Research Article

Contractive Mapping in Generalized, Ordered Metric Spaces with Application in Integral Equations

L. Gholizadeh, R. Saadati, W. Shatanawi, and S. M. Vaezpour

1 Department of Mathematics, Science and Research Branch, Islamic Azad University (IAU), Tehran, Iran
2 Department of Mathematics, Hashemite University, P.O. Box 150459, Zarqa 13115, Jordan
3 Department of Mathematics, Amirkabir University of Technology, Tehran, Iran

Correspondence should be addressed to R. Saadati, rsaadati@eml.cc

Received 22 June 2011; Revised 1 October 2011; Accepted 3 October 2011

1. Introduction

The Banach fixed point theorem for contraction mapping has been generalized and extended in many directions [1–11]. Nieto and Rodriguez-López [10], Ran and Reurings [12], and Petrusel and Rus [13] presented some new results for contractions in partially ordered metric spaces. The main idea in [10, 12, 14] involves combining the ideas of an iterative technique in the contraction mapping principle with those in the monotone technique. Also, Mustafa and Sims [15] introduced the concept of G-metric. Some authors [16, 17] have proved some fixed point theorems in these spaces. Recently, Saadati et al. [18], using the concept of G-metric, defined an Ω-distance on complete G-metric space and generalized the concept of w-distance due to Kada et al. [19].

In this paper, we extend some recent fixed point theorems by using this concept and prove various fixed point theorems in generalized partially ordered G-metric spaces.

At first we recall some definitions and lemmas. For more information see [15–18, 20–23].
**Definition 1** (see [15]). Let $X$ be a nonempty set. A function $G : X \times X \times X \to [0, \infty)$ is called a G-metric if the following conditions are satisfied:

(i) $G(x,y,z) = 0$ if $x = y = z$ (coincidence),

(ii) $G(x,x,y) > 0$ for all $x, y \in X$, where $x \neq y$,

(iii) $G(x,x,z) \leq G(x,y,z)$ for all $x, y, z \in X$, with $z \neq y$,

(iv) $G(x,y,z) = G(p(x,y,z))$, where $p$ is a permutation of $x, y, z$ (symmetry),

(v) $G(x,y,z) \leq G(x,a,a) + G(a,y,z)$ for all $x, y, z, a \in X$ (rectangle inequality).

A G-metric is said to be symmetric if $G(x,y,y) = G(y,x,x)$ for all $x, y \in X$.

**Definition 2.** Let $(X, G)$ be a G-metric space,

1. a sequence $\{x_n\}$ in $X$ is said to be G-Cauchy sequence if, for each $\varepsilon > 0$, there exists a positive integer $n_0$ such that for all $m, n, l \geq n_0$, $G(x_n, x_m, x_l) < \varepsilon$;

2. a sequence $\{x_n\}$ in $X$ is said to be G-convergent to a point $x \in X$ if, for each $\varepsilon > 0$, there exists a positive integer $n_0$ such that for all $m, n \geq n_0$, $G(x_n, x_m, x) < \varepsilon$.

**Definition 3** (see [15]). Let $(X, G)$ be a G-metric space. Then a function $\Omega : X \times X \times X \to [0, \infty)$ is called an $\Omega$-distance on $X$ if the following conditions are satisfied:

(a) $\Omega(x,y,z) \leq \Omega(x,a,a) + \Omega(a,y,z)$ for all $x, y, z, a \in X$,

(b) for any $x, y \in X$, $\Omega(x,y, \cdot), \Omega(x, \cdot, y) : X \to [0, \infty)$ are lower semicontinuous,

(c) for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\Omega(x,a,a) \leq \delta$ and $\Omega(a,y,z) \leq \delta$ imply $G(x,y,z) \leq \varepsilon$.

**Example 1** (see [18]). Let $(X, d)$ be a metric space and $G : X^3 \to [0, \infty)$ defined by

$$G(x,y,z) = \max\{d(x,y), d(y,z), d(x,z)\},$$

for all $x, y, z \in X$. Then $\Omega = G$ is an $\Omega$-distance on $X$.

**Example 2** (see [18]). In $X = \mathbb{R}$ we consider the G-metric $G$ defined by

$$G(x,y,z) = \frac{1}{3}(|x-y| + |y-z| + |x-z|),$$

for all $x, y, z \in \mathbb{R}$. Then $\Omega : \mathbb{R}^3 \to [0, \infty)$ defined by

$$\Omega(x,y,z) = \frac{1}{3}(|z-x| + |x-y|),$$

for all $x, y, z \in \mathbb{R}$ is an $\Omega$-distance on $\mathbb{R}$.

For more example see [18].
Lemma 1.1 (see [18]). Let $X$ be a metric space with metric $G$ and $\Omega$ be an $\Omega$-distance on $X$. Let $x_n, y_n$ be sequences in $X$, $\alpha_n, \beta_n$ be sequences in $[0, \infty)$ converging to zero and let $x, y, z, a \in X$. Then one has the following.

1. If $\Omega(y, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y, z) \leq \beta_n$ for $n \in \mathbb{N}$, then $G(y, y, z) < \varepsilon$ and hence $y = z$.
2. If $\Omega(y_n, x_n, x_n) \leq \alpha_n$ and $\Omega(x_n, y_m, z) \leq \beta_n$ for $m > n$ then $G(y_n, y_m, z) \to 0$ and hence $y_n \to z$.
3. If $\Omega(x_n, x_m, x_l) \leq \alpha_n$ for any $l, m, n \in \mathbb{N}$ with $n \leq m \leq l$, then $x_n$ is a $G$-Cauchy sequence.

Definition 4 (see [18]). $G$-metric space $X$ is said to be $\Omega$-bounded if there is a constant $M > 0$ such that $\Omega(x, y, z) \leq M$ for all $x, y, z \in X$.

2. Fixed Point Theorems on Partially Ordered $G$-Metric Spaces

Definition 5. Suppose $(X, \leq)$ is a partially ordered space and $T : X \to X$ is a mapping of $X$ into itself. We say that $T$ is nondecreasing if for $x, y \in X$,

$$x \leq y \implies T(x) \leq T(y).$$

Theorem 2.1. Let $(X, \leq)$ be a partially ordered space. Suppose that there exists a $G$-metric on $X$ such that $(X, G)$ is a complete $G$-metric space and $\Omega$ is an $\Omega$-distance on $X$ such that $X$ is $\Omega$-bounded. Let $f : X \to X$ and $g : X \to X$ weakly compatible and $f, g$ be non-decreasing mapping such that

(a) $g(X) \subseteq f(X)$;
(b) $\Omega(gx, gy, gz) \leq k \max\{\Omega(fx, fy, fz), \Omega(fx, gx, fz), \Omega(fy, gy, fz), \Omega(fx, gy, fz), \Omega(fy, gx, fz)\}$ for all $x, y, z \in X$ and $0 \leq k < 1$,
(c) for every $x \in X$ and $y \in X$ with $f(y) \neq g(y)$, $\inf\{\Omega(fx, y, fx) + \Omega(fx, y, gx) + \Omega(fx, gx, y) : f(x) \leq g(x)\} > 0$;
(d) there exist $x_0 \in X$ that $f(x_0) \leq g(x_0)$; then $f$ and $g$ have a unique common fixed point $u$ in $X$ and $\Omega(u, u, u) = 0$.

Proof. Let $x_0 \in X$ that $f(x_0) \leq g(x_0)$. By part (a), we can choose $x_1 \in X$ such that $f(x_1) = g(x_0)$. Again from part (a), we can choose $x_2 \in X$ such that $f(x_2) = g(x_1)$. Continuing this process we can construct sequences $\{x_n\}$ in $X$ such that,

$$y_n = gx_n = fx_{n+1}, \quad \forall \ n \geq 0,$$

$$x_n \leq x_{n+1}. \quad (2.2)$$

Now, since $g$ is non-decreasing mapping then,

$$gx_n \leq gx_{n+1}, \quad \forall \ n \geq 0, \quad (2.3)$$
so, for all \( s \geq 0 \),

\[
\Omega(y_n, y_{n+1}, y_{n+s}) = \Omega(g x_n, g x_{n+1}, g x_{n+s})
\leq k \max\{\Omega(f x_n, f x_{n+1}, f x_{n+s}), \Omega(f x_n, g x_n, f x_{n+s}), \Omega(f x_{n+1}, g x_{n+1}, f x_{n+s})\}
\leq k \max\{\Omega(y_{n-1}, y_n, y_{n+s-1}), \Omega(y_{n-1}, y_n, y_{n+s-1}), \Omega(y_{n-1}, y_{n+1}, y_{n+s-1}), \Omega(y_{n-1}, y_{n+1}, y_{n+s-1})\}.
\tag{2.4}
\]

Then,

\[
\Omega(y_n, y_{n+1}, y_{n+s}) \leq k \max\{\Omega(y_{n-1}, y_n, y_{n+s-1}), \Omega(y_{n-1}, y_{n+1}, y_{n+s-1}), \\
\Omega(y_{n-1}, y_{n+1}, y_{n+s-1}), \Omega(y_n, y_{n+s-1})\}.
\tag{2.5}
\]

Now since,

\[
\Omega(y_{n-1}, y_{n+1}, y_{n+s-1}) \leq k \max\{\Omega(y_{n-2}, y_n, y_{n+s-2}), \Omega(y_{n-2}, y_{n-1}, y_{n+s-2}), \Omega(y_n, y_{n+1}, y_{n+s-2})\}
\]

\[
\Omega(y_n, y_{n+s-1}) \leq k \max\{\Omega(y_{n-1}, y_{n-1}, y_{n+s-2}), \Omega(y_{n-1}, y_n, y_{n+s-2}), \Omega(y_{n-1}, y_n, y_{n+s-2})\}
\]

\[
\Omega(y_{n-1}, y_{n+1}, y_{n+s-1}) \leq k \max\{\Omega(y_{n-1}, y_{n-1}, y_{n+s-2}), \Omega(y_{n-1}, y_{n}, y_{n+s-2}), \Omega(y_{n-1}, y_{n}, y_{n+s-2}), \Omega(y_{n-1}, y_{n}, y_{n+s-2})\}.
\tag{2.6}
\]

thus,

\[
\Omega(y_n, y_{n+1}, y_{n+s}) \leq k^2 \max\{\Omega(y_i, y_j, y_l), \quad n-2 \leq i \leq n, n-1 \leq j \leq n+1, n+s-2 \leq t \leq n+s-1\}
\]

\[
\vdots
\]

\[
\leq k^{n-1} \max\{\Omega(y_i, y_j, y_l), \quad 1 \leq i \leq n, 2 \leq j \leq n+1, s+1 \leq t \leq n+s-1\}.
\tag{2.7}
\]

So \( \Omega(y_n, y_{n+1}, y_{n+s}) \leq k^{n-1} M_{n,s} \) where

\[
M_{n,s} := \max\{\Omega(y_i, y_j, y_l), \quad 1 \leq i \leq n, 2 \leq j \leq n+1, s+1 \leq t \leq n+s-1\}.
\tag{2.8}
\]

Now, for any \( l > m > n \) with \( m = n + k \) and \( l = m + t \ (k, t \in \mathbb{N}) \), we have,

\[
\lim_{m,n,l \to \infty} \Omega(y_n, y_m, y_l) = 0.
\tag{2.9}
\]
Since $X$ is $\Omega$-bounded and

\begin{align*}
\Omega(y_n, y_m, y_l) &\leq \Omega(y_n, y_{n+1}, y_{n+1}) + \Omega(y_{n+1}, y_m, y_l) \\
&\leq \Omega(y_n, y_{n+1}, y_{n+1}) + \Omega(y_{n+1}, y_{n+2}, y_{n+2}) + \cdots + \Omega(y_{m-1}, y_m, y_l) \\
&\leq k^{n-1}M_{n,1} + k^nM_{n+1,2} + \cdots + k^{m-2}M_{m-1,l+1} \\
&\leq \sum_{j=1}^{m-n+2} k^{n-j}M \leq \frac{k^{n-1}}{1-k}M,
\end{align*}

so, by Part (3) of Lemma 1.1, \{$y_n$\} is a $G$-Cauchy sequence. Since $X$ is $G$-complete, \{$y_n$\} converges to a point $y \in X$. Thus, for $\varepsilon > 0$ and by the lower semicontinuity of $\Omega$, we have

\begin{align*}
\Omega(y_n, y_m, y) &\leq \liminf_{p \to \infty} \Omega(y_n, y_m, y_p) \leq \varepsilon, \quad m \geq n \\
\Omega(y_n, y, y_l) &\leq \liminf_{p \to \infty} \Omega(y_n, y_p, y_l) \leq \varepsilon, \quad l \geq n.
\end{align*}

Assume that $fy \neq gy$. Since

\begin{align*}
y_n = f_{x_n+1} = g_{x_n} \leq g_{x_{n+1}} = f_{x_{n+2}} = y_{n+1},
\end{align*}

so, $y_n \leq y_{n+1}$, and,

\begin{align*}
0 < \inf \{\Omega(y_n, y, y_n) + \Omega(y_n, y_{n+1}, y) + \Omega(y_n, y, y_{n+1})\} \leq 3\varepsilon,
\end{align*}

for every $\varepsilon > 0$, that is a contraction. So, we have $fy = gy$. Then, by (b),

\begin{align*}
\Omega(gy, gy, gy) &\leq k\Omega(gy, gy, gy),
\end{align*}

so, $\Omega(gy, gy, gy) = 0$. Similarly, $\Omega(g^2y, g^2y, gy) = 0$.

Now,

\begin{align*}
\Omega(gy, g^2y, gy) &\leq k \max \{\Omega(gy, g^2y, gy), \Omega(g^2y, gy, gy), \Omega(g^2y, g^2y, gy)\} \\
&\leq k \max \{\Omega(gy, g^2y, gy), \Omega(g^2y, gy, gy)\} \\
&\leq k \max \{\Omega(gy, g^2y, gy), \Omega(g^2y, gy, gy)\}.
\end{align*}

Thus,

\begin{align*}
\Omega(gy, g^2y, gy) = 0, \quad \Omega(g^2y, gy, gy) = 0.
\end{align*}
By Part (c) of Definition 3, $G(g^2y, g^2y, gy) = 0$ and consequently $g^2y = gy$ which implies that $gy$ is a fixed point for $g$. Now,

$$f(gy) = g(fy) = g^2y = gy.$$  \hspace{1cm} (2.17)

So, it is enough to put $gy = u$, then $u$ is a common fixed point of $f$ and $g$.

**Uniqueness:** Assume that there exist $v \in X$ such that $fv = gv = v$. Hence, we have,

$$\Omega(v, v, v) \leq k\Omega(v, v, v),$$  \hspace{1cm} (2.18)

and so $\Omega(v, v, v) = 0$. Also, $\Omega(v, v, u) = 0$. On the other hand,

$$\Omega(v, u, u) \leq k \max\{\Omega(v, u, u), \Omega(u, v, u)\},$$

$$\Omega(u, u, u) \leq k \max\{\Omega(u, v, u), \Omega(v, u, u)\},$$  \hspace{1cm} (2.19)

which follows that, $\Omega(v, u, u) = \Omega(u, v, u) = 0$. Then by Part (c) of Definition 3, $u = v$ and $\Omega(u, u, u) = 0$. \hfill \Box

The following corollary is a generalization of [24, Theorem 2.1].

**Corollary 2.2.** Let $(X, \preceq)$ be a partially ordered space. Suppose that there exists a $G$-metric on $X$ such that $(X, G)$ is a $G$-metric space and $\Omega$ is an $\Omega$-distance on $X$ such that $X$ be $\Omega$-bounded. Let $f : X \to X$ and $g : X \to X$ be weakly compatible and $f, g$ be a non-decreasing mapping such that

(a) $g(X) \subseteq f(X)$ and either $f(X)$ or $g(X)$ is complete;

(b) for all $x, y, z \in X$ and $0 \leq k < 1$, $\Omega(gx, gy, gz) \leq k\Omega(fx, fy, fz)$;

(c) for every $x \in X$ and $y \in X$ with $f(y) \neq g(y)$, $\inf\{\Omega(fx, y, fx) + \Omega(fx, y, gx) + \Omega(fx, gx, y) : f(x) \leq g(x)\} > 0$;

(d) there exist $x_0 \in X$ that $f(x_0) \leq g(x_0)$;

then $f$ and $g$ have a unique common fixed point $y$ in $X$ and $\Omega(y, y, y) = 0$.

**Definition 6** (see [25]). Let $\Phi$ be the set of all functions $\varphi$ such that $\varphi : [0, \infty) \to [0, \infty)$ is a continuous and nondecreasing function with $\varphi(t) < t$ for all $t \in \mathbb{R}^+$ and $\sum_{n=1}^{\infty}\varphi^n(t) < \infty$ for each $t \in \mathbb{R}^+$. The function $\varphi$ is called a growth or control function of $T : X \to X$.

It is clear that

$$\lim_{n \to \infty}\varphi^n(t) = 0, \quad \forall t \in \mathbb{R}^+, \varphi^n(0) = 0.$$  \hspace{1cm} (2.20)

**Theorem 2.3.** Let $(X, \preceq)$ be a partially ordered space. Suppose that there exists a $G$-metric on $X$ such that $(X, G)$ is a complete $G$-metric space and $\Omega$ is an $\Omega$-distance on $X$ and $T$ is a non-decreasing mapping from $X$ into itself. Let $X$ be $\Omega$-bounded. Suppose that $\varphi \in \Phi$ and

$$\Omega(Tx, T^2x, Tw) \leq \varphi(\Omega(x, Tx, w)) \quad \forall x \leq Tx, \ w \in X.$$  \hspace{1cm} (2.21)
Also, for every \( x \in X \)

\[
\inf \left\{ \Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, T^2x, y) : x \leq Tx \right\} > 0, 
\tag{2.22}
\]

for every \( y \in X \) with \( y \neq Ty \). If there exists an \( x_0 \in X \) with \( x_0 \leq Tx_0 \), then \( T \) has a unique fixed point. Moreover, if \( v = Tv \), then \( \Omega(v, v, v) = 0 \).

**Proof.** If \( x_0 = Tx_0 \), then the proof is finished. Suppose that \( Tx_0 \neq x_0 \). Since \( x_0 \leq Tx_0 \) and \( T \) is non-decreasing, we obtain

\[
x_0 \leq Tx_0 \leq T^2x_0 \leq \cdots \leq T^{n+1}x_0 \leq \cdots 
\tag{2.23}
\]

For all \( n \in \mathbb{N} \) and \( t \geq 0 \),

\[
\Omega\left( T^n x_0, T^{n+1} x_0, T^{n+1} x_0 \right) \leq \psi^n \left( \Omega\left( T^{n-1} x_0, T^n x_0, T^{n+1} x_0 \right) \right)
\leq \psi^n \left( \Omega\left( T^{n-2} x_0, T^{n-1} x_0, T^{n+1} x_0 \right) \right)
\leq \cdots
\leq \psi^n \left( \Omega(x_0, Tx_0, Tx_0) \right).
\tag{2.24}
\]

We claim that for \( m = n + k \) and \( l = m + t \) \((k, t \in \mathbb{N})\) with \( l > m > n \),

\[
\lim_{m,n,l \to \infty} \Omega\left( T^n x_0, T^m x_0, T^l x_0 \right) = 0. 
\tag{2.25}
\]

We prove by,

\[
\Omega\left( T^n x_0, T^m x_0, T^l x_0 \right) \leq \Omega\left( T^n x_0, T^{n+1} x_0, T^{n+1} x_0 \right) + \Omega\left( T^{n+1} x_0, T^m x_0, T^l x_0 \right)
\leq \Omega\left( T^n x_0, T^{n+1} x_0, T^{n+1} x_0 \right) + \Omega\left( T^{n+1} x_0, T^{n+2} x_0, T^{n+2} x_0 \right)
+ \cdots + \Omega\left( T^{m-1} x_0, T^m x_0, T^l x_0 \right)
\leq \psi^n \left( \Omega(x_0, Tx_0, Tx_0) \right) + \psi^{n+1} \left( \Omega(x_0, Tx_0, Tx_0) \right)
+ \cdots + \psi^{m-2} \left( \Omega(x_0, Tx_0, Tx_0) \right) + \psi^{m-1} \left( \Omega(x_0, Tx_0, T^{l+1} x_0) \right)
\leq \psi^{m-1} (M) \left( \sum_{n=1}^{\infty} \psi^n (M) \right). 
\tag{2.26}
\]

Since \( \sum_{n=1}^{\infty} \psi^n (M) < \infty \), so,

\[
\lim_{m,n,l \to \infty} \Omega\left( T^n x_0, T^m x_0, T^l x_0 \right) = 0. 
\tag{2.27}
\]
By Part (c) of Lemma 1.1 \(\{T^nx_0\}\) is a G-Cauchy sequence. Since \(X\) is G-complete, \(\{T^nx_0\}\) converges to a point \(u \in X\). Let \(n \in \mathbb{N}\) be fixed. By lower semicontinuity of \(\Omega\),

\[
\Omega(T^nx_0, T^m x_0, u) \leq \liminf_{p \to \infty} \Omega(T^nx_0, T^m x_0, T^p x_0) \leq \varepsilon, \quad m > n,
\]

\[
\Omega(T^nx_0, u, T^l x_0) \leq \liminf_{p \to \infty} \Omega(T^nx_0, T^p x_0, T^m x_0) \leq \varepsilon, \quad l \geq n.
\]  

(2.28)

Assume that \(u \neq Tu\). Since \(T^n x_0 \leq T^{n+1} x_0\),

\[
0 < \inf \left\{ \Omega(T^n x_0, u, T^n x_0) + \Omega(T^n x_0, u, T^{n+1} x_0) + \Omega(T^n x_0, T^{n+2} x_0, u) : n \in \mathbb{N} \right\} \leq 3\varepsilon,
\]  

(2.29)

for every \(\varepsilon > 0\), which is a contraction. Therefore, we have \(u = Tu\).

**Uniqueness:** let \(v\) be another fixed point of \(T\), then

\[
\Omega(u, u, v) = \Omega(Tu, T^2 u, Tv) \leq \phi(\Omega(u, Tu, v)) < \Omega(u, u, v),
\]  

(2.30)

which is a contraction. Therefore, fixed point \(u\) is unique. Now, if \(v = Tv\), we have,

\[
\Omega(v, v, v) = \Omega(Tv, T^2 v, T^3 v) \leq \phi(\Omega(v, Tv, T^2 v)) = \phi(\Omega(v, v, v)).
\]  

(2.31)

So \(\Omega(v, v, v) = 0\).

\[\square\]

**Corollary 2.4.** Let the assumptions of Theorem 2.3 hold and

\[
\Omega(T^m x, T^{m+1} x, T^m w) \leq \phi(\Omega(x, Tx, w))) \quad \forall m \in \mathbb{N}, \ x \leq Tx, \ w \in X,
\]  

(2.32)

then \(T\) has a unique fixed point.

**Proof.** From Theorem 2.3, \(T^m\) has a unique fixed point \(u\). However,

\[
Tu = T(T^m u) = T^{m+1} u = T^m Tu,
\]  

(2.33)

so \(Tu\) is also a fixed point of \(T^m\). Since the fixed point of \(T^m\) is unique, it must be the case that \(Tu = u\).

\[\square\]

**Corollary 2.5.** Let the assumptions of Theorem 2.3 hold and \(T : X \to X\) satisfies,

\[
\Omega(Tx, T^2 x, Tx) \leq \phi(\Omega(x, Tx, x)) \quad \forall x \leq Tx.
\]  

(2.34)

Then \(T\) has a unique fixed point.

**Proof.** Take \(w = x\), and apply Theorem 2.3.

\[\square\]
Theorem 2.6. Let $(X, \preceq)$ be a partially ordered space. Suppose that there exists a $G$-metric on $X$ such that $(X, G)$ is a complete $G$-metric space, $\Omega$ is an $\Omega$-distance on $X$, and $T$ is a non-decreasing mapping from $X$ into itself. Let $X$ be $\Omega$-bounded. Suppose that

$$\Omega(Tx, T^2x, Tw) \leq k\left(\Omega(x, T^2x, Tw) + \Omega(x, Tx, Tx)\right),$$

(2.35)

where $x \preceq Tx, w \in X, k \in [0, 1/3)$. Also for every $x \in X$,

$$\inf\{\Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, T^2x, y) : x \leq Tx\} > 0,$$

(2.36)

for every $y \in X$ with $y \neq Ty$. If there exists an $x_0 \in X$ with $x_0 \leq Tx_0$, then $T$ has a unique fixed point say $u$ and $\Omega(u, u, u) = 0$.

Proof. Let $x_0 \in X$ be an arbitrary point, and define the sequence $x_n$ by $x_n = T^n x_0$. By (2.35) and for all $t \geq 0$,

$$\Omega(x_n, x_{n+1}, x_{n+t}) \leq k(\Omega(x_{n-1}, x_{n+1}, x_{n+t}) + \Omega(x_{n-1}, x_n, x_n)).$$

(2.37)

But by Part (a) of Definition 3,

$$\Omega(x_{n-1}, x_{n+1}, x_{n+t}) \leq \Omega(x_{n-1}, x_n, x_n) + \Omega(x_n, x_{n+1}, x_{n+t}).$$

(2.38)

Hence,

$$\Omega(x_n, x_{n+1}, x_{n+t}) \leq k[2 \Omega(x_{n-1}, x_n, x_n) + \Omega(x_n, x_{n+1}, x_{n+t})],$$

(2.39)

which implies,

$$\Omega(x_n, x_{n+1}, x_{n+t}) \leq \frac{2k}{1-k} \Omega(x_{n-1}, x_n, x_n).$$

(2.40)

Let $r = 2k/(1 - k)$, then $r < 1$ and by repeated application of (2.40), we have

$$\Omega(x_n, x_{n+1}, x_{n+t}) \leq r^n \Omega(x_0, x_1, x_1).$$

(2.41)

Now, for any $l > m > n$ with $m = n + k$ and $l = m + t$ ($k, t \in \mathbb{N}$), we have,

$$\Omega(x_n, x_m, x_l) \leq \Omega(x_n, x_{n+1}, x_{n+1}) + \Omega(x_{n+1}, x_{n+2}, x_{n+2})$$

$$+ \Omega(x_{n+2}, x_{n+3}, x_{n+3}) + \cdots + \Omega(x_{m-1}, x_m, x_l)$$

$$\leq \left(r^n + r^{n+1} + \cdots + r^{m-1}\right) \Omega(x_0, x_1, x_1)$$

$$\leq \frac{r^n}{1 - r} \Omega(x_0, x_1, x_1).$$

(2.42)
So,

$$\lim_{m,n,l \to \infty} \Omega(x_n, x_m, x_l) = 0. \quad (2.43)$$

By Part (3) of Lemma 1.1, $x_n$ is a G-Cauchy sequence. Since $X$ is G-complete, $x_n$ converges to a point $u \in X$. Now, similar to proving Theorem 2.1, $T$ has a unique fixed point and $\Omega(u, u, u) = 0$. □

**Corollary 2.7.** Let the assumptions of Theorem 2.6 hold and

$$\Omega(T^m x, T^{m+2} x, T^m w) \leq k \left( \Omega(x, T^{m+2} x, T^m w) + \Omega(x, T^m x, T^m x) \right) \quad (2.44)$$

where $k \in [0, 1/3]$, then $T$ has a unique fixed point.

**Proof.** The argument is similar to that used in the proof of Corollary 2.4. □

### 3. Applications

In this section, we give an existence theorem for a solution of a class of integral equations. Denote by $\Lambda$ the set of all functions $\lambda : [0, +\infty) \to [0, +\infty)$ satisfying the following hypotheses:

(i) $\lambda$ is a Lebesgue-integrable mapping on each compact of $[0, +\infty)$,

(ii) for every $\varepsilon > 0$, we have $\int_0^\varepsilon \lambda(s) ds > 0$,

(iii) $\|\lambda\| < 1$, where $\|\lambda\|$ denotes to the norm of $\lambda$.

Now, we have the following results.

**Theorem 3.1.** Let $(X, \leq)$ be a partially ordered space. Suppose that there exists a G-metric on $X$ such that $(X, G)$ is a complete G-metric space and $\Omega$ is an $\Omega$-distance on $X$ and $T$ is a non-decreasing mapping from $X$ into itself. Let $X$ be $\Omega$-bounded. Suppose that

$$\Omega(T x, T^2 x, Tw) \leq \int_0^{\Omega(x, Tx, w)} \alpha(s) ds, \quad (3.1)$$

where $\alpha \in \Lambda$. Also, suppose that for every $x \in X$

$$\inf \left\{ \Omega(x, y, x) + \Omega(x, y, T x) + \Omega(x, T^2 x, y) : x \leq T x \right\} > 0, \quad (3.2)$$

for every $y \in X$ with $y \neq Ty$. If there exists an $x_0 \in X$ with $x_0 \leq Tx_0$, then $T$ has a unique fixed point.

**Proof.** Define $\phi : [0, +\infty) \to [0, +\infty)$ by $\phi(t) = \int_0^t \alpha(s) ds$. It is clear that $\phi$ is nondecreasing and continuous. From (iii), we have

$$\phi(t) = \int_0^t \lambda(s) ds \leq \int_0^t |\lambda(s)| ds \leq \|\lambda\| t < t. \quad (3.3)$$
Also, note that
\[
\phi'(t) = \phi(\phi(t)) \leq \| \lambda \| \phi(t) \leq \| \lambda \|^2 t.
\] (3.4)

In general, we have \( \phi^n(t) \leq \| \lambda \|^n t. \) Thus, we have
\[
\sum_{n=1}^{\infty} \phi^n(t) \leq \sum_{n=1}^{\infty} \| \lambda \|^n t = \frac{\| \lambda \| t}{1 - \| \lambda \|} < +\infty.
\] (3.5)

Therefore \( \phi \) satisfies all the hypotheses of Definition 6. By inequality (3.1), we have
\[
\Omega(Tx, T^2x, Tw) \leq \phi(\Omega(x, Tx, w)).
\]
Therefore by Theorem 2.3, \( T \) has a unique fixed point. \( \square \)

Now, our aim is to give an existence theorem for a solution of the following integral equation:
\[
u(t) = \int_0^t K(t, s, u(s))ds + g(t), \quad t \in [0, 1].
\] (3.6)

Let \( X = C([0, 1]) \) be the set of all continuous functions defined on \([0, 1]\). Define
\[
G : X \times X \times X \to \mathbb{R}^+
\] (3.7)

by
\[
G(x, y, z) = \max\{\|x - y\|, \|x - z\|, \|y - z\|\},
\] (3.8)

where \( \|x\| = \sup\{|x(t)| : t \in [0, 1]\} \). Then \((X, G)\) is a complete G-metric space. Let \( \Omega = G \).

Then \( \Omega \) is an \( \Omega \)-distance on \( X \).

Define an ordered relation \( \preceq \) on \( X \) by
\[
x \preceq y \quad \text{iff} \quad x(t) \leq y(t), \quad \forall t \in [0, 1].
\] (3.9)

Then \((X, \preceq)\) is a partially ordered set. Now, we prove the following result.

**Theorem 3.2.** Suppose the following hypotheses hold.

(a) \( K : [0, 1] \times [0, 1] \times \mathbb{R}^+ \to \mathbb{R}^+ \) and \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) are continuous.

(b) \( K \) is nondecreasing in its first coordinate and \( g \) is nondecreasing.

(c) There exist a continuous function \( G : [0, 1] \times [0, 1] \to [0, +\infty] \) such that
\[
|K(t, s, u) - K(t, s, v)| \leq G(t, s)|u - v|,
\] (3.10)

for each comparable \( u, v \in \mathbb{R}^+ \) and each \( t, s \in [0, 1] \).

(d) \( \sup_{t \in [0, 1]} \int_0^1 G(t, s)ds \leq r \) for some \( r < 1 \).

Then the integral equation (3.6) has a solution \( u \in C([0, 1]) \).
**Proof.** Define $T : C([0, 1]) \to C([0, 1])$ by

$$Tx(t) = \int_0^1 K(t, s, x(s))ds + g(t), \quad t \in [0, 1]. \quad (3.11)$$

By hypothesis (b), we have that $T$ is nondecreasing. Now, if

$$\inf \{ \Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, T^2x, y) : x \leq Tx \} = 0, \quad (3.12)$$

for $y \in C([0, 1])$ with $y \neq Ty$, then for each $n \in \mathbb{N}$ there exists $x_n \in C([0, 1])$ with $x_n \leq Tx_n$ such that

$$\Omega(x_n, y, x_n) + \Omega(x_n, y, Tx_n) + \Omega(x_n, T^2x_n, y) \leq \frac{1}{n}. \quad (3.13)$$

So, we have

$$\Omega(x_n, y, Tx_n) = \max \{ \|x_n - y\|, \|x_n - Tx_n\|, \|y - Tx_n\| \} \leq \frac{1}{n}. \quad (3.14)$$

Therefore, for each $t \in [0, 1]$, we have

$$\lim_{n \to +\infty} x_n(t) = y(t),$$

$$\lim_{n \to +\infty} Tx_n(t) = y(t). \quad (3.15)$$

By the continuity of $K$, we have

$$y(t) = \lim_{n \to +\infty} Tx_n(t)$$

$$= \int_0^1 K(t, s, \lim_{n \to +\infty} x_n(s))ds + g(t) \quad (3.16)$$

$$= \int_0^1 K(t, s, y(s))ds + g(t) = Ty(t).$$

Thus, we have $y = Ty$, a contradiction. Thus,

$$\inf \{ \Omega(x, y, x) + \Omega(x, y, Tx) + \Omega(x, T^2x, y) : x \leq Tx \} > 0. \quad (3.17)$$
Define $\phi : [0, +\infty) \rightarrow [0, +\infty)$ by $\phi(t) = rt$. For $x \in C([0, T])$ with $x \leq Tx$, we have

$$\Omega(Tx, T^2x, Tx) = \sup_{t \in [0, 1]} \left| Tx(t) - T^2x(t) \right|$$

$$= \sup_{t \in [0, 1]} \left| \int_0^1 K(t, s, x(s)) - K(t, s, Tx(s)) \, ds \right|$$

$$\leq \sup_{t \in [0, 1]} \int_0^1 |K(t, s, x(s)) - K(t, s, Tx(s))| \, ds$$

$$\leq \sup_{t \in [0, 1]} \int_0^1 G(t, s)|x(s) - Tx(s)| \, ds$$

$$\leq \sup_{t \in [0, 1]} |x(t) - Tx(t)| \sup_{t \in [0, 1]} \int_0^T G(t, s) \, ds$$

$$= \Omega(x, Tx, x) \sup_{t \in [0, 1]} \int_0^1 G(t, s) \, ds$$

$$\leq r\Omega(x, Tx, x)$$

$$= \phi(\Omega(x, Tx, x)).$$

Moreover, take $x_0 = 0$, then $x_0 \leq Tx_0$. Thus all the required hypotheses of Corollary 2.5 are satisfied. Thus there exists a solution $u \in C([0, T])$ of the integral equation (3.6). \qed

**Acknowledgment**

The authors would like to thank the referee and area editor Professor Cristian Toma for providing useful suggestions and comments for the improvement of this paper.

**References**


