Research Article
On the General Solution of the Ultrahyperbolic Bessel Operator

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Abstract

We study the general solution of equation $\square^k_{B,c} u(x) = f(x)$, where $\square^k_{B,c}$ is the ultrahyperbolic Bessel operator iterated $k$-times and is defined by

$$\square^k_{B,c} = \left( \sum_{\beta} \frac{\partial^{2\beta_i}}{\partial x_i^{2\beta_i}} \right)^k, \quad 1 \leq \beta \leq q, \quad p + q = n,$$

where $p + q = n$, $n$ is the dimension of $\mathbb{R}^n$, and $k$ is a nonnegative integer. $f(x)$ is a given generalized function, $u(x)$ is an unknown generalized function, $k$ is a nonnegative integer, and $x \in \mathbb{R}^n$.

1. Introduction

The $n$-dimensional ultrahyperbolic operator $\square^k_{B,c}$ iterated $k$-times is defined by

$$\square^k_{B,c} = \left( \sum_{\beta} \frac{\partial^{2\beta_i}}{\partial x_i^{2\beta_i}} \right)^k,$$

where $p + q = n$, $n$ is the dimension of space $\mathbb{R}^n$, and $k$ is a nonnegative integer.

Consider the linear differential equation of the form

$$\square^k_{B,c} u(x) = f(x),$$

where $u(x)$ and $f(x)$ are generalized functions and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$. 

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Gel’fand and Shilov [1] first introduced the fundamental solution of (1.2), which is a complicated form. Later, Trione [2] has shown that the generalized function $R_{2k}(x)$, defined by (2.8) with $|\sigma| = 0$, is a unique fundamental solution of (1.2) and Téllez [3] also proved that $R_{2k}(x)$ exists only in the case when $p$ is odd with $n$ odd or even and $p+q = n$. A wealth of some effective works on the fundamental solution of the $n$-dimensional classical ultrahyperbolic operator have been presented by Kananthai and Sritanratana [4–9].

In 2004, Yıldırım et al. [10] have introduced the Bessel ultrahyperbolic operator iterated $k$-times with $x \in \mathbb{R}^+_n = \{ x : x = (x_1,x_2,\ldots,x_n), x_1 > 0, \ldots, x_n > 0 \}$,

$$
\Box^k_{B,c} = \left( B_{x_1} + B_{x_2} + \cdots + B_{x_p} - B_{x_{p+1}} - \cdots - B_{x_{p+q}} \right)^k,
$$

(1.3)

where $p + q = n$, $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + (2\nu_i/x_i)(\partial/\partial x_i)$, $2\nu_i = 2\beta_i + 1$, $\beta_i > -1/2$ [11], $k$ is a nonnegative integer, and $n$ is the dimension of $\mathbb{R}^+_n$. They also have studied the fundamental solution of Bessel ultrahyperbolic operator.

In 2007, Sarıkaya and Yıldırım [12] have studied the weak solution of the compound Bessel ultrahyperbolic equation and also studied the Bessel ultrahyperbolic heat equation [13].

In 2009, Saglam et al. [14] have developed the operator of (1.3), defined by (1.6), and it is called the ultrahyperbolic Bessel operator iterated $k$-times. They have also studied the product of the ultrahyperbolic Bessel operator related to elastic waves.

Next, Srisombat and Nonlaopon [15] have studied the weak solution of

$$
\Box^k_{B,c} u(x) = f(x),
$$

(1.4)

where $u(x)$ and $f(x)$ are some generalized functions. They have developed (1.4) into the form

$$
\sum_{k=0}^{m} C_k \Box^k_{B,c} u(x) = f(x),
$$

(1.5)

which is called the compound ultrahyperbolic Bessel equation. In finding the solution of (1.5), they have used the properties of $B$-convolution for the generalized functions.

The purpose of this study is to find the general solution of equation $\Box^k_{B,c} u(x) = f(x)$, where $\Box^k_{B,c}$ is the ultrahyperbolic Bessel operator iterated $k$-times and is defined by

$$
\Box^k_{B,c} = \left[ \frac{1}{c^2} \left( B_{x_1} + B_{x_2} + \cdots + B_{x_p} \right) - \left( B_{x_{p+1}} + \cdots + B_{x_{p+q}} \right) \right]^k
$$

(1.6)

$p + q = n$, $n$ is the dimension of $\mathbb{R}^+_n = \{ x : x = (x_1,x_2,\ldots,x_n), x_1 > 0, \ldots, x_n > 0 \}$, $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + (2\nu_i/x_i)(\partial/\partial x_i)$, $2\nu_i = 2\beta_i + 1$, $\beta_i > -1/2$, $x_i > 0$ ($i = 1,2,\ldots,n$), $f(x)$ is a given generalized function, $u(x)$ is an unknown generalized function, $k$ is a nonnegative integer, $c$ is a positive constant, and $x \in \mathbb{R}^+_n$.
2. Preliminaries

Let \( T^y_x \) be the generalized shift operator acting on the function \( \varphi \), according to the law [11, 16]:

\[
T^y_x \varphi(x) = C_v \int_0^\pi \cdots \int_0^\pi \varphi \left( \sqrt{x^2 + y^2} - 2x_1y_1 \cos \vartheta_1, \ldots, \sqrt{x^2_n + y^2_n} - 2x_ny_n \cos \vartheta_n \right)
\times \left( \prod_{i=1}^n \sin^{2v_i-1} \vartheta_i \right) d\vartheta_1 \cdots d\vartheta_n, 
\]

where \( x, y \in \mathbb{R}_n^+ \) and \( C_v = \prod_{i=1}^n (\Gamma(v_i + 1)/\Gamma(1/2)\Gamma(v_i)) \). We remark that this shift operator is closely connected to the Bessel differential operator [11]:

\[
\frac{d^2 U}{dx^2} + \frac{2v}{x} \frac{dU}{dx} = \frac{d^2 U}{dy^2} + \frac{2v}{y} \frac{dU}{dy},
\]

\( U(x, 0) = f(x), \)

\( U_y(x, 0) = 0. \)

The convolution operator is determined by the \( T^y_x \) as follows:

\[
(f \ast \varphi)(y) = \int_{\mathbb{R}_n^+} f(y) T^y_x \varphi(x) \left( \prod_{i=1}^n y_i^{2v_i} \right) dy.
\]

The convolution (2.3) is known as a \( B \)-convolution. We note the following properties of the \( B \)-convolution and the generalized shift operator.

(a) \( T^y_x \cdot 1 = 1 \).

(b) \( T^y_x \cdot f(x) = f(x) \).

(c) If \( f(x), g(x) \in C(\mathbb{R}_n^+) \), \( g(x) \) is a bounded function all \( x > 0 \), and

\[
\int_{\mathbb{R}_n^+} |f(x)| \left( \prod_{i=1}^n x_i^{2v_i} \right) dx < \infty,
\]

then

\[
\int_{\mathbb{R}_n^+} T^y_x f(x) g(y) \left( \prod_{i=1}^n y_i^{2v_i} \right) dy = \int_{\mathbb{R}_n^+} f(y) T^y_x g(x) \left( \prod_{i=1}^n y_i^{2v_i} \right) dy.
\]

(d) From (c), we have the following equality for \( g(x) = 1 \):

\[
\int_{\mathbb{R}_n^+} T^y_x f(x) \left( \prod_{i=1}^n y_i^{2v_i} \right) dy = \int_{\mathbb{R}_n^+} f(y) \left( \prod_{i=1}^n y_i^{2v_i} \right) dy.
\]

(e) \((f \ast g)(x) = (g \ast f)(x)\).
Definition 2.1. Let \( x = (x_1, x_2, \ldots, x_n) \) be a point of the \( n \)-dimensional space \( \mathbb{R}^n_+ \). Denote the nondegenerated quadratic form by

\[
V = c^2 \left( x_1^2 + x_2^2 + \cdots + x_p^2 \right) - x_{p+1}^2 - x_{p+2}^2 - \cdots - x_{p+q}^2,
\]

where \( p + q = n \). The interior of the forward cone is defined by \( \Gamma_+ = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n_+ : x_i > 0, \ i = 1, \ldots, n \} \), where \( \overline{\Gamma}_+ \) designates its closure. For any complex number \( \alpha \), we define

\[
R_{\alpha,c}^H(x) = \begin{cases} 
\frac{\sqrt{v(n-2|\alpha|)/2}}{\Gamma((2+\alpha-n-2|\alpha|)/2)\Gamma((1-\alpha)/2)\Gamma(\alpha)} V^{(\alpha-n-2|\alpha|)/2} K_n(\alpha), & \text{for } x \in \Gamma_+, \\
0, & \text{for } x \notin \Gamma_+,
\end{cases}
\]

where

\[
K_n(\alpha) = \frac{\pi^{(n+2|\alpha|-1)/2}}{\Gamma((2+\alpha-n-2|\alpha|)/2)\Gamma((1-\alpha)/2)\Gamma(\alpha)}.
\]

The function \( R_{\alpha,c}^H(x) \) is introduced by [10, 12, 17, 18]. It is well known that \( R_{\alpha,c}^H(x) \) is an ordinary function if \( \text{Re}(\alpha) \geq n \) and is the distribution of \( \alpha \) if \( \text{Re}(\alpha) < n \). Let \( \text{supp} \ R_{\alpha,c}^H(x) \subset \overline{\Gamma}_+ \), where \( \text{supp} \ R_{\alpha,c}^H(x) \) denotes the support of \( R_{\alpha,c}^H(x) \).

By putting \( p = c = 1 \) into (2.7), (2.8), and (2.9), and using the Legendre’s duplication of \( \Gamma(z) \),

\[
\Gamma(2z) = 2^{2z-1}\pi^{-1/2}\Gamma(z)\Gamma\left(z + \frac{1}{2}\right),
\]

the formula (2.8) is reduced to

\[
M_{\alpha}^H(x) = \begin{cases} 
\frac{\sqrt{v(n-2|\alpha|)/2}}{\Gamma((2+\alpha-n-2|\alpha|)/2)\Gamma((1-\alpha)/2)\Gamma(\alpha)} V^{(\alpha-n-2|\alpha|)/2} H_n(\alpha), & \text{for } x \in \Gamma_+, \\
0, & \text{for } x \notin \Gamma_+,
\end{cases}
\]

where \( V = x_1^2 - x_2^2 - \cdots - x_n^2 \) and

\[
H_n(\alpha) = \pi^{(n+2|\alpha|-1)/2} 2^{n-1} \Gamma\left(\frac{2+\alpha-n-2|\alpha|}{2}\right) \Gamma\left(\frac{\alpha}{2}\right).
\]

Note that the function \( M_{\alpha}^H(x) \) is precisely the Bessel hyperbolic kernel of Marcel Riesz.

Lemma 2.2. Given the equation

\[
\Box^k_{B,\alpha} u(x) = \delta(x),
\]

where \( \Box^k_{B,\alpha} \) is the Bessel hyperbolic operator of order \( k \) and \( \delta(x) \) is the Dirac delta function.
Lemma 2.3. \( \square_{B,c}^k \) is defined by (1.6) and \( x \in \mathbb{R}_+^n \), then we obtain \( u(x) = R^H_{2k,c}(x) \) as a fundamental solution of (2.13), where \( R^H_{2k,c}(x) \) is defined by (2.8).

The proof of this Lemma is given in [14].

Lemma 2.4. \( \square_{B,c}^k \) is defined by (1.6) and \( x \in \mathbb{R}_+^n \), then we obtain \( u(x) = R^H_{2k,c}(x) \) as a fundamental solution of (2.13), where \( R^H_{2k,c}(x) \) is defined by (2.8).

The proof of this Lemma is given in [15].

Lemma 2.5. Given the equation

\[
\square_{B,c}^k u(x) = 0,
\]

where \( \square_{B,c}^k \) is the ultrahyperbolic Bessel operator iterated \( k \)-times, as defined by (1.6), and \( x \in \mathbb{R}_+^n \), then

\[
u(x) = \left[ R^H_{2(k-1),c}(x) \right]^{(m)},
\]

defined by (2.8) with \( m \) derivatives, as a solution of (2.15) with \( m = \frac{(n+2|\nu| - 4)}{2} \), \( n + 2|\nu| \geq 4 \) and \( n \) is an even dimension.

Proof. We first show that the generalized function \( \delta^{(m)}(c^2 r^2 - s^2) \), where \( r^2 = x_1^2 + x_2^2 + \cdots + x_p^2, \ s^2 = x_{p+1}^2 + x_{p+2}^2 + \cdots + x_{p+q}^2, \ p + q = n, \) is a solution of

\[
\square_{B,c} u(x) = 0,
\]

and \( \square_{B,c} \) is defined by (1.6) with \( k = 1 \) and \( x \in \mathbb{R}_+^n \). Now for \( 1 \leq i \leq p \), we have

\[
\frac{\partial}{\partial x_i} \delta^{(m)}(c^2 r^2 - s^2) = 2c^2 x_i \delta^{(m+1)}(c^2 r^2 - s^2),
\]

\[
\frac{\partial^2}{\partial x_i^2} \delta^{(m)}(c^2 r^2 - s^2) = 2c^2 \delta^{(m+1)}(c^2 r^2 - s^2) + 4c^4 x_i^2 \delta^{(m+2)}(c^2 r^2 - s^2).
\]
Thus, we have

\[
\frac{1}{c^2} \sum_{i=1}^{p} \left[ \frac{\partial^2}{\partial x_i^2} \delta^{(m)}(c^2 r^2 - s^2) + \frac{2 v_i}{x_i} \frac{\partial}{\partial x_i} \delta^{(m)}(c^2 r^2 - s^2) \right] \\
= 2p \delta^{(m+1)}(c^2 r^2 - s^2) + 4c^2 r^2 \delta^{(m+2)}(c^2 r^2 - s^2) + 4 |v'| \delta^{(m+1)}(c^2 r^2 - s^2) \\
= (2p + 4 |v'|) \delta^{(m+1)}(c^2 r^2 - s^2) + 4 \left( c^2 r^2 - s^2 \right) \delta^{(m+2)}(c^2 r^2 - s^2) + 4s^2 \delta^{(m+2)}(c^2 r^2 - s^2) \\
= \left[ 2p + 4 |v'| - 4(m + 2) \right] \delta^{(m+1)}(c^2 r^2 - s^2) + 4s^2 \delta^{(m+2)}(c^2 r^2 - s^2)
\]

(2.19)

by applying Lemma 2.4 with \( P = c^2 r^2 - s^2 \), where \( |v'| = v_1 + v_2 + \cdots + v_p \).

Similarly, we have

\[
\sum_{i=p+1}^{p+q} \left[ \frac{\partial^2}{\partial x_i^2} \delta^{(m)}(c^2 r^2 - s^2) + \frac{2 v_i}{x_i} \frac{\partial}{\partial x_i} \delta^{(m)}(c^2 r^2 - s^2) \right] \\
= \left[ -(2q + 4 |v''|) + 4(m + 2) \right] \delta^{(m+1)}(c^2 r^2 - s^2) + 4c^2 r^2 \delta^{(m+2)}(c^2 r^2 - s^2)
\]

(2.20)

by applying Lemma 2.4 with \( P = c^2 r^2 - s^2 \), where \( |v''| = v_{p+1} + v_{p+2} + \cdots + v_{p+q} \).

Thus, we have

\[
\square_{B_c} \delta^{(m)}(c^2 r^2 - s^2) = \frac{1}{c^2} \sum_{i=1}^{p} \left[ \frac{\partial^2}{\partial x_i^2} + \frac{2 v_i}{x_i} \frac{\partial}{\partial x_i} \right] \delta^{(m)}(c^2 r^2 - s^2) \\
- \sum_{i=p+1}^{p+q} \left[ \frac{\partial^2}{\partial x_i^2} + \frac{2 v_i}{x_i} \frac{\partial}{\partial x_i} \right] \delta^{(m)}(c^2 r^2 - s^2) \\
= \left[ 2(p + q + 2 |v|) - 8(m + 2) \right] \delta^{(m+1)}(c^2 r^2 - s^2) \\
- 4 \left( c^2 r^2 - s^2 \right) \delta^{(m+2)}(c^2 r^2 - s^2) \\
= \left[ 2(n + 2 |v|) - 8(m + 2) \right] \delta^{(m+1)}(c^2 r^2 - s^2) + 4(m + 2) \delta^{(m+1)}(c^2 r^2 - s^2) \\
= \left[ 2(n + 2 |v|) - 4(m + 2) \right] \delta^{(m+1)}(c^2 r^2 - s^2)
\]

(2.21)

by applying Lemma 2.4 with \( P = c^2 r^2 - s^2 \), where \( |v| = |v'| + |v''| \).

If \( [2(n + 2 |v|) - 4(m + 2)] = 0 \), we obtain

\[
\square_{B_c} \delta^{(m)}(c^2 r^2 - s^2) = 0.
\]

(2.22)
That is, \( u(x) = \delta^{(m)}(c^2x^2 - s^2) \) is a solution of (2.15) with \( m = (n + 2|v| - 4)/2, \ n + 2|v| \geq 4, \) and \( n \) is an even dimension. Now \( \Box_{B,c}^k u(x) = 0 \) can be written in the form
\[
\Box_{B,c}^k \left( \Box_{B,c}^{k-1} u(x) \right) = 0. 
\] (2.23)

From (2.17), we have
\[
\Box_{B,c}^{k-1} u(x) = \delta^{(m)} \left( c^2x^2 - s^2 \right) 
\] (2.24)
with \( m = (n + 2|v| - 4)/2, \ n + 2|v| \geq 4, \) and \( n \) being an even dimension. By Lemma 2.3(a), we can write (2.24) in the form
\[
\Box_{B,c}^{k-1} \delta \ast u(x) = \delta^{(m)} \left( c^2x^2 - s^2 \right). 
\] (2.25)

\( B \)-convolving both sides of the above equation with the function \( R_{2(k-1),c}^H(x) \), we obtain
\[
R_{2(k-1),c}^H(x) \ast \Box_{B,c}^{k-1} \delta \ast u(x) = R_{2(k-1),c}^H(x) \ast \delta^{(m)} \left( c^2x^2 - s^2 \right), 
\]
\[
\Box_{B,c}^{k-1} \left[ R_{2(k-1),c}^H(x) \right] \ast u(x) = \left[ R_{2(k-1),c}^H(x) \right]^{(m)}, 
\] (2.26)

by Lemma 2.2.

It follows that \( u(x) = \left[ R_{2(k-1),c}^H(x) \right]^{(m)} \) is a solution of (2.15) with \( m = (n+2|v| - 4)/2, \ n+2|v| \geq 4 \) and \( n \) is an even dimension.

The generalized function \( \delta^{(m)}(c^2x^2 - s^2) \) mentioned in Lemma 2.5 has been also studied on the aspect of multiplicative product, distributional product and applications, for more details, see [19–23].

3. Main Result

Theorem 3.1. Given the equation
\[
\Box_{B,c}^k u(x) = f(x), 
\] (3.1)
where \( \Box_{B,c}^k \) is the ultrahyperbolic Bessel operator iterated \( k \)-times and is defined by (1.6), \( f(x) \) is a generalized function, \( u(x) \) is an unknown generalized function, \( x \in \mathbb{R}_n^+ \), and \( n \) is an even, then (3.1) has the general solution
\[
u(x) = \left[ R_{2(k-1),c}^H(x) \right]^{(m)} + R_{2k,c}^H(x) \ast f(x), 
\] (3.2)

where \( \left[ R_{2k,c}^H(x) \right]^{(m)} \) is a function defined by (2.8) with \( m \) derivatives.
Proof. $B$-convolving both sides of (3.1) with $R_{2k,c}^H(x)$, we obtain

$$R_{2k,c}^H(x) \ast \left( \Box_{B,c}^k u(x) \right) = R_{2k,c}^H(x) \ast f(x).$$

(3.3)

By Lemma 2.2, we have

$$\Box_{B,c}^k \left( R_{2k,c}^H(x) \right) \ast u(x) = \delta \ast u(x) = R_{2k,c}^H(x) \ast f(x).$$

(3.4)

So, we obtain that

$$u(x) = R_{2k,c}^H(x) \ast f(x)$$

(3.5)

is the solution of (3.1).

For a homogeneous equation $\Box_{B,c}^k u(x) = 0$, we have a solution

$$u(x) = \left[ R_{2(k-1),c}^H(x) \right]^{(m)}$$

(3.6)

by Lemma 2.5. Thus the general solution of (3.1) is

$$u(x) = \left[ R_{2(k-1),c}^H(x) \right]^{(m)} + R_{2k,c}^H(x) \ast f(x).$$

(3.7)

This completes the proof.

By putting $c = 1$, (3.1) becomes the Bessel ultrahyperbolic equation

$$\Box_{B}^k w(x) = f(x),$$

(3.8)

where $\Box_{B}^k$ is the Bessel ultrahyperbolic operator iterated $k$-times, and is defined by (1.3), $f(x)$ is a generalized function and $w(x)$ is an unknown generalized function. From (3.5) we have that

$$w(x) = R_{2k}^H(x) \ast f(x)$$

(3.9)

is a solution of (3.8), where $R_{2k}^H(x) = R_{2k,1}^H(x)$ defined by (2.8).

From (3.2), we obtain that the general solution of the Bessel ultrahyperbolic equation is

$$w(x) = \left[ R_{2(k-1),1}^H(x) \right]^{(m)} + R_{2k}^H(x) \ast f(x).$$

(3.10)
Moreover, if we put \( k = 1, \ p = 1 \) and \( x_1 = t(t) \) times), then (3.8) is reduced to the Bessel wave equation

\[
\Box_B w(x) = \left( B_t - \sum_{i=2}^{n} B_{x_i} \right) w(x) = f(x), \tag{3.11}
\]

where

\[
\Box_B = B_t - \sum_{i=2}^{n} B_{x_i} \tag{3.12}
\]

is the Bessel wave operator and \( B_{x_i} = \partial^2 / \partial x_i^2 + (2v_i / x_i)(\partial / \partial x_i) \).

Thus, we obtain \( w(x) = M_2(x) \ast f(x) \) as a solution of the Bessel wave equation, since \( R^H_2(x) \) becomes \( M^H_2(x) \), where \( M^H_2(x) \) is the Bessel ultrahyperbolic kernel of Marcel Riesz, and is defined by (2.11) with \( \alpha = 2 \). And from (3.2), we obtain the general solution of Bessel wave equation as

\[
w(x) = \delta^{(m)}(x) + M^H_2(x) \ast f(x), \tag{3.13}
\]

where \( \delta^{(m)}(x) \) is a solution of

\[
\left( B_t - \sum_{i=2}^{n} B_{x_i} \right) w(x) = 0. \tag{3.14}
\]

Now we put \( V = t^2 - x_2^2 - x_3^2 - \cdots - x_n^2 \) and \( s^2 = x_2^2 + x_3^2 + \cdots + x_n^2 \). By [24], we obtain that

\[
w(x, t) = \delta^{(m)}(t^2 - s^2) \tag{3.15}
\]

is the solution of (3.14) with the initial conditions \( w(x, 0) = 0 \) and \( \partial w(x, 0) / \partial t = (-1)^m 2^{m+1} \pi^{m+1} \delta(x) \) at \( t = 0 \) and \( x = (x_2, x_3, \ldots, x_n) \in \mathbb{R}^n_{+1} \).

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