and $F$ are defined as

$$\left(\nabla_X T\right) Y = \nabla_X TY - T\nabla_X Y, \quad (2.10)$$
$$\left(\nabla_X F\right) Y = \nabla^\perp_X FY - F\nabla_X Y \quad (2.11)$$

for all $X, Y \in TM$.

Let $M$ be a Riemannian manifold isometrically immersed in an almost contact metric manifold $\overline{M}$, then for every $x \in M$ there exists a maximal invariant subspace denoted by $\mathfrak{D}_x$ of the tangent space $T_x M$ of $M$. If the dimension of $\mathfrak{D}_x$ is the same for all values of $x \in M$, then $\mathfrak{D}_x$ gives an invariant distribution $\mathfrak{D}$ on $M$.

A submanifold $M$ of an almost contact metric manifold $\overline{M}$ is called semi-invariant submanifold if there exists on $M$ a differentiable invariant distribution $\mathfrak{D}$ whose orthogonal complementary distribution $\mathfrak{D}^\perp$ is anti-invariant, that is,

(i) $TM = \mathfrak{D} \oplus \mathfrak{D}^\perp \oplus \langle \xi \rangle,$
for any \( x \in M \), where \( T_x^\perp M \) denotes the orthogonal space of \( T_xM \) in \( T_x\overline{M} \). A semi-invariant submanifold is called anti-invariant if \( \mathcal{D}_x = \{0\} \) and invariant if \( \mathcal{D}_x^\perp = \{0\} \), respectively, for any \( x \in M \). It is called the proper semi-invariant submanifold if neither \( \mathcal{D}_x = \{0\} \) nor \( \mathcal{D}_x^\perp = \{0\} \), for every \( x \in M \).

Let \( M \) be a semi-invariant submanifold of an almost contact metric manifold \( \overline{M} \). Then, \( F(T_xM) \) is a subspace of \( T_x^\perp M \). Then for every \( x \in M \), there exists an invariant subspace \( \nu_x \) of \( T_x\overline{M} \) such that

\[
T_x^\perp M = F(T_xM) \oplus \nu_x. \tag{2.12}
\]

A semi-invariant submanifold \( M \) of an almost contact metric manifold \( \overline{M} \) is called Riemannian product if the invariant distribution \( \mathcal{D} \) and anti-invariant distribution \( \mathcal{D}^\perp \) are totally geodesic distributions in \( M \).

Let \((M_1, g_1)\) and \((M_2, g_2)\) be two Riemannian manifolds, and let \( f \) be a positive differentiable function on \( M_1 \). The warped product of \( M_1 \) and \( M_2 \) is the product manifold \( M_1 \times_f M_2 = (M_1 \times M_2, g) \), where

\[
g = g_1 + f^2 g_2, \tag{2.13}
\]

where \( f \) is called the warping function of the warped product. The warped product \( N_1 \times_f N_2 \) is said to be trivial or simply Riemannian product if the warping function \( f \) is constant. This means that the Riemannian product is a special case of warped product.

We recall the following general results obtained by Bishop and O’Neill [10] for warped product manifolds.

**Lemma 2.2.** Let \( M = M_1 \times_f M_2 \) be a warped product manifold with the warping function \( f \). Then

(i) \( \nabla_X Y \in TM_1 \), for each \( X, Y \in TM_1 \),

(ii) \( \nabla_X Z = \nabla_Z X = (X \ln f) Z \), for each \( X \in TM_1 \) and \( Z \in TM_2 \),

(iii) \( \nabla_Z W - (g(Z, W) / f) \text{grad} f \),

where \( \nabla \) and \( \nabla^{M_2} \) denote the Levi-Civita connections on \( M \) and \( M_2 \), respectively.

In the above lemma \( \text{grad} f \) is the gradient of the function \( f \) defined by \( g(\text{grad} f, U) = Uf \), for each \( U \in TM \). From the Lemma 2.2, we have that on a warped product manifold \( M = M_1 \times_f M_2 \)

(i) \( M_1 \) is totally geodesic in \( M \);

(ii) \( M_2 \) is totally umbilical in \( M \).

Now, we denote by \( P_X Y \) and \( Q_X Y \) the tangential and normal parts of \( (\nabla_X \phi)Y \), that is,

\[
(\nabla_X \phi)Y = P_X Y + Q_X Y \tag{2.14}
\]
for all $X, Y \in TM$. Making use of (2.5), (2.6), and (2.8)-(2.11), the following relations may easily be obtained

\begin{align}
\rho_X Y &= (\nabla_X T) Y - A_{FY} X - Bh(X, Y), \\
Q_X Y &= \left(\nabla_X F\right) Y + h(X, TY) - C h(X, Y).
\end{align}

(2.15)
(2.16)

It is straightforward to verify the following properties of $\rho$ and $Q$, which we enlist here for later use:

1. $\rho_{X+Y} W = \rho_X W + \rho_Y W$, (ii) $Q_{X+Y} W = Q_X W + Q_Y W,$
2. $\rho_X (Y + W) = \rho_X Y + \rho_X W, (ii) \ Q_X (Y + W) = Q_X Y + Q_X W,$
3. $g(\rho_X Y, W) = -g(Y, \rho_X W)$

for all $X, Y, W \in TM$.

On a submanifold $M$ of a nearly cosymplectic manifold $\overline{M}$, we obtain from (2.4) and (2.14) that

\begin{align}
(i) \rho_X Y + \rho_Y X &= 0, \quad (ii) \ Q_X Y + Q_Y X = 0
\end{align}

(2.17)

for any $X, Y \in TM$.

3. Warped Product Semi-Invariant Submanifolds

Throughout the section we consider the submanifold $M$ of a nearly cosymplectic manifold $\overline{M}$ such that the structure vector field $\xi$ is tangent to $M$. First, we prove that the warped product $M = M_1 \times_f M_2$ is trivial when $\xi$ is tangent to $M_2$, where $M_1$ and $M_2$ are Riemannian submanifolds of a nearly cosymplectic manifold $\overline{M}$. Thus, we consider the warped product $M = M_1 \times_f M_2$, when $\xi$ is tangent to the submanifold $M_1$. We have the following nonexistence theorem.

**Theorem 3.1.** A warped product submanifold $M = M_1 \times_f M_2$ of a nearly cosymplectic manifold $\overline{M}$ is a usual Riemannian product if the structure vector field $\xi$ is tangent to $M_2$, where $M_1$ and $M_2$ are the Riemannian submanifolds of $\overline{M}$.

**Proof.** For any $X \in TM_1$ and $\xi$ tangent to $M_2$, we have

\begin{equation}
\nabla_X \xi = \nabla_X \xi + h(X, \xi).
\end{equation}

Using the fact that $\xi$ is Killing on a nearly cosymplectic manifold (see Proposition 2.1) and Lemma 2.2(ii), we get

\begin{equation}
0 = (X \ln f) \xi + h(X, \xi).
\end{equation}

Equating the tangential component of (3.2), we obtain $X \ln f = 0$, for all $X \in TM_1$, that is, $f$ is constant function on $M_1$. Thus, $M$ is Riemannian product. This proves the theorem. □
Now, the other case of warped product $M = M_1 \times_f M_2$ when $\xi \in TM_1$, where $M_1$ and $M_2$ are the Riemannian submanifolds of $\overline{M}$. For any $X \in TM_2$, we have
\[
\overline{\nabla}_X \xi = \nabla_X \xi + h(X, \xi).
\] (3.3)

By Proposition 2.1, and Lemma 2.2(ii), we obtain
\[
(i) \xi \ln f = 0, \quad (ii) h(X, \xi) = 0.
\] (3.4)

Thus, we consider the warped product semi-invariant submanifolds of a nearly cosymplectic manifold $\overline{M}$ of the types:

(i) $M = M_\perp \times_f M_T,$
(ii) $M = M_T \times_f M_\perp$,

where $M_T$ and $M_\perp$ are invariant and anti-invariant submanifolds of $\overline{M}$, respectively. In the following theorem we prove that the warped product semi-invariant submanifold of the type (i) is CR-product.

**Theorem 3.2.** The warped product semi-invariant submanifold $M = M_\perp \times_f M_T$ of a nearly cosymplectic manifold $\overline{M}$ is a usual Riemannian product of $M_\perp$ and $M_T$, where $M_\perp$ and $M_T$ are anti-invariant and invariant submanifolds of $\overline{M}$, respectively.

**Proof.** When $\xi \in TM_T$, then by Theorem 3.1, $M$ is a Riemannian product. Thus, we consider $\xi \in TM_\perp$. For any $X \in TM_T$ and $Z \in TM_\perp$, we have
\[
g(h(X, \phi X), FZ) = g(h(X, \phi X), \phi Z) = g\left(\overline{\nabla}_X \phi X, \phi Z\right)
= g\left(\phi \overline{\nabla}_X X, \phi Z\right) + g\left(\overline{\nabla}_X \phi X, \phi Z\right).
\] (3.5)

From the structure equation of nearly cosymplectic, the second term of right hand side vanishes identically. Thus from (2.2), we derive
\[
g(h(X, \phi X), FZ) = g\left(\overline{\nabla}_X X, Z\right) - \eta(Z)g\left(\overline{\nabla}_X \xi, \xi\right)
= -g\left(X, \overline{\nabla}_X Z\right) + \eta(Z)g\left(X, \overline{\nabla}_X \xi\right).
\] (3.6)

Then from (2.5), Lemma 2.2(ii), and Proposition 2.1, we obtain
\[
g(h(X, \phi X), FZ) = -(Z \ln f)\|X\|^2.
\] (3.7)

Interchanging $X$ by $\phi X$ in (3.7) and using the fact that $\xi \in TM_\perp$, we obtain
\[
g(h(X, \phi X), FZ) = (Z \ln f)\|X\|^2.
\] (3.8)
It follows from (3.7) and (3.8) that $Z \ln f = 0$, for all $Z \in TM_{\bot}$. Also, from (3.4) we have $\xi \ln f = 0$. Thus, the warping function $f$ is constant. This completes the proof of the theorem.

From the above theorem we have seen that the warped product of the type $M = M_{\bot} \times f M_{T}$ is a usual Riemannian product of an anti-invariant submanifold $M_{\bot}$ and an invariant submanifold $M_{T}$ of a nearly cosymplectic manifold $\overline{M}$. Since both $M_{\bot}$ and $M_{T}$ are totally geodesic in $M$, then $M$ is CR-product. Now, we study the warped product semi-invariant submanifold $M = M_{T} \times f M_{\bot}$ of a nearly cosymplectic manifold $\overline{M}$.

**Theorem 3.3.** Let $M = M_{T} \times f M_{\bot}$ be a warped product semi-invariant submanifold of a nearly cosymplectic manifold $\overline{M}$. Then the invariant distribution $\mathfrak{D}$ and the anti-invariant distribution $\mathfrak{D}_{\bot}$ are always integrable.

*Proof.* For any $X, Y \in \mathfrak{D}$, we have

$$F[X, Y] = F\nabla_{X}Y - F\nabla_{Y}X. \quad (3.9)$$

Using (2.11), we obtain

$$F[X, Y] = \left(\nabla_{X}F\right)Y - \left(\nabla_{Y}F\right)X. \quad (3.10)$$

Then by (2.16), we derive

$$F[X, Y] = Q_{X}Y - h(X, TY) + Ch(X, Y) - Q_{Y}X + h(Y, TX) - Ch(X, Y). \quad (3.11)$$

Thus from (2.17)(ii), we get

$$F[X, Y] = 2Q_{X}Y + h(Y, TX) - h(X, TY). \quad (3.12)$$

Now, for any $X, Y \in D$, we have

$$h(X, TY) + \nabla_{X}TY = \nabla_{X}TY = \nabla_{X}\phi Y. \quad (3.13)$$

Using the covariant derivative property of $\nabla_{\phi}$, we obtain

$$h(X, TY) + \nabla_{X}TY = \left(\nabla_{X}\phi\right)Y + \phi \nabla_{X}Y. \quad (3.14)$$

Then by (2.5) and (2.14), we get

$$h(X, TY) + \nabla_{X}TY = P_{X}Y + Q_{X}Y + \phi(\nabla_{X}Y + h(X, Y)). \quad (3.15)$$
Since $M_T$ is totally geodesic in $M$ (see Lemma 2.2(i)), then using (2.8) and (2.9), we obtain

$$h(X,TY) + \nabla_X TY = \mathcal{D}_X Y + Q_X Y + T\nabla_X Y + Bh(X,Y) + Ch(X,Y). \tag{3.16}$$

Equating the normal components of (3.16), we get

$$h(X,TY) = Q_X Y + Ch(X,Y). \tag{3.17}$$

Similarly, we obtain

$$h(Y,TX) = Q_Y X + Ch(X,Y). \tag{3.18}$$

Then from (3.17) and (3.18), we arrive at

$$h(Y,TX) - h(X,TY) = Q_Y X - Q_X Y. \tag{3.19}$$

Hence, using (2.17)(ii), we get

$$h(Y,TX) - h(X,TY) = -2Q_X Y. \tag{3.20}$$

Thus, it follows from (3.12) and (3.20) that $F[X,Y] = 0$, for all $X,Y \in D$. This proves the integrability of $D$. Now, for the integrability of $D^\perp$, we consider any $X \in D$ and $Z,W \in D^\perp$, and we have

$$g([Z,W],X) = g(\nabla_Z W - \nabla_W Z, X).$$

$$= -g(\nabla_Z X, W) + g(\nabla_W X, Z). \tag{3.21}$$

Using Lemma 2.2(ii), we obtain

$$g([Z,W],X) = -(X \ln f)g(Z,W) + (X \ln f)g(Z,W) = 0. \tag{3.22}$$

Thus from (3.22), we conclude that $[Z,W] \in \mathfrak{d}^\perp$, for each $Z,W \in \mathfrak{d}^\perp$. Hence, the theorem is proved completely.

**Lemma 3.4.** Let $M = M_T \times {}_f M_L$ be a warped product submanifold of a nearly cosymplectic manifold $\overline{M}$. If $X,Y \in TM_T$ and $Z,W \in TM_L$, then

(i) $g(\mathcal{D}_X Y, Z) = g(h(X,Y), FZ) = 0,$

(ii) $g(\mathcal{D}_X Z, W) = g(h(X,Z), FW) - g(h(X,W), FZ) = -(\phi X \ln f)g(Z,W) - g(h(X,Z), FW),$

(iii) $g(h(\phi X, Z), FZ) = (X \ln f)\|Z\|^2.$
Proof. For a warped product manifold $M = M_T \times f M_L$, we have that $M_T$ is totally geodesic in $M$; then by (2.10), $(\nabla_X T) Y \in TM_T$, for any $X, Y \in TM_T$, and therefore from (2.15), we get
\begin{equation}
 g(\rho_X Y, Z) = -g(Bh(X, Y), Z) = g(h(X, Y), FZ). \tag{3.23}
\end{equation}

The left-hand side of (3.23) is skew symmetric in $X$ and $Y$ whereas the right hand side is symmetric in $X$ and $Y$, which proves (i). Now, from (2.10) and (2.15), we have
\begin{equation}
 \rho_X Z = -T \nabla_X Z - A_{FZ} X - Bh(X, Z) \tag{3.24}
\end{equation}
for any $X \in TM_T$ and $Z \in TM_L$. Using Lemma 2.2 (ii), the first term of right-hand side is zero. Thus, taking the product with $W \in TM_L$, we obtain
\begin{equation}
 g(\rho_X Z, W) = -g(A_{FZ} X, W) - g(Bh(X, Z), W), \tag{3.25}
\end{equation}
Then by (2.2) and (2.7), we get
\begin{equation}
 g(\rho_X Z, W) = -g(h(X, W), FZ) + g(h(X, Z), FW). \tag{3.26}
\end{equation}
which proves the first equality of (ii). Again, from (2.10) and (2.15), we have
\begin{equation}
 \rho_Z X = \nabla_Z TX - T \nabla_Z X - Bh(X, Z). \tag{3.27}
\end{equation}
Thus using Lemma 2.2(ii), we derive
\begin{equation}
 \rho_Z X = (TX \ln f) Z - Bh(X, Z). \tag{3.28}
\end{equation}
Taking inner product with $W \in TM_L$ and using (2.2), we obtain
\begin{equation}
 g(\rho_Z X, W) = (\phi X \ln f) g(Z, W) + g(h(X, Z), FW). \tag{3.29}
\end{equation}
Then from (2.17)(i), we get
\begin{equation}
 g(\rho_X Z, W) = -(\phi X \ln f) g(Z, W) - g(h(X, Z), FW). \tag{3.30}
\end{equation}
This is the second equality of (ii). Now, from (3.24) and (3.28), we have
\begin{equation}
 \rho_X Z + \rho_Z X = -T \nabla_X Z - A_{FZ} X + (TX \ln f) Z - 2Bh(X, Z). \tag{3.31}
\end{equation}
Left-hand side and the first term of right-hand side are zero on using (2.17)(i) and Lemma 2.2(i), respectively. Thus the above equation takes the form
\begin{equation}
 (TX \ln f) Z = A_{FZ} X + 2Bh(X, Z). \tag{3.32}
\end{equation}
Taking the product with $Z$ and on using (2.2) and (2.7), we get
\[
(\phi X \ln f)\|Z\|^2 = g(h(X, Z), FZ) - 2g(h(X, Z), FZ) = -g(h(X, Z), FZ).
\] (3.33)

Interchanging $X$ by $\phi X$ and using (2.1), we obtain
\[
\{-X + \eta(X)\xi\} \ln f\|Z\|^2 = -g(h(\phi X, Z), FZ).
\] (3.34)

Thus by (3.4)(i), the above equation reduces to
\[
(X \ln f)\|Z\|^2 = g(h(\phi X, Z), FZ).
\] (3.35)

This proves the lemma completely. \(\square\)

**Theorem 3.5.** A proper semi-invariant submanifold $M$ of a nearly cosymplectic manifold $\overline{M}$ is locally a semi-invariant warped product if and only if the shape operator of $M$ satisfies
\[
A_{\phi^Z}X = - (\phi X \mu) Z, \quad X \in \mathcal{D} \oplus (\xi), \quad Z \in \mathcal{D}^\perp
\] (3.36)

for some function $\mu$ on $M$ satisfying $V(\mu) = 0$ for each $V \in \mathcal{D}^\perp$.

**Proof.** If $M = M_T \times f M_L$ is a warped product semi-invariant submanifold, then by Lemma 3.4 (iii), we obtain (3.36). In this case $\mu = \ln f$.

Conversely, suppose $M$ is a semi-invariant submanifold of a nearly cosymplectic manifold $\overline{M}$ satisfying (3.36). Then
\[
g(h(X, Y), \phi Z) = g(A_{\phi^Z}X, Y) = - (\phi X \mu) g(Y, Z) = 0.
\] (3.37)

Now, from (2.5) and the property of covariant derivative of $\overline{\nabla}$, we have
\[
g(h(X, Y), \phi Z) = g(\overline{\nabla}_X Y, \phi Z) = - g(\phi\overline{\nabla}_X Y, Z) = - g(\nabla_X \phi Y, Z) + g(\nabla_X \phi Y, Z).
\] (3.38)

Then from (2.5), (2.14), and (3.37), the above equation takes the form
\[
g(\nabla_X TY, Z) = g(P_X Y, Z).
\] (3.39)

Using (2.10) and (2.15), we obtain
\[
g(\nabla_X TY, Z) = g(\nabla_X TY, Z) - g(T \nabla_X Y, Z) - g(Bh(X, Y), Z).
\] (3.40)
Thus by (2.2), the above equation reduces to

\[ g(T\nabla_X Y, Z) = g(h(X, Y), \phi Z). \] (3.41)

Hence using (2.7) and (3.36), we get

\[ g(T\nabla_X Y, Z) = g(A_{\phi} X, Y) = 0, \] (3.42)

which implies \( \nabla_X Y \in \mathfrak{D} \oplus \langle \xi \rangle \), that is, \( \mathfrak{D} \oplus \langle \xi \rangle \) is integrable and its leaves are totally geodesic in \( M \). Now, for any \( Z, W \in D^\perp \) and \( X \in \mathfrak{D} \oplus \langle \xi \rangle \), we have

\[ g(\nabla_Z W, \phi X) = g(\nabla_Z W, \phi X) = -g(\phi \nabla_Z W, X) \]
\[ = g\left( (\nabla_Z \phi) W, X \right) - g(\nabla_Z \phi W, X). \] (3.43)

Then, using (2.6) and (2.14), we obtain

\[ g(\nabla_Z W, \phi X) = g(\partial_Z W, X) + g(A_{\phi} W, X). \] (3.44)

Thus from (2.7) and the property \((p_3)\), we arrive at

\[ g(\nabla_Z W, \phi X) = -g(W, \partial_Z X) + g(h(Z, X), \phi W). \] (3.45)

Again using (2.7) and (2.17)(i), we get

\[ g(\nabla_Z W, \phi X) = g(\partial_X Z, W) + g(A_{\phi} W, Z). \] (3.46)

On the other hand, from (2.10) and (2.15), we have

\[ P_X Z = -T\nabla_X Z - A_{F} Z - B h(X, Z). \] (3.47)

Taking the product with \( W \in D^\perp \) and using (3.36), we obtain

\[ g(\partial_X Z, W) = -g(T\nabla_X Z, W) + (\phi X \mu) g(Z, W) + g(h(X, Z), F W). \] (3.48)

The first term of right-hand side of above equation is zero using the fact that \( TW = 0 \), for any \( W \in \mathfrak{D}^\perp \). Again using (2.7), we get

\[ g(\partial_X Z, W) = (\phi X \mu) g(Z, W) + g(A_{\phi} W, Z). \] (3.49)

Thus from (3.36), we derive

\[ g(\partial_X Z, W) = (\phi X \mu) g(Z, W) - (\phi X \mu) g(Z, W) = 0. \] (3.50)
Then from (3.36), (3.46), and (3.50), we obtain
\[ g(\nabla_ZW, \phi X) = -(\phi X \mu) g(Z, W). \quad (3.51) \]

Let \( M_\perp \) be a leaf of \( D_\perp \), and let \( h_\perp \) be the second fundamental form of the immersion of \( M_\perp \) into \( M \). Then for any \( Z, W \in D_\perp \), we have
\[ g(h_\perp(Z, W), \phi X) = g(\nabla_ZW, \phi X). \quad (3.52) \]
Hence, from (3.51) and (3.52), we conclude that
\[ g(h_\perp(Z, W), \phi X) = -(\phi X \mu) g(Z, W). \quad (3.53) \]

This means that integral manifold \( M_\perp \) of \( D_\perp \) is totally umbilical in \( M \). Since the anti-invariant distribution \( D_\perp \) of a semi-invariant submanifold \( M \) is always integrable (Theorem 3.3) and \( V(\mu) = 0 \) for each \( V \in D_\perp \), which implies that the integral manifold of \( D_\perp \) is an extrinsic sphere in \( M \); that is, it is totally umbilical and its mean curvature vector field is nonzero and parallel along \( M_\perp \). Hence by virtue of results obtained in [11], \( M \) is locally a warped product \( M_T \times f M_\perp \), where \( M_T \) and \( M_\perp \) denote the integral manifolds of the distributions \( D_\oplus (\xi) \) and \( D_\perp \), respectively and \( f \) is the warping function. Thus the theorem is proved. \( \square \)

References
