Letter to the Editor

Analyzing “Homotopy Perturbation Method for Solving Fourth-Order Boundary Value Problem”

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We analyze a previous paper by S. T. Mohyud-Din and M. A. Noor (2007) and show the mistakes in it. Then, we demonstrate a more efficient method for solving fourth-order boundary value problems.

1. Problem

Let us consider the fifth-order boundary problem of the type

$$u^{(4)}(x) = f(u, u', u'', u''') + g(x) \quad (1.1)$$

with the boundary conditions

$$u(0) = \alpha_1, \quad u'(0) = \alpha_2, \quad u(1) = \beta_1, \quad u'(1) = \beta_2, \quad (1.2)$$

where f and g are continuous functions and $\alpha_1, \alpha_2, \beta_1$, and β_2 are real constants.

The homotopy perturbation method (HPM) is employed in [1] for solving such problems. The purpose of this paper is to point out the mistakes in paper [1] and demonstrate more efficient method for solving the problems of type (1.1)-(1.2).

First of all, we show the mistakes.

(1) In Example 3.2 the approximate solution

$$u_{\text{approx}}(x) = 512 + 480x + 224.00000000226055x^2 + \dots \quad (1.3)$$

of the problem

$$\begin{aligned} u^{(4)}(x) &= u(x) + u''(x) + e^x(x-3), \\ u(0) &= 1, \quad u'(0) = 0, \quad u(1) = 0, \quad u'(1) = -e \end{aligned} \quad (1.4)$$

(see formula (3.19) in [1]) has no relationship with the exact solution $u_{\text{exact}}(x) = (1-x)e^x$. And "error" is at least $u_{\text{approx}}(0) - u_{\text{exact}}(0) = 512 - e = 509.28$.

It is strange that the authors demonstrated the reliability of the errors in the table (see Table 3.2 in [1]).

(2) In Example 3.3 the approximate solution

$$u_{\text{approx}}(x) = 11.3472 - 262.827x - 3.40768x^3 + \dots \quad (1.5)$$

of the problem

$$\begin{aligned} u^{(4)} &= \sin x + \sin^2 x - (u''(x))^2, \\ u(0) &= 0, \quad u'(0) = 1, \quad u(1) = \sin 1, \quad u'(1) = \cos 1 \end{aligned} \quad (1.6)$$

(see formula (3.28) in [1]) has no relationship with the exact solution $u = \sin x$, since $u_{\text{approx}}(0) = 11.3472$, $u'_{\text{approx}}(0) = -262.827$.

2. Homotopy Perturbation Method

The basic ideas of the standard HPM were given by He [2, 3], and a new interpretation of HPM was given by He [4]. We introduce a new reliable procedure for choosing the initial approximation in HPM. To do so, we consider the following general nonlinear differential equation

$$Lu + Nu = f(u, x) \quad (2.1)$$

with some initial boundary conditions, where L and N are, respectively, the linear and nonlinear operators.

According to HPM, we construct a homotopy which satisfies the following relations:

$$H(u, p) = Lu - Lv_0 + pLv_0 + p[Nu - f(u, x)] = 0, \quad (2.2)$$

where $p \in [0, 1]$ is an embedding parameter and v_0 is an initial approximation. When we put $p = 0$ and $p = 1$ in (2.2), we obtain

$$H(u, 0) = Lu - Lv_0, \quad H(u, 1) = Lu + Nu - f(u, x), \quad (2.3)$$

respectively. In topology, this is called deformation and $Lu - Lv_0$ and $Lu + Nu - f(u, x)$ are called homotopics.

The solution of (2.2) is expressed as

$$u(x) = u_0(x) + pu_1(x) + p^2u_2(x) + \dots \quad (2.4)$$

Hence, the approximate solution of (1.5) can be expressed as

$$u(x, t) = u_0(x) + u_1(x) + u_2(x) + \dots \quad (2.5)$$

Mohyud-Din and Noor [1] tried to rewrite the problem as a system of integral equations and then HPM applied for each equation. In the use of HPM, what we are mainly concerned about are the auxiliary operator L and the initial guess v_0 . We take $L = (d/dx^4)(\cdot)$, and

$$v_0 = \alpha_1 + \alpha_2 x + Ax^3 + Bx^4 + L^{-1}(g(x)), \quad (2.6)$$

where A and B are yet to be determined. Then using the boundary conditions $u(1) = \beta_1$, $u'(1) = \beta_2$ we determine A and B .

3. Applications

Here we apply the HPM to solve correctly the problems in [1].

Example 3.1 (see [1, Example 3.2]). We have

$$u^{(4)}(x) = u(x) + u''(x) + e^x(x - 3) \quad (3.1)$$

with boundary conditions

$$u(0) = 1, \quad u'(0) = 0, \quad u(1) = 0, \quad u'(1) = -e. \quad (3.2)$$

We construct a homotopy which satisfies the relation

$$u^{(4)}(x) - v_0^{(4)}(x) + p[v_0^{(4)}(x) - u(x) - u''(x) - e^x(x - 3)] = 0, \quad (3.3)$$

where

$$v_0 = 1 + Ax^2 + Bx^3 + L^{-1}(e^x(x - 3)). \quad (3.4)$$

Now substituting (3.4) into (3.3), we obtain

$$\begin{aligned} & u_0^{(4)} + pu_1^{(4)} + p^2u_2^{(4)} + \dots - v_0^{(4)}(x) \\ & + p[v_0^{(4)}(x) - u_0 - pu_1 - p^2u_2 - \dots - u_0'' - pu_1'' - p^2u_2'' - \dots - e^x(x - 3)] = 0, \end{aligned} \quad (3.5)$$

and, equating the coefficients of a like powers of p , we get a system of equations:

$$\begin{aligned}
 u_0^{(4)}(x) - v_0^{(4)}(x) &= 0, & u_0(0) &= 1, & u_0'(0) &= 0, & u_0''(0) &= A, & u_0'''(0) &= B, \\
 u_1^{(4)} + v_0^{(4)}(x) - u_0 - u_0'' - e^x(x-3) &= 0, & u_1(0) &= 0, \\
 u_1'(0) &= 0, & u_1''(0) &= 0, & u_1'''(0) &= 0, \\
 u_2^{(4)} - u_1 - u_1'' &= 0, & u_2(0) &= 0, & u_2'(0) &= 0, & u_2''(0) &= 0, & u_2'''(0) &= 0, \\
 u_3^{(4)} - u_2 - u_2'' &= 0, & u_3(0) &= 0, & u_3'(0) &= 0, & u_3''(0) &= 0, & u_3'''(0) &= 0, \dots
 \end{aligned} \tag{3.6}$$

Solving (3.6), we get

$$\begin{aligned}
 u_0 &= 1 + Ax^2 + Bx^3 + L^{-1}(e^x(x-3)) \\
 &= 1 + Ax^2 + Bx^3 + 6x - 7e^x + xe^x + \frac{5}{2}x^2 + \frac{2}{3}x^3 + 7 \\
 &= 6x - 7e^x + Ax^2 + Bx^3 + xe^x + \frac{5}{2}x^2 + \frac{2}{3}x^3 + 8, \\
 u_1 &= 20 + x^5\left(\frac{1}{20}B + \frac{1}{10}\right) + x^4\left(\frac{1}{12}A + \frac{13}{24}\right) + x^6\left(\frac{1}{360}A + \frac{1}{144}\right) \\
 &\quad + x^7\left(\frac{1}{840}B + \frac{1}{1260}\right) + \frac{1}{6}14x^3 + \frac{1}{2}16x^2 + e^x(2x-20) + 18x, \\
 u_2 &= L^{-1}(u_1 + u_1'') \\
 &= L^{-1}\left(32x - 36e^x + Ax^2 + \frac{A}{6}x^4 + Bx^3 + \frac{A}{360}x^6 + \frac{B}{10}x^5 + \frac{B}{840}x^7 \right. \\
 &\quad \left. + 4xe^x + \frac{29}{2}x^2 + \frac{13}{3}x^3 + \frac{3}{4}x^4 + \frac{2}{15}x^5 + \frac{1}{144}x^6 + \frac{1}{1260}x^7 + 36\right) \\
 &= 52 + x^6\left(\frac{1}{360}A + \frac{29}{720}\right) + x^7\left(\frac{1}{840}B + \frac{13}{2520}\right) + x^8\left(\frac{1}{10\,080}A + \frac{1}{2240}\right) \\
 &\quad + x^9\left(\frac{B}{30\,240} + \frac{1}{22\,680}\right) + x^{10}\left(\frac{A}{1814\,400} + \frac{1}{725\,760}\right) + x^{11}\left(\frac{B}{6652\,800} + \frac{1}{9979\,200}\right) \\
 &\quad + \frac{1}{6}40x^3 + \frac{1}{2}44x^2 + e^x(4x-52) + 48x + \frac{3}{2}x^4 + \frac{4}{15}x^5, \dots
 \end{aligned} \tag{3.7}$$

Using only three-term approximation, we have

$$\begin{aligned}
 u &= u_0 + u_1 + u_2 = 6x - 7e^x + Ax^2 + Bx^3 + xe^x + \frac{5}{2}x^2 + \frac{2}{3}x^3 + 8 \\
 &\quad + 20 + x^5\left(\frac{1}{20}B + \frac{1}{10}\right) + x^4\left(\frac{1}{12}A + \frac{13}{24}\right) + x^6\left(\frac{1}{360}A + \frac{1}{144}\right)
 \end{aligned}$$

$$\begin{aligned}
& + x^7 \left(\frac{1}{840} B + \frac{1}{1260} \right) + \frac{1}{6} 14x^3 + \frac{1}{2} 16x^2 + e^x (2x - 20) + 18x \\
& + 52 + x^6 \left(\frac{A}{360} + \frac{29}{720} \right) + x^7 \left(\frac{B}{840} + \frac{13}{2520} \right) + x^8 \left(\frac{A}{10080} + \frac{1}{2240} \right) \\
& + x^9 \left(\frac{B}{30240} + \frac{1}{22680} \right) + x^{10} \left(\frac{A}{1814400} + \frac{1}{725760} \right) + x^{11} \left(\frac{B}{6652800} + \frac{1}{9979200} \right) \\
& + \frac{1}{6} 40x^3 + \frac{1}{2} 44x^2 + e^x (4x - 52) + 48x + \frac{3}{2} x^4 + \frac{4}{15} x^5.
\end{aligned} \tag{3.8}$$

Now it follows from conditions $u(1) = 0$, $u'(1) = -e$ that $A = -0.46717$ and $B = -0.38354$ and, therefore,

$$\begin{aligned}
u = u_0 + u_1 + u_2 = & 80 + 72x - 79e^x + 7xe^x + 32.033x^2 + 9.2832x^3 + 2.0027x^4 \\
& + 0.34749x^5 + 4.4627 \times 10^{-2}x^6 + 5.0392 \times 10^{-3}x^7 + 4.0008 \times 10^{-4}x^8 \\
& + 3.1409 \times 10^{-5}x^9 + 1.1204 \times 10^{-6}x^{10} + 4.2558 \times 10^{-8}x^{11},
\end{aligned} \tag{3.9}$$

or, in power series form,

$$\begin{aligned}
u = & 1 - 0.467x^2 - 0.38347x^3 - 0.1223x^4 - 0.01918x^5 - 6.762 \times 10^{-3}x^6 \\
& - 9.132 \times 10^{-4}x^7 + 4.0008 \times 10^{-4}x^8 + 3.1409 \times 10^{-5}x^9.
\end{aligned} \tag{3.10}$$

Higher accuracy level can be attained by evaluating some more terms of $u(x)$.

Example 3.2 (see [1, Example 3.3]). We have

$$u^{(4)}(x) = \sin x + \sin^2 x - (u''(x))^2 \tag{3.11}$$

with boundary conditions

$$u(0) = 0, \quad u'(0) = 1, \quad u(1) = \sin 1, \quad u'(1) = \cos 1 \tag{3.12}$$

(the exact solution of the problem is $u = \sin x$).

We construct a homotopy which satisfies the relation

$$u^{(4)}(x) - v_0^{(4)}(x) + p \left[v_0^{(4)}(x) + (u''(x))^2 - \sin x - \sin^2 x \right] = 0, \tag{3.13}$$

where

$$v_0 = x + Ax^2 + Bx^3 + L^{-1}(\sin x + \sin^2 x). \tag{3.14}$$

Substituting (3.14) into (3.13), we obtain

$$u_0^{(4)} + pu_1^{(4)} + p^2u_2^{(4)} + \cdots - v_0^{(4)}(x) + p \left[v_0^{(4)}(x) + (u_0'' + pu_1'' + p^2u_2'' + \cdots)^2 - \sin x - \sin^2 x \right] = 0, \quad (3.15)$$

and, equating the coefficients of a like powers of p , we get a system of equations:

$$\begin{aligned} u_0^{(4)}(x) - v_0^{(4)}(x) &= 0, & u_0(0) &= 1, & u_0'(0) &= 0, \\ u_0'(0) &= 1, & u_0''(0) &= A, & u_0'''(0) &= B, \\ u_1^{(4)} + v_0^{(4)}(x) + (u_0'')^2 - \sin x - \sin^2 x &= 0, & u_1(0) &= 0, \\ u_1'(0) &= 0, & u_1''(0) &= 0, & u_1'''(0) &= 0, \\ u_2^{(4)} + 2u_0''u_1'' &= 0, & u_2(0) &= 0, & u_2'(0) &= 0, & u_2''(0) &= 0, & u_2'''(0) &= 0, \\ u_3^{(4)} + (u_1'')^2 + 2u_0''u_2'' &= 0, & u_3(0) &= 0, & u_3'(0) &= 0, & u_3''(0) &= 0, & u_3'''(0) &= 0, \dots \end{aligned} \quad (3.16)$$

Solving (3.16) we get

$$\begin{aligned} u_0 &= x + Ax^2 + Bx^3 + L^{-1}(\sin x + \sin^2 x) \\ &= x + Ax^2 + Bx^3 + \frac{1}{32} + \sin x - \frac{1}{32} \cos 2x + \frac{1}{6}x^3 - \frac{1}{16}x^2 - x + \frac{1}{48}x^4, \\ u_1 &= -\frac{1}{1260}Ax^7 - x^8 \left(\frac{1}{5040}A + \frac{1}{1680}B \right) - x^9 \left(\frac{1}{6048}B - \frac{1}{90720}A \right) \\ &\quad + x^{10} \left(\frac{A}{113400} + \frac{B}{100800} - \frac{1}{181440} \right) - \frac{1}{6}A^2x^4 - \frac{1}{40}B^2x^6 - \frac{1}{10}ABx^5 + O(x^{11}), \\ u_2 &= x^{14} \left(\frac{1}{7567560}A^2 + \frac{1}{30270240}AB + \frac{1}{23284800}A - \frac{1}{1441440}B^2 + \frac{1}{2522520}B \right) \\ &\quad + x^{13} \left(\frac{1}{21621600}A^2 - \frac{17}{5405400}AB + \frac{1}{1853280}A - \frac{1}{739200}B^2 + \frac{43}{43243200}B \right) \\ &\quad + x^{11} \left(-\frac{23}{4989600}A^2 + \frac{1}{15400}AB + \frac{13}{158400}B^2 \right) + x^{10} \left(\frac{17}{226800}A^2 + \frac{1}{4200}BA \right) \\ &\quad + x^{12} \left(-\frac{1}{249480}A^2 - \frac{13}{3326400}AB + \frac{1}{907200}A + \frac{1}{44352}B^2 \right) \\ &\quad + x^9 \left(\frac{1}{3780}A^2 + \frac{1}{336}B^3 \right) + \frac{1}{45}A^3x^6 + \frac{4}{105}A^2Bx^7 + \frac{9}{560}AB^2x^8. \end{aligned} \quad (3.17)$$

Using only three-term approximation we have

$$\begin{aligned}
 u = & x + Ax^2 + Bx^3 + \frac{1}{32} + \sin x - \frac{1}{32} \cos 2x + \frac{1}{6}x^3 - \frac{1}{16}x^2 - x + \frac{1}{48}x^4 \\
 & - \frac{1}{1260}Ax^7 - x^8 \left(\frac{1}{5040}A + \frac{1}{1680}B \right) - x^9 \left(\frac{1}{6048}B - \frac{1}{90720}A \right) \\
 & + x^{10} \left(\frac{1}{113400}A + \frac{1}{100800}B - \frac{1}{181440} \right) - \frac{1}{6}A^2x^4 - \frac{1}{40}B^2x^6 - \frac{1}{10}ABx^5 \\
 & + x^9 \left(\frac{1}{3780}A^2 + \frac{1}{336}B^3 \right) + \frac{1}{45}A^3x^6 + \frac{4}{105}A^2Bx^7 + \frac{9}{560}AB^2x^8.
 \end{aligned} \tag{3.18}$$

Now by using the conditions $u(1) = \sin 1$, $u'(1) = \cos 1$, we have a system of equations of degree three. Solving this system numerically (applying some standard computer programs) we have that $A = 5.8611 \times 10^{-3}$, $B = -0.17455$ and the series solution

$$\begin{aligned}
 u = & x + 5.8611 \times 10^{-3}x^2 - 0.17455x^3 - 6.3333 \times 10^{-6}x^4 + 8.4356 \times 10^{-3}x^5 \\
 & + 2.0161 \times 10^{-3}x^6 - 2.0329 \times 10^{-4}x^7 - 9.2803 \times 10^{-5}x^8 + O(x^9).
 \end{aligned} \tag{3.19}$$

4. Conclusion

In this paper we have used the homotopy perturbation method for finding the solution of fourth-order linear and nonlinear boundary value problems. We presented a simple way to choose L and v_0 when we use the homotopy perturbation method. In most cases, our simple choice yields very good approximation of exact solution.

References

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