Research Article

A Sextuple Product Identity with Applications

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We get a new proof of a sextuple product identity depending on the Laurent expansion of an analytic function in an annulus. Many identities, including an identity for \((q; q)_\infty^4\), are obtained from this sextuple product identity.

1. Introduction

For convenience, we let \(|q| < 1\) throughout the paper. We employ the standard notation

\[
(a; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n), \quad (a, b, \ldots, c; q)_\infty = (a; q)_\infty (b; q)_\infty \cdots (c; q)_\infty.
\] (1.1)

Series product has been an interesting topic. The Jacobi triple product is one of the most famous series-product identity. We announce it in the following (see, e.g, [1, page 35, Entry 19] or [2, Equation (2.1)]):

\[
(q, z, \frac{q}{z}; q)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{(1/2)n(n-1)} z^n, \quad z \neq 0.
\] (1.2)

It is well known that an analytic function has a unique Laurent expansion in an annulus. Bailey [3] used this property to prove the quintuple product identity. By this approach, Cooper [4, 5] and Kongsiriwong and Liu [2] proved many types of the Macdonald identities and some other series-product identities. In this paper, we use this method to deal with a sextuple product identity.
In Section 2, we present the sextuple product identity (2.1 below) and its proof. Our identity is equivalent to [2, Equation (8.16)] by Kongsiriwong and Liu, which is the simplification of [2, Equation (6.13)]. Kongsiriwong and Liu got [2, Equation (8.16)] from a more general identity. In this section, we give it a direct proof.

In Section 3, we get many identities from this sextuple product identity.

To simplify notation, we often write $\sum_n$ for $\sum_{n=-\infty}^{\infty}$ in the following when no confusion occurs.

### 2. A New Proof of the Sextuple Product Identity

The starting point of our investigation in this section is the identity in the following theorem.

**Theorem 2.1.** For any complex number $z$ with $z \neq 0$, one has

$$
\left( q, z, q^2 \right)_\infty \left( q^3, z^3, q^9 \right)_\infty = \left( q^{12}, -q^6, -q^6 \right)_\infty \sum_{n} q^{2n^2-2n} z^{4n} + 2 \left( q^{12}, -q^{12}, -q^{12} \right)_\infty \sum_{n} q^{2n^2+1} z^{4n+2} \tag{2.1}
$$

Before the proof of Theorem 2.1, we need some preparations. The two identities in the following lemma are from [6]. We write them in this version.

**Lemma 2.2.** One has

$$
\left( q^8, q^3, q^5 \right)_\infty \left( q^{24}, q^9, q^{15} \right)_\infty = q^2 \left( q^8, q, q^7 \right)_\infty \left( q^{24}, q^3, q^{21} \right)_\infty \tag{2.2}
$$

$$
\left( q^8, q, q^7 \right)_\infty \left( q^{24}, q^9, q^{15}, q^{24} \right)_\infty - q \left( q^8, q^3, q^5 \right)_\infty \left( q^{24}, q^3, q^{21}, q^{24} \right)_\infty = \left( q^2, q, q \right)_\infty \left( q^6, -q^6, -q^6 \right)_\infty \tag{2.3}
$$

*Proof.** For (2.2), see [6, Equation (3.18)]. Equation (2.3) is from [6, Equation (3.21)]. Its proof is similar to that of [6, Equation (3.18)].

The lemma above is used to prove the following two identities.
Lemma 2.3. One has

\[
(q, q, q; q)_{\infty} \left( q^3, -q^3, -q^3; q^3 \right)_{\infty} + (q, iq, -iq; q)_{\infty} \left( q^3, -iq^3, iq^3; q^3 \right)_{\infty} = 2 \left( q^3, -q^3, -q^3; q^3 \right)_{\infty}
\]

(2.4)

\[
(q, q, q; q)_{\infty} \left( q^3, -q^3, -q^3; q^3 \right)_{\infty} - (q, iq, -iq; q)_{\infty} \left( q^3, -iq^3, iq^3; q^3 \right)_{\infty} = 2q \left( q^4, -q^2, -q^2; q^4 \right)_{\infty} \left( q^{12}, -q^{12}, -q^{12}; q^{12} \right)_{\infty}.
\]

(2.5)

Proof. By (1.2), we have

\[
(q, q, q; q)_{\infty} \left( q^3, -q^3, -q^3; q^3 \right)_{\infty} = \frac{1}{4} (q, -1, q; q)_{\infty} \left( q^3, -1, -q^3; q^3 \right)_{\infty} = \frac{1}{4} \sum_m q^{(1/2)(m^2-n)} \sum_n q^{(3/2)(n^2-n)} = \sum_m q^{2m^2+m} \sum_n q^{6n^2+3n}
\]

\[
= \sum_m q^{8m^2+2m} \sum_n q^{24n^2+6n} + q^4 \sum_m q^{8m^2+6m} \sum_n q^{24n^2+18n}
\]

(2.6)

\[
(q, iq, -iq; q)_{\infty} \left( q^3, -iq^3, iq^3; q^3 \right)_{\infty} = \frac{1}{2} (q, i, -iq; q)_{\infty} \left( q^3, -i, iq^3; q^3 \right)_{\infty} = \frac{1}{2} \sum_m (-1)^m q^{(1/2)(m^2-m)} i^m \sum_n (-1)^n q^{(3/2)(n^2-n)} i^{3n}
\]

\[
= \sum_m (-1)^m q^{2m^2+m} \sum_n (-1)^n q^{6n^2+3n}
\]

\[
= \sum_m q^{8m^2+2m} \sum_n q^{24n^2+6n} + q^4 \sum_m q^{8m^2+6m} \sum_n q^{24n^2+18n}
\]

(2.7)

\[
(q, iq, -iq; q)_{\infty} \left( q^3, -iq^3, iq^3; q^3 \right)_{\infty} = \frac{1}{2} (q, i, -iq; q)_{\infty} \left( q^3, -i, iq^3; q^3 \right)_{\infty} = \frac{1}{2} \sum_m (-1)^m q^{(1/2)(m^2-m)} i^m \sum_n (-1)^n q^{(3/2)(n^2-n)} i^{3n}
\]

\[
= \sum_m (-1)^m q^{2m^2+m} \sum_n (-1)^n q^{6n^2+3n}
\]

\[
= \sum_m q^{8m^2+2m} \sum_n q^{24n^2+6n} + q^4 \sum_m q^{8m^2+6m} \sum_n q^{24n^2+18n}
\]

(2.7)

Adding (2.6) and (2.7), we have

\[
(q, q, q; q)_{\infty} \left( q^3, -q^3, -q^3; q^3 \right)_{\infty} + (q, iq, -iq; q)_{\infty} \left( q^3, -iq^3, iq^3; q^3 \right)_{\infty} = 2 \sum_m q^{8m^2+2m} \sum_n q^{24n^2+6n} + 2q^4 \sum_m q^{8m^2+6m} \sum_n q^{24n^2+18n}
\]

(2.8)

\[
= 2 \left( q^{16}, -q^6, -q^{10}; q^{16} \right)_{\infty} \left( q^{48}, -q^{18}, -q^{30}; q^{48} \right)_{\infty}
\]

\[
+ 2q^4 \left( q^{16}, -q^2, -q^{14}; q^{16} \right) \left( q^{48}, -q^6, -q^{12}; q^{48} \right)_{\infty}.
\]

By (2.2), we have (2.4).
Subtracting (2.7) from (2.6), we obtain
\[
(q, -q, -q; q)_\infty \left( q^3, -q^3, -q^3; q^3 \right)_\infty - (q, iq, iq; q)_\infty \left( q^3, -iq^3, iq^3; q^3 \right)_\infty
\]
\[= 2q^3 \sum_m q^{8m^2 + 2m} \sum_n q^{24n^2 + 18n} + 2q \sum_m q^{8m^2 + 6m} \sum_n q^{24n^2 + 6n}\]
\[= 2q^3 \left( q^{16}, -q^6, -q^{10}; q^{16} \right)_\infty \left( q^{48}, -q^6, -q^{42}; q^{48} \right)_\infty
\]
\[+ 2q \left( q^{16}, -q^2, -q^{14}; q^{16} \right)_\infty \left( q^{48}, -q^{16}, -q^{32}; q^{48} \right)_\infty. \]

Replacing \( q \) in (2.3) by \( -q^2 \) and, then, applying the resulting identity to the above equation, we get (2.5). This completes the proof.

\[\square\]

Proof of Theorem 2.1. Set
\[
f(z, q) = (q, z, \frac{q}{z}, q)_\infty \left( q^3, z^3, \frac{q^3}{z^3}; q^3 \right)_\infty. \tag{2.10}\]

Then \( f \) is an analytic function of \( z \) in the annulus \( 0 < |z| < \infty \). Put
\[
f(z, q) = \sum_n a_n(q) z^n, \quad 0 < |z| < \infty. \tag{2.11}\]

By (2.10), we can easily verify
\[
f(z, q) = z^4 f(zq, q). \tag{2.12}\]

Combining (2.11) and (2.12) gives
\[
\sum_m a_m(q) z^m = \sum_m q^{m-4} a_{m-4}(q) z^m. \tag{2.13}\]

Equate the coefficients of \( z^m \) on both sides to get
\[
a_m(q) = q^{m-4} a_{m-4}(q). \tag{2.14}\]

Using the above relation, we obtain
\[
a_{4m-1}(q) = q^{2m^2-3m} a_{-1}(q), \quad a_{4m}(q) = q^{2m^2-2m} a_0(q), \tag{2.15}
\]
\[
a_{4m+1}(q) = q^{2m^2-m} a_1(q), \quad a_{4m+2}(q) = q^{2m^2} a_2(q). \tag{2.15}
\]
Substituting the above four identities into (2.11), we have

\[
f(z, q) = a_{-1}(q) \sum_m q^{2m^2 - 3m - 4m - 1} + a_0(q) \sum_m q^{2m^2 - 2m - 4m} \\
+ a_1(q) \sum_m q^{2m^2 - m - 4m + 1} + a_2(q) \sum_m q^{2m^2 - 4m + 2}.
\]  
(2.16)

By (2.10), we also have

\[
f(z, q) = f\left(\frac{q}{z}, q\right).
\]  
(2.17)

This gives

\[
\sum_m a_m(q) z^m = \sum_m q^{-m} a_{-m}(q) z^m.
\]  
(2.18)

Then we have

\[
a_m(q) = q^{-m} a_{-m}(q).
\]  
(2.19)

Set \( m = 1 \) to get

\[
a_1(q) = q^{-1} a_{-1}(q).
\]  
(2.20)

By this relation, (2.16) reduces to

\[
f(z, q) = a_0(q) \sum_m q^{2m^2 - 2m - 4m + 1} + a_1(q) \sum_m q^{(1/2)(m^2 - m)} z^{2m + 1} \\
+ a_2(q) \sum_m q^{2m^2 - 4m + 2}.
\]  
(2.21)

Now, it remains to determine \( a_0(q), a_1(q), \) and \( a_2(q) \).

Putting \( z = 1 \) in (2.21) gives

\[
0 = a_0(q) \sum_m q^{2m^2 - 2m} + a_1(q) \sum_m q^{(1/2)(m^2 - m)} + a_2(q) \sum_m q^{2m^2}.
\]  
(2.22)

Set \( z = -1 \) in (2.21) to get

\[
4(q, -q, -q; q)_\infty \left( q^3, -q^3, -q^3; q^3 \right)_\infty \\
= a_0(q) \sum_m q^{2m^2 - 2m} - a_1(q) \sum_m q^{(1/2)(m^2 - m)} + a_2(q) \sum_m q^{2m^2}.
\]  
(2.23)
Taking \( z = i \) in (2.21) and noting that \( \sum_m (-1)^m q^{(1/2)(m^2-m)} = 0 \), we have

\[
(q, i, -iq; q)_\infty \left( q^3, -i, iq^3; q^3 \right)_\infty = a_0(q) \sum_m q^{2m^2-2m} - a_2(q) \sum_m q^{2m^2}. \tag{2.24}
\]

Subtracting (2.23) from (2.22) and noting that \( \sum_m q^{(1/2)(m^2-m)} = 2(q, -q, q)_\infty \), we obtain

\[
a_1(q) = -\left( q^3, -q^3, -q^3; q^3 \right)_\infty. \tag{2.25}
\]

Add (2.22) and (2.23) to get

\[
2(q, -q, -q; q)_\infty \left( q^3, -q^3, -q^3; q^3 \right)_\infty = a_0(q) \sum_m q^{2m^2-2m} + a_2(q) \sum_m q^{2m^2}. \tag{2.26}
\]

Adding (2.24) and (2.26) and, then, using (1.2) in the resulting equation, we obtain

\[
(q, -q, -q; q)_\infty \left( q^3, -q^3, -q^3; q^3 \right)_\infty + (q, iq, -iq; q)_\infty \left( q^3, -iq^3, iq^3; q^3 \right)_\infty = 2a_0(q) \left( q^4, -q^4, -q^4; q^4 \right)_\infty. \tag{2.27}
\]

By (2.4), we have

\[
a_0(q) = \left( q^{12}, -q^6, -q^6; q^{12} \right)_\infty. \tag{2.28}
\]

Similarly, subtracting (2.24) from (2.26) and, then using (1.2), we have

\[
(q, -q, -q; q)_\infty \left( q^3, -q^3, -q^3; q^3 \right)_\infty - (q, iq, -iq; q)_\infty \left( q^3, -iq^3, iq^3; q^3 \right)_\infty = a_2(q) \left( q^4, -q^2, -q^2; q^4 \right)_\infty. \tag{2.29}
\]

Applying (2.5) to this equation gives

\[
a_2(q) = 2q \left( q^{12}, -q^{12}, -q^{12}; q^{12} \right)_\infty, \tag{2.30}
\]

which completes the proof.
Corollary 3.2. One has

\[ 3(q;q)_\infty^3 \left( q^3 ; q^3 \right)_\infty^3 = \left( q^{12}, -q^6, -q^6 ; q^{12} \right)_\infty \sum_n 2n(4n-1)q^{2n^2-2n} \]

\[ + 2 \left( q^{12}, -q^{12}, -q^{12} ; q^{12} \right)_\infty \sum_n (2n+1)(4n+1)q^{2n^2+1} \]

\[ - \left( q^3, -q^3, -q^3 ; q^3 \right)_\infty \sum_n n(2n+1)q^{1/2(n^2-n)}. \]

**Proof.** Dividing both sides of (2.1) by \((1-z)^2\), letting \(z \to 1\), and then using L’Hospital’s rule twice on the right-hand side gives (3.1).

\[ \square \]

Corollary 3.1. One has

\[ \left( q^{24}, -q^{12}, -q^{12}, q^{24} \right)_\infty \left( q^8, -q^4, -q^4, q^8 \right)_\infty + 4q^4 \left( q^{24}, -q^{24}, -q^{24}, q^{24} \right)_\infty \left( q^8, -q^8, -q^8, q^8 \right)_\infty \]

\[ + 2q \left( q^6, -q^6, -q^6, q^6 \right)_\infty \left( q^2, -q^2, -q^2, q^2 \right)_\infty \]

\[ = \left( q^2, -q, -q, q^2 \right)_\infty \left( q^6, -q^3, -q^3, q^6 \right)_\infty, \]

(3.2)

\[ \left( q^{36}, -q^{18}, -q^{18}, q^{36} \right)_\infty \left( q^{12}, -q^4, -q^4, q^{12} \right)_\infty + 2q^5 \left( q^{36}, -q^{36}, -q^{36}, q^{36} \right)_\infty \left( q^{12}, -q^2, -q^{10}, q^{12} \right)_\infty \]

\[ - q \left( q^9, -q^9, -q^9, q^9 \right)_\infty \left( q^3, -q, -q^3, q^3 \right)_\infty \]

\[ = \left( q^2 ; q^2 \right)_\infty \left( q^3 ; q^3 \right)_\infty, \]

(3.3)

\[ \left( q^{12}, -q^6, -q^6, q^{12} \right)_\infty \left( q^4, -q^4, -q^4, q^4 \right)_\infty - q \left( q^{12}, -q^{12}, -q^{12}, q^{12} \right)_\infty \left( q^4, -q^2, -q^2, q^4 \right)_\infty \]

\[ = \left( q^2 ; q^2 \right)_\infty \left( q^6 ; q^6 \right)_\infty \frac{ \left( q^6 ; q^6 \right)_\infty }{ \left( -q ; q^2 \right)_\infty \left( -q^3 ; q^6 \right)_\infty }, \]

(3.4)

3. Some Applications

In this section, we deduce many modular identities from Theorem 2.1.

Corollary 3.1. One has
Proof. Replace $q$ in (2.1) by $q^2$ and, then, $z$ by $-q$. Using (1.2) in the resulting identity gives (3.2).

Replace $q$ in (2.1) by $q^3$ and, then, $z$ by $q$. Using (1.2) in the resulting identity gives (3.3).

Replace $q$ in (2.1) by $q^4$ and, then, $z$ by $q$. Using (1.2) and the fact that $(q^4, -q, -q^3; q^4)_\infty = (q, -q, -q; q)_\infty$ in the resulting identity, we obtain

\[
\left(\left(q^{48}, -q^{24}, -q^{24}, q^{48}\right)_\infty + 2q^6\left(q^{48}, -q^{48}, -q^{48}, q^{48}\right)_\infty\right)\left(q^4, -q^4, -q^4, q^4\right)_\infty
-q\left(q^{12}, -q^{12}, -q^{12}, q^{12}\right)_\infty \left(q^4, -q^2, -q^2, q^4\right)_\infty = \frac{(q^2; q^2)_\infty (q^6; q^6)_\infty}{(-q; q^2)_\infty (-q^5; q^5)_\infty}.\]

By (1.2), we have

\[
\left(q^{12}, -q^{6}, -q^{6}, q^{12}\right)_\infty = \sum_n q^{6n^2} = \sum_n q^{6(2n)^2} + \sum_n q^{6(2n+1)^2} = \left(q^{48}, -q^{24}, -q^{24}, q^{48}\right)_\infty + 2q^6\left(q^{48}, -q^{48}, -q^{48}, q^{48}\right)_\infty.
\]

Combining (3.7) and (3.8) gives (3.4).

Replace $q$ in (2.1) by $q^5$ and, then, $z$ by $q^2$. Using (1.2) in the resulting identity gives (3.5).

Replace $q$ in (2.1) by $q^5$ and, then, $z$ by $q$. Using (1.2) in the resulting identity gives (3.6).

Obviously, using the same method above, we can get more identities from (2.1). Now, we deduce a formula for $(q; q)_\infty$ from (2.1).
Corollary 3.3. One has

\[ (q, q)_\infty^4 = 2 \sum_m q^{2m} \sum_n 2nq^{2n^2 + 2n} + 2q \sum_m q^{2m^2 + 2m} \sum_n (2n - 1)q^{6n^2 - 4n} + \sum_m q^{2m^2 + m} \sum_n (2n + 1)q^{(1/2)(3n^2 + n)}. \] (3.9)

Proof. Denote the left-hand side of (2.1) by \( f(z) \) and the right-hand side of (2.1) by \( g(z) \). Let \( z_0 \) be a zero point of \( f(z) \). Because (2.1) holds in \( 0 < |z| < \infty \), \( z_0 \) is also a zero point of \( g(z) \). If \( az_0 = 1 \), we have

\[ \lim_{z \to z_0} \frac{f(z)}{1 - az} = \lim_{z \to z_0} \frac{g(z)}{1 - az}. \] (3.10)

Setting \( z_0 = a = 1 \) in (3.10) and by L’Hospital’s rule on the right-hand side, we have

\[ 0 = \left( q^3, -q^3, -q^3, q^3 \right)_\infty \sum_n (2n + 1)q^{(1/2)(n^2 - n)} - 2\left( q^{12}, -q^{12}, -q^{12}, q^{12} \right)_\infty \sum_n (4n + 2)q^{2n^2 + 1} - \left( q^{12}, -q^6, -q^6, q^{12} \right)_\infty \sum_n 4nq^{2n^2 - 2n}. \] (3.11)

Let \( \omega = e^{(2/3)\pi i} \). Putting \( z_0 = \omega \) and \( a = \omega^2 \) in (3.10) and noting \( \omega^{3n} = 1 \) for any integer \( n \), we have

\[ 3(1 - \omega) \left( q^3, q^3 \right)_\infty^4 = \left( q^3, -q^3, -q^3, q^3 \right)_\infty \sum_n (2n + 1)q^{(1/2)(n^2 - n)} \omega^{2n(n-1)} - 2\left( q^{12}, -q^{12}, -q^{12}, q^{12} \right)_\infty \sum_n (4n + 2)q^{2n^2 + 1} \omega^{n-1} - \left( q^{12}, -q^6, -q^6, q^{12} \right)_\infty \sum_n 4nq^{2n^2 - 2n} \omega^n. \] (3.12)

Taking \( z_0 = \omega^2 \) and \( a = \omega \) in (3.10), we obtain

\[ 3 \left( 1 - \omega^2 \right) \left( q^3, q^3 \right)_\infty^4 = \left( q^3, -q^3, -q^3, q^3 \right)_\infty \sum_n (2n + 1)q^{(1/2)(n^2 - n)} \omega^{n-1} - 2\left( q^{12}, -q^{12}, -q^{12}, q^{12} \right)_\infty \sum_n (4n + 2)q^{2n^2 + 1} \omega^{2(n-1)} - \left( q^{12}, -q^6, -q^6, q^{12} \right)_\infty \sum_n 4nq^{2n^2 - 2n} \omega^{2n}. \] (3.13)
Adding the above three identities together gives

\[
9(q^3;q^3)_\infty^4 = (q^3,-q^3,-q^3;q^3)_\infty \sum_n (2n+1)q^{(1/2)(n^2-n)} \left( 1 + \omega^n + \omega^{2(n-1)} \right) \\
- 2(q^{12},-q^{12},-q^{12};q^{12})_\infty \sum_n (4n+2)q^{2n^2+1} \left( 1 + \omega^n + \omega^{2(n-1)} \right) \\
- (q^{12},-q^6,-q^6;q^{12})_\infty \sum_n 4nq^{3n^2-2n} \left( 1 + \omega^n + \omega^{2n} \right).
\]

Using the fact

\[
1 + \omega^n + \omega^{2n} = \begin{cases} 
3, & n \equiv 0 \pmod 3, \\
0, & n \not\equiv 0 \pmod 3
\end{cases}
\]

in the above identity and, then, replacing \( q^3 \) by \( q \), we get

\[
(q;q)_\infty^4 = (q,-q,-q;q)_\infty \sum_n (2n+1)q^{(1/2)(3n^2+n)} \\
- 4q(q^4,-q^4,-q^4;q^4)_\infty \sum_n (2n+1)q^{6n^2+4n} \\
- 2(q^4,-q^2,-q^2;q^4)_\infty \sum_n 2nq^{3n^2-2n}.
\]

Replacing \( n \) in the last two sums on the right-hand side of the above identity by \(-n\) and, then, applying (1.2) to the resulting equation, we get Corollary 3.3.

\[
\square
\]

4. Conclusion

Besides the Jacobi triple product (1.2), well-known series-product identities are known as such an identity. Recently, we also obtain some other identities of this kind, including the simplifications of the formulae [2, Equations (6.12) and (6.14)], with a different method. These identities are widely used in number theory, combinatorics, and many other fields. Literature on this topic abounds. In (2.1), if we replace \( z \) by \( e^{2iz} \), then the right-hand side of (2.1) turns into Fourier series. For recent papers on the applications of Fourier analysis, we refer the readers to [7–9].

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