Research Article

Totally Umbilical Proper Slant and Hemislant Submanifolds of an LP-Cosymplectic Manifold

Siraj Uddin,¹ Meraj Ali Khan,² and Khushwant Singh³

¹ Institute of Mathematical Sciences, Faculty of Science, University of Malaya, Kuala Lumpur 50603, Malaysia
² Department of Mathematics, University of Tabouk, Tabouk, Saudi Arabia
³ School of Mathematics and Computer Applications, Thapar University, Patiala 147 004, India

Correspondence should be addressed to Siraj Uddin, siraj.ch@gmail.com

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In the present note, we study slant and hemi-slant submanifolds of an LP-cosymplectic manifold which are totally umbilical. We prove that every totally umbilical proper slant submanifold of an LP-cosymplectic manifold is either totally geodesic or if the manifold is not totally geodesic then we derive a formula for slant angle of the submanifold. Also, we obtain the integrability conditions of the distributions of a hemi-slant submanifold, and then we give a result on its classification.

1. Introduction

A manifold \( \overline{M} \) with Lorentzian paracontact metric structure \((\phi, \xi, \eta, g)\) satisfying \((\nabla_X \phi)Y = 0\) is called an LP-cosymplectic manifold, where \(\nabla\) is the Levi-Civita connection corresponding to the Lorentzian metric \(g\) on \(\overline{M}\). The study of slant submanifolds was initiated by Chen [1]. Since then, many research papers have appeared in this field. Slant submanifolds are the natural generalization of both holomorphic and totally real submanifolds. Lotta [2] defined and studied these submanifolds in contact geometry. Later on, Cabrerizo et al. studied slant, semi-slant, and bislant submanifolds in contact geometry [3, 4]. In particular, totally umbilical proper slant submanifold of a Kaehler manifold has also been studied in [5]. Recently, Khan et al. [6] studied these submanifolds in the setting of Lorentzian paracontact manifolds.

The idea of hemi-slant submanifolds was introduced by Carriazo as a particular class of bislant submanifolds, and he called them antislant submanifolds [7]. Recently, these submanifolds are studied by Sahin for their warped products [8]. In this paper, we study slant and hemi-slant submanifolds of an LP-cosymplectic manifold. We prove that a
2. Preliminaries

Let $\mathcal{M}$ be a $n$-dimensional paracontact manifold with the Lorentzian paracontact metric structure $(\phi, \xi, \eta, g)$, that is, $\phi$ is a $(1,1)$ tensor field, $\xi$ is a contravariant vector field, $\eta$ is a 1-form, and $g$ is a Lorentzian metric with signature $(-,+,\ldots,+)$ on $\mathcal{M}$, satisfying [9],

$$\phi^2 = X + \eta(X)\xi, \quad \eta(\xi) = -1, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \text{rank}(\phi) = n - 1, \quad (2.1)$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi), \quad (2.2)$$

for all $X, Y \in T\mathcal{M}$.

A Lorentzian paracontact metric structure on $\mathcal{M}$ is called a Lorentzian para-cosymplectic structure if $\nabla\phi = 0$, where $\nabla$ denotes the Levi-Civita connection with respect to $g$. The manifold $\mathcal{M}$ in this case is called a Lorentzian para-cosymplectic (in brief, an LP-cosymplectic) manifold [10]. From formula $\nabla\phi = 0$, it follows that $\nabla_X\xi = 0$.

Let $M$ be a submanifold of a Lorentzian almost paracontact manifold $\mathcal{M}$ with Lorentzian almost paracontact structure $(\phi, \xi, \eta, g)$. Let the induced metric on $M$ also be denoted by $g$, then Gauss and Weingarten formulae are given by

$$\nabla_X Y = \nabla_X Y + h(X, Y), \quad (2.3)$$

$$\nabla_X N = -A_N X + \nabla_X^\perp N, \quad (2.4)$$

for any $X, Y \in TM$ and $N \in T^\perp M$, where $TM$ is the Lie algebra of vector field in $M$ and $T^\perp M$ is the set of all vector fields normal to $M$. $\nabla^\perp$ is the connection in the normal bundle, $h$ is the second fundamental form, and $A_N$ is the Weingarten endomorphism associated with $N$. It is easy to see that

$$g(A_N X, Y) = g(h(X, Y), N). \quad (2.5)$$

For any $X \in TM$, we write

$$\phi X = PX + FX, \quad (2.6)$$

where $PX$ is the tangential component and $FX$ is the normal component of $\phi X$. Similarly for $N \in T^\perp M$, we write

$$\phi N = BN + CN, \quad (2.7)$$

where $BN$ is the tangential component and $CN$ is the normal component of $\phi N$. 

Totally umbilical proper slant submanifold $M$ is either totally geodesic in $\mathcal{M}$ or if it is not totally geodesic, then the slant angle $\theta = \tan^{-1}(\sqrt{g(X, Y)/\eta(X)\eta(Y)})$. Also, we define hemi-slan submanifolds of an LP-contact manifold. After we find integrability conditions of the distributions, we investigate a classification of totally umbilical hemi-slan submanifolds of an LP-cosymplectic manifold.
The covariant derivatives of the tensor fields $\phi$, $P$, and $F$ are defined as

\begin{align}
\nabla_X\phi Y &= \nabla_X\phi Y - \phi \nabla_X Y, \quad \forall X, Y \in TM, \\
\nabla_X P Y &= \nabla_X P Y - P \nabla_X Y, \quad \forall X, Y \in TM, \\
\nabla_X F Y &= \nabla_X F Y - F \nabla_X Y, \quad \forall X, Y \in TM.
\end{align}

(2.8)

(2.9)

(2.10)

Moreover, for an LP-cosymplectic manifold, one has

\begin{align}
\nabla_X P Y &= A_F Y X + Bh(X, Y), \\
\nabla_X F Y &= Ch(X, Y) - h(X, PY).
\end{align}

(2.11)

(2.12)

A submanifold $M$ is said to be totally umbilical if

$$h(X, Y) = g(X, Y)H,$$

(2.13)

where $H$ is the mean curvature vector. Furthermore, if $h(X, Y) = 0$ for all $X, Y \in TM$, then $M$ is said to be totally geodesic, and if $H = 0$, then $M$ is minimal in $\overline{M}$.

A submanifold $M$ of a paracontact manifold $\overline{M}$ is said to be a slant submanifold if for any $x \in M$ and $X \in T_x M - \langle \xi \rangle$, the angle between $\phi X$ and $T_x M$ is constant. The constant angle $\theta \in [0, \pi/2]$ is then called slant angle of $M$. The tangent bundle $TM$ of $M$ is decomposed as

$$TM = D \oplus \langle \xi \rangle,$$

(2.14)

where the orthogonal complementary distribution $D$ of $\langle \xi \rangle$ is known as the slant distribution on $M$. If $\mu$ is $\phi$-invariant subspace of the normal bundle $T^\perp M$, then

$$T^\perp M = FTM \oplus \mu.$$

(2.15)

Khan et al. [6] proved the following theorem for a slant submanifold $M$ of a Lorentzian paracontact manifold $\overline{M}$ with slant angle $\theta$.

**Theorem 2.1.** Let $M$ be a submanifold of an LP-contact manifold $\overline{M}$ such that $\xi \in TM$, then $M$ is slant submanifold if and only if there exists a constant $\lambda \in [0, 1]$ such that

$$P^2 = \lambda (I + \eta \otimes \xi).$$

(2.16)

Furthermore, if $\theta$ is slant angle of $M$, then $\lambda = \cos^2 \theta$. 
Thus, one has the following consequences of formula (2.16):

\[
g(\mathbb{P}X, \mathbb{P}X) = \cos^2 \theta [g(X, Y) + \eta(X)\eta(Y)], \tag{2.17}
g(FX, FY) = \sin^2 \theta [g(X, Y) + \eta(X)\eta(Y)], \tag{2.18}
\]

for any \(X, Y \in TM\).

### 3. Totally Umbilical Proper Slant Submanifold

In this section, we consider \(M\) as a totally umbilical proper slant submanifold of an LP-cosymplectic manifold \(\overline{M}\). Such submanifolds we always consider tangent to the structure vector field \(\xi\).

**Theorem 3.1.** A nontrivial totally umbilical proper slant submanifold \(M\) of an LP-cosymplectic manifold \(\overline{M}\) is either totally geodesic or if it is not totally geodesic in \(\overline{M}\), then the slant angle \(\theta = \tan^{-1} \left(\sqrt{\frac{g(X, Y)\eta(X)\eta(Y)}}\right)\), for any \(X, Y \in TM\).

**Proof.** For any \(X, Y \in TM\), (2.11) gives

\[
\left(\nabla_X P\right)Y = A_{FY} X + Bh(X, Y). \tag{3.1}
\]

Taking the product with \(\xi\) and using (2.9), we obtain

\[
g(\nabla_X PY, \xi) = g(A_{FY} X, \xi) + g(Bh(X, Y), \xi). \tag{3.2}
\]

Using (2.5) and the fact that \(M\) is totally umbilical, the above equation takes the form

\[
-g(\xi, \nabla_X \xi) = g(H, FY)\eta(X) + g(X, Y)g(BH, \xi). \tag{3.3}
\]

Then, from the characteristic equation of LP-cosymplectic manifold, we obtain

\[
0 = g(H, FY)\eta(X). \tag{3.4}
\]

Thus, from (3.4), it follows that either \(H \in \mu\) or \(M\) is trivial.

Now, for an LP-cosymplectic manifold, one has, from (2.8),

\[
\nabla_X \phi Y = \phi \nabla_X Y, \tag{3.5}
\]

for any \(X, Y \in TM\). From (2.3) and (2.6), we obtain

\[
\nabla_X P Y + \nabla_X F Y = \phi (\nabla_X Y + h(X, Y)). \tag{3.6}
\]
Again using (2.3), (2.4), and (2.6), we get

$$\nabla_XPY + h(X, PY) - A_{FY}X + \nabla_X^\perp FY = P\nabla_XY + F\nabla_XY + \phi h(X, Y). \quad (3.7)$$

As $M$ is totally umbilical, then

$$\nabla_XPY + h(X, PY) - A_{FY}X + \nabla_X^\perp FY = P\nabla_XY + F\nabla_XY + g(X, Y)\phi H. \quad (3.8)$$

Taking the inner product with $\phi H$ and using the fact that $H \in \mu$, we obtain

$$g(h(X, PY), \phi H) + g\left(\nabla_X^\perp FY, \phi H\right) = g(F\nabla_XY, \phi H) + g(X, Y)g(\phi H, \phi H). \quad (3.9)$$

Then from (2.2) and (2.13), we get

$$g(X, PY)g(H, \phi H) + g\left(\nabla_X^\perp FY, \phi H\right) = g(F\nabla_XY, \phi H) + g(X, Y)\|H\|^2. \quad (3.10)$$

Again, using (2.2) and the fact that $H \in \mu$, then $\phi H$ is also lies in $\mu$; thus, we obtain

$$g\left(\nabla_X^\perp FY, \phi H\right) = g(X, Y)\|H\|^2. \quad (3.11)$$

Then, from (2.4), we derive

$$g\left(\nabla_X FY, \phi H\right) = g(X, Y)\|H\|^2. \quad (3.12)$$

Now, for any $X \in TM$, one has

$$\left(\nabla_X \phi\right)H = \overline{\nabla}_X \phi H - \phi \overline{\nabla}_X H. \quad (3.13)$$

Using the fact that as $\overline{M}$ is an LP-cosymplectic manifold, we obtain

$$\nabla_X \phi H = \phi \overline{\nabla}_X H. \quad (3.14)$$

Using (2.4), (2.6), and (2.7), we obtain

$$-A_{\phi H}X + \nabla_X^\perp \phi H = -PA_{\phi}X - FA_{\phi}X + B\nabla_X^\perp H + C\nabla_X H. \quad (3.15)$$

Taking the product in (3.15) with $FY$ for any $Y \in TM$ and using the fact $C\nabla_X H \in \mu$, the above equation gives

$$g\left(\nabla_X^\perp \phi H, FY\right) = -g(FA_{\phi}X, FY). \quad (3.16)$$
Using (2.18), we obtain

\[ g\left(\nabla_X FY, \phi H\right) = \sin^2 \theta \left[ g(A_H X, Y) + \eta(A_H X) \eta(Y) \right], \quad (3.17) \]

then, from (2.5) and (2.13), we get

\[ g\left(\nabla_X FY, \phi H\right) = \sin^2 \theta \left[ g(X, Y) + \eta(X) \eta(Y) \right] \|H\|^2. \quad (3.18) \]

Thus, from (3.12) and (3.18), we derive

\[ \left[ \cos^2 \theta g(X, Y) - \sin^2 \theta \eta(X) \eta(Y) \right] \|H\|^2 = 0. \quad (3.19) \]

Hence, (3.19) gives either \( H = 0 \) or if \( H \neq 0 \), then the slant angle of \( M \) is \( \theta = \tan^{-1}(\sqrt{g(X, Y)/\eta(X)\eta(Y)}) \). This proves the theorem completely. \( \square \)

### 4. Hemislant Submanifolds

In the following section, we assume that \( M \) is a hemi-slant submanifold of an LP-cosymplectic manifold \( \overline{M} \) such that the structure vector field \( \xi \) tangent to \( M \). First, we define a hemi-slant submanifold, and then we obtain the integrability conditions of the involved distributions \( D_1 \) and \( D_2 \) in the definition of a hemi-slant submanifold \( M \) of an LP-cosymplectic manifold \( \overline{M} \).

**Definition 4.1.** A submanifold \( M \) of an LP-contact manifold \( \overline{M} \) is said to be a hemi-slant submanifold if there exist two orthogonal complementary distributions \( D_1 \) and \( D_2 \) satisfying

(i) \( TM = D_1 \oplus D_2 \oplus \langle \xi \rangle \),
(ii) \( D_1 \) is a slant distribution with slant angle \( \theta \neq \pi/2 \),
(iii) \( D_2 \) is totally real that is, \( \phi D_2 \subseteq T^\perp M \).

If \( \mu \) is \( \phi \)-invariant subspace of the normal bundle \( T^\perp M \), then in case of hemi-slant submanifold, the normal bundle \( T^\perp M \) can be decomposed as

\[ T^\perp M = F D_1 \oplus F D_2 \oplus \mu. \quad (4.1) \]

In the following, we obtain the integrability conditions of involved distributions in the definition of hemi-slant submanifold.

**Proposition 4.2.** Let \( M \) be a hemi-slant submanifold of an LP-cosymplectic manifold \( \overline{M} \), then the anti-invariant distribution \( D_2 \) is integrable if and only if

\[ A_{FZ}W = A_{FW}Z, \quad (4.2) \]

for any \( Z, W \in D_2 \).
Proof. For any \( Z, W \in D_2 \), one has
\[
\phi[Z, W] = \phi\nabla_Z W - \phi\nabla_W Z. \tag{4.3}
\]
Using (2.8), we obtain
\[
\phi[Z, W] = \nabla_Z \phi W - \nabla_W \phi Z. \tag{4.4}
\]
Then, from (2.4), we derive
\[
\phi[Z, W] = -A_{FW} Z + \nabla^\perp_Z FW + A_{FZ} W - \nabla^\perp_W FZ. \tag{4.5}
\]
As \( D_2 \) is an anti-invariant distribution, then the tangential part of (4.5) should be identically zero; hence, we obtain the required result. \( \square \)

**Proposition 4.3.** Let \( M \) be a hemi-slant submanifold of an LP-cosymplectic manifold \( \overline{M} \), then the invariant distribution \( D_1 \oplus \langle \xi \rangle \) is integrable if and only if
\[
g\left( h(X, PY) - h(Y, PX) + \nabla^\perp_X FY - \nabla^\perp_Y FX, FZ \right) = 0, \tag{4.6}
\]
for any \( X, Y \in D_1 \oplus \langle \xi \rangle \) and \( Z \in D_2 \).

Proof. For any \( X, Y \in D_1 \oplus \langle \xi \rangle \), one has
\[
\phi[X, Y] = \phi\nabla_X Y - \phi\nabla_Y X. \tag{4.7}
\]
Then, from (2.8) and the fact that \( \overline{M} \) is LP-cosymplectic, we obtain
\[
\phi[X, Y] = \nabla_X \phi Y - \nabla_Y \phi X. \tag{4.8}
\]
Using (2.6), we get
\[
\phi[X, Y] = \nabla_X PY + \nabla_X FY - \nabla_Y PX - \nabla_Y FX. \tag{4.9}
\]
Thus, from (2.3) and (2.4), we derive
\[
\phi[X, Y] = \nabla_X PY + h(X, PY) - A_{FY} X + \nabla^\perp_X FY - \nabla_Y PX - h(Y, PX) + A_{FX} Y - \nabla^\perp_Y FX. \tag{4.10}
\]
Taking the product in (4.10) with \( FZ \), for any \( Z \in D_2 \), we obtain
\[
g(\phi[X, Y], FZ) = g\left( h(X, PY) + \nabla^\perp_X FY - h(Y, PX) - \nabla^\perp_Y FX, FZ \right). \tag{4.11}
\]
Thus, the assertion follows from (4.11) after using (2.2) and the fact that $\xi$ is tangential to $D_1$. 

Now, we consider $M$ as a totally umbilical hemi-slant submanifold of an LP-cosymplectic manifold $\overline{M}$. For any $X, Y \in TM$, one has

$$\overline{\nabla}_X \phi Y = \phi \overline{\nabla}_X Y.$$  \hfill (4.12)

Using this fact, if we take for any $Z, W \in D_2$, then from (2.3) and (2.4), the above equation takes the form

$$-A_{FW}Z + \nabla^2_Z FW = \phi(\nabla_Z W + h(Z, W)).$$ \hfill (4.13)

Thus, on using (2.6) and (2.7), we obtain

$$-A_{FW}Z + \nabla^2_Z FW = P\nabla_Z W + F\nabla_Z W + Bh(Z, W) + Ch(Z, W).$$ \hfill (4.14)

Equating the tangential components, we get

$$P\nabla_Z W = -A_{FW}Z - Bh(Z, W).$$ \hfill (4.15)

Taking the product with $V \in D_2$, we obtain

$$g(P\nabla_Z W, V) = -g(A_{FW}Z, V) - g(Bh(Z, W), V).$$ \hfill (4.16)

Using (2.2), (2.5), and the fact that $PW = 0$, for any $W \in D_2$, thus, the above equation takes the form

$$0 = g(h(Z, V), FW) + g(Bh(Z, W), V).$$ \hfill (4.17)

As $M$ is totally umbilical, we derive

$$0 = g(Z, V)g(H, FW) + g(Z, W)g(BH, V).$$ \hfill (4.18)

Thus, (4.18) has a solution if either $Z = W = V = \xi$, that is, $\dim D_2 = 1$ or $H \in \mu$ or $D_2 = \{0\}$. Hence, we state the following theorem.

**Theorem 4.4.** Let $M$ be a totally umbilical hemi-slant submanifold of an LP-cosymplectic manifold $\overline{M}$, then at least one of the following statements is true:

(i) the dimension of anti-invariant distribution is one, that is, $\dim D_2 = 1$,

(ii) the mean curvature vector $H \in \mu$,

(iii) $M$ is proper slant submanifold of $\overline{M}$. 

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References
