Research Article

Existence Results for a Nonlinear Semipositone Telegraph System with Repulsive Weak Singular Forces

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Using the fixed point theorem of cone expansion/compression, we consider the existence results of positive solutions for a nonlinear semipositone telegraph system with repulsive weak singular forces.

1. Introduction

In this paper, we are concerned with the existence of positive solutions for the nonlinear telegraph system:

$$\begin{aligned} u_{tt} - u_{xx} + c_1 u_t + a_1(t, x)u &= f(t, x, v), \\ v_{tt} - v_{xx} + c_2 v_t + a_2(t, x)v &= g(t, x, u), \end{aligned} \quad (1.1)$$

with doubly periodic boundary conditions

$$\begin{aligned} u(t + 2\pi, x) &= u(t, x + 2\pi) = u(t, x), \quad (t, x) \in \mathbb{R}^2, \\ v(t + 2\pi, x) &= v(t, x + 2\pi) = v(t, x), \quad (t, x) \in \mathbb{R}^2. \end{aligned} \quad (1.2)$$

In particular, the function $f(t, x, v)$ may be singular at $v = 0$ or superlinear at $v = +\infty$, and $g(t, x, u)$ may be singular at $u = 0$ or superlinear at $u = +\infty$.

In the latter years, the periodic problem for the semilinear singular equation

$$x'' + a(t)x = \frac{b(t)}{x^\lambda} + c(t), \quad (1.3)$$

with $a, b, c \in L^1[0, T]$ and $\lambda > 0$, has received the attention of many specialists in differential equations. The main methods to study (1.3) are the following three common techniques:

- (i) the obtainment of a priori bounds for the possible solutions and then the applications of topological degree arguments;
- (ii) the theory of upper and lower solutions;
- (iii) some fixed point theorems in a cone.

We refer the readers to see [1–7] and the references therein.

Equation (1.3) is related to the stationary version of the telegraph equation

$$u_{tt} - u_{xx} + cu_t + \lambda u = f(t, x, u), \quad (1.4)$$

where $c > 0$ is a constant and $\lambda \in \mathbb{R}$. Because of its important physical background, the existence of periodic solutions for a single telegraph equation or telegraph system has been studied by many authors; see [8–16]. Recently, Wang utilize a weak force condition to enable the achievement of new existence criteria for positive doubly periodic solutions of nonlinear telegraph system through a basic application of Schauder's fixed point theorem in [17]. Inspired by these papers, here our interest is in studying the existence of positive doubly periodic solutions for a semipositone nonlinear telegraph system with repulsive weak singular forces by using the fixed point theorem of cone expansion/compression.

Lemma 1.1 (see [18]). *Let E be a Banach space, and let $K \subset E$ be a cone in E . Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$, and let $T : K \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow K$ be a completely continuous operator such that either*

- (i) $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_2$; or
- (ii) $\|Tu\| \geq \|u\|$, $u \in K \cap \partial\Omega_1$ and $\|Tu\| \leq \|u\|$, $u \in K \cap \partial\Omega_2$.

Then, T has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

This paper is organized as follows: in Section 2, some preliminaries are given; in Section 3, we give the main results.

2. Preliminaries

Let $\mathbb{T}^2$ be the torus defined as

$$\mathbb{T}^2 = \left(\frac{\mathbb{R}}{2\pi\mathbb{Z}} \right) \times \left(\frac{\mathbb{R}}{2\pi\mathbb{Z}} \right). \quad (2.1)$$

Doubly 2π -periodic functions will be identified to be functions defined on $\mathbb{T}^2$. We use

the notations

$$L^p(\mathbb{T}^2), C(\mathbb{T}^2), C^\alpha(\mathbb{T}^2), D(\mathbb{T}^2) = C^\infty(\mathbb{T}^2), \dots \quad (2.2)$$

to denote the spaces of doubly periodic functions with the indicated degree of regularity. The space $D'(\mathbb{T}^2)$ denotes the space of distributions on $\mathbb{T}^2$.

By a doubly periodic solution of (1.1)-(1.2) we mean that a $(u, v) \in L^1(\mathbb{T}^2) \times L^1(\mathbb{T}^2)$ satisfies (1.1)-(1.2) in the distribution sense; that is,

$$\begin{aligned} \int_{\mathbb{T}^2} u(\varphi_{tt} - \varphi_{xx} - c_1 \varphi_t + a_1(t, x) \varphi) dt dx &= \int_{\mathbb{T}^2} f(t, x, v) \varphi dt dx, \\ \int_{\mathbb{T}^2} v(\varphi_{tt} - \varphi_{xx} - c_2 \varphi_t + a_2(t, x) \varphi) dt dx &= \int_{\mathbb{T}^2} g(t, x, u) \varphi dt dx, \end{aligned} \quad \forall \varphi \in D(\mathbb{T}^2). \quad (2.3)$$

First, we consider the linear equation

$$u_{tt} - u_{xx} + c_i u_t - \lambda_i u = h_i(t, x), \quad \text{in } D'(\mathbb{T}^2), \quad (2.4)$$

where $c_i > 0$, $\lambda_i \in \mathbb{R}$, and $h_i(t, x) \in L^1(\mathbb{T}^2)$, ($i = 1, 2$).

Let $\mathcal{E}_{\lambda_i}$ be the differential operator

$$\mathcal{E}_{\lambda_i} = u_{tt} - u_{xx} + c_i u_t - \lambda_i u, \quad (2.5)$$

acting on functions on $\mathbb{T}^2$. Following the discussion in [14], we know that if $\lambda_i < 0$, then $\mathcal{E}_{\lambda_i}$ has the resolvent R_{λ_i} :

$$R_{\lambda_i} : L^1(\mathbb{T}^2) \longrightarrow C(\mathbb{T}^2), \quad h_i \longmapsto u_i, \quad (2.6)$$

where u_i is the unique solution of (2.4), and the restriction of R_{λ_i} on $L^p(\mathbb{T}^2)$ ($1 < p < \infty$) or $C(\mathbb{T}^2)$ is compact. In particular, $R_{\lambda_i} : C(\mathbb{T}^2) \rightarrow C(\mathbb{T}^2)$ is a completely continuous operator.

For $\lambda_i = -c_i^2/4$, the Green function $G_i(t, x)$ of the differential operator $\mathcal{E}_{\lambda_i}$ is explicitly expressed; see lemma 5.2 in [14]. From the definition of $G_i(t, x)$, we have

$$\begin{aligned} \underline{G}_i &:= \text{ess inf } G_i(t, x) = \frac{e^{-3c_i\pi/2}}{(1 - e^{-c_i\pi})^2}, \\ \overline{G}_i &:= \text{ess sup } G_i(t, x) = \frac{(1 + e^{-c_i\pi})}{2(1 - e^{-c_i\pi})^2}. \end{aligned} \quad (2.7)$$

Let E denote the Banach space $C(\mathbb{T}^2)$ with the norm $\|u\| = \max_{(t,x) \in \mathbb{T}^2} |u(t, x)|$, then E is an ordered Banach space with cone

$$K_0 = \{u \in E \mid u(t, x) \geq 0, \forall (t, x) \in \mathbb{T}^2\}. \quad (2.8)$$

For convenience, we assume that the following condition holds throughout this paper:

(H1) $a_i(t, x) \in C(\mathbb{T}^2, \mathbb{R}^+)$, $0 < a_i(t, x) \leq c_i^2/4$ for $(t, x) \in \mathbb{T}^2$, and $\int_{\mathbb{T}^2} a_i(t, x) dt dx > 0$.

Next, we consider (2.4) when $-\lambda_i$ is replaced by $a_i(t, x)$. In [10], Li has proved the following unique existence and positive estimate result.

Lemma 2.1. *Let $h_i(t, x) \in L^1(\mathbb{T}^2)$; E is the Banach space $C(\mathbb{T}^2)$. Then; (2.4) has a unique solution $u_i = P_i h_i$; $P_i : L^1(\mathbb{T}^2) \rightarrow C(\mathbb{T}^2)$ is a linear bounded operator with the following properties;*

- (i) $P_i : C(\mathbb{T}^2) \rightarrow C(\mathbb{T}^2)$ is a completely continuous operator;
- (ii) if $h_i(t, x) > 0$, then a.e. $(t, x) \in \mathbb{T}^2$, $P_i[h_i(t, x)]$ has the positive estimate

$$\underline{G}_i \|h_i\|_{L^1} \leq P_i[h_i(t, x)] \leq \frac{\overline{G}_i}{\underline{G}_i \|a_i\|_{L^1}} \|h_i\|_{L^1}. \quad (2.9)$$

3. Main Result

In this section, we establish the existence of positive solutions for the telegraph system

$$\begin{aligned} v_{tt} - v_{xx} + c_1 v_t + a_1(t, x)v &= f(t, x, u), \\ v_{tt} - v_{xx} + c_2 v_t + a_2(t, x)v &= g(t, x, u). \end{aligned} \quad (3.1)$$

where $a_i \in C(R^2, R^+)$ and $f(t, x, v)$ may be singular at $v = 0$. In particular, $f(t, x, v)$ may be negative or superlinear at $v = +\infty$. $g(t, x, u)$ has the similar assumptions. Our interest is in working out what weak force conditions of $f(t, x, v)$ at $v = 0$, $g(t, x, u)$ at $u = 0$ and what superlinear growth conditions of $f(t, x, v)$ at $v = +\infty$, $g(t, x, u)$ at $u = +\infty$ are needed to obtain the existence of positive solutions for problem (1.1)-(1.2).

We assume the following conditions throughout.

(H2) $f, g : \mathbb{T}^2 \times (0, \infty) \rightarrow R$ is continuous, and there exists a constant $M > 0$ such that

$$f_1(t, x, u) + M \geq 0, \quad f_2(t, x, u) + M \geq 0, \quad \forall (t, x) \in \mathbb{T}^2 \text{ and } u, v \in (0, \infty). \quad (3.2)$$

(H3) $F(t, x, v) = f(t, x, v) + M \leq j_1(v) + h_1(v)$ for $(t, x, v) \in \mathbb{T}^2 \times (0, \infty)$ with $j_1 > 0$ continuous and nonincreasing on $(0, \infty)$, $h_1 \geq 0$ continuous on $(0, \infty)$ and h_1/j_1 nondecreasing on $(0, \infty)$.

$G(t, x, u) = g(t, x, u) + M \leq j_2(u) + h_2(u)$ for $(t, x, u) \in \mathbb{T}^2 \times (0, \infty)$ with $j_2 > 0$ continuous and nonincreasing on $(0, \infty)$, $h_2 \geq 0$ continuous on $(0, \infty)$ and h_2/j_2 nondecreasing on $(0, \infty)$.

(H4) $F(t, x, v) = f(t, x, v) + M \geq j_3(v) + h_3(v)$ for all $(t, x, v) \in \mathbb{T}^2 \times (0, \infty)$ with $j_3 > 0$ continuous and nonincreasing on $(0, \infty)$, $h_3 \geq 0$ continuous on $(0, \infty)$ with h_3/j_3 nondecreasing on $(0, \infty)$;

$G(t, x, u) = g(t, x, u) + M \geq j_4(u) + h_4(u)$ for all $(t, x, u) \in \mathbb{T}^2 \times (0, \infty)$ with $j_4 > 0$ continuous and nonincreasing on $(0, \infty)$, $h_4 \geq 0$ continuous on $(0, \infty)$ with h_4/j_4 nondecreasing on $(0, \infty)$.

(H5) There exists

$$r > \frac{M \|\omega_1\|}{\delta_1}, \quad (3.3)$$

such that

$$r \geq \frac{4\pi^2 \overline{G_1}}{\underline{G_1} \|a_1\|_{L^1}} I_1 \cdot I_2, \quad (3.4)$$

here

$$\begin{aligned} I_1 &= j_1 \left(\underline{G_2} j_4(r) \left\{ 1 + \frac{h_4(\delta_1 r - M\|\omega_1\|)}{j_4(\delta_1 r - M\|\omega_1\|)} \right\} 4\pi^2 - M\|\omega_2\| \right), \\ I_2 &= 1 + \frac{h_1 \left(\left(4\pi^2 \overline{G_2} / \underline{G_2} \|a_2\|_{L^1} \right) j_2(\delta_1 r - M\|\omega_1\|) \{ 1 + h_2(r) / j_2(r) \} \right)}{j_1 \left(\left(4\pi^2 \overline{G_2} / \underline{G_2} \|a_2\|_{L^1} \right) j_2(\delta_1 r - M\|\omega_1\|) \{ 1 + h_2(r) / j_2(r) \} \right)}, \end{aligned} \quad (3.5)$$

where $\delta_i = (\underline{G_i} \|a_i\|_{L^1} / \overline{G_i}) \in (0, 1)$, and $\omega_i(t, x)$ is the unique solution to problem:

$$\begin{aligned} u_{tt} - u_{xx} + c_i u_t + a_i(t, x) u &= 1, \\ u(t + 2\pi, x) = u(t, x + 2\pi) &= u(t, x), \end{aligned} \quad (t, x) \in \mathbb{R}^2. \quad (3.6)$$

(H6) There exists $R > r$, such that

$$\begin{aligned} 4\pi^2 \underline{G_1} I_3 \cdot I_4 &\geq R, \\ \delta_2 j_4(R) \left\{ 1 + \frac{h_4(\delta_1 R - M\|\omega_1\|)}{j_4(\delta_1 R - M\|\omega_1\|)} \right\} &> M, \end{aligned} \quad (3.7)$$

where

$$\begin{aligned} I_3 &= \underline{G_1} j_3 \left(\frac{4\pi^2 \overline{G_2}}{\underline{G_2} \|a_2\|_{L^1}} j_2(\delta_1 R - M\|\omega_1\|) \left\{ 1 + \frac{h_2(R)}{j_2(R)} \right\} \right), \\ I_4 &= 1 + \frac{h_3 \left(\underline{G_2} j_4(R) \{ 1 + h_4(\delta_1 R - M\|\omega_1\|) / j_4(\delta_1 R - M\|\omega_1\|) \} 4\pi^2 - M\|\omega_2\| \right)}{j_3 \left(\underline{G_2} j_4(R) \{ 1 + h_4(\delta_1 R - M\|\omega_1\|) / j_4(\delta_1 R - M\|\omega_1\|) \} 4\pi^2 - M\|\omega_2\| \right)}. \end{aligned} \quad (3.8)$$

Theorem 3.1. Assume that (H1)–(H6) hold. Then, the problem (1.1)–(1.2) has a positive doubly periodic solution (u, v) .

Proof. To show that (1.1)–(1.2) has a positive solution, we will proof that

$$\begin{aligned} u_{tt} - u_{xx} + c_1 u_t + a_1(t, x) u &= F(t, x, v - M\omega_2), \\ v_{tt} - v_{xx} + c_2 v_t + a_2(t, x) v &= G(t, x, u - M\omega_1) \end{aligned} \quad (3.9)$$

has a solution $(\tilde{u}, \tilde{v}) = (u + M\omega_1, v + M\omega_2)$ with $\tilde{u} > M\omega_1, \tilde{v} > M\omega_2$ for $(t, x) \in \mathbb{T}^2$. In addition, by Lemma 2.1, it is clear to see that $(u, v) \in C^2(\mathbb{T}^2) \times C^2(\mathbb{T}^2)$ is a solution of (3.9) if and only if $(u, v) \in C(\mathbb{T}^2) \times C(\mathbb{T}^2)$ is a solution of the following system:

$$\begin{aligned} u &= P_1(F(t, x, v - M\omega_2)), \\ v &= P_2(G(t, x, u - M\omega_1)). \end{aligned} \quad (3.10)$$

Evidently, (3.10) can be rewritten as the following equation:

$$u = P_1(F(t, x, P_2(G(t, x, u - M\omega_1)) - M\omega_2)). \quad (3.11)$$

Define a cone $K \subset E$ as

$$K = \{u \in E : u \geq 0, u \geq \delta_1 \|u\|\}. \quad (3.12)$$

We define an operator $T : E \rightarrow K$ by

$$(Tu)(t, x) = P_1(F(t, x, P_2(G(t, x, u - M\omega_1)) - M\omega_2)) \quad (3.13)$$

for $u \in E$ and $(t, x) \in \mathbb{T}^2$. We have the conclusion that $T : E \rightarrow E$ is completely continuous and $T(K) \subseteq K$. The complete continuity is obvious by Lemma 2.1. Now, we show that $T(K) \subseteq K$.

For any $u \in K$, we have

$$Tu = P_1(F(t, x, P_2(G(t, x, u - M\omega_1)) - M\omega_2)). \quad (3.14)$$

From (H1)–(H3) and Lemma 2.1, we have

$$\begin{aligned} Tu &= P_1(F(t, x, P_2(G(t, x, u - M\omega_1)) - M\omega_2)) \\ &\geq \underline{G}_1 \|F(t, x, P_2(G(t, x, u - M\omega_1)) - M\omega_2)\|_{L^1}, \\ \|Tu\| &= \|P(F(t, x, P_2(G(t, x, u - M\omega_1)) - M\omega_2))\| \\ &\leq \frac{\overline{G}_1}{\underline{G}_1 \|a_1\|_{L^1}} \|F(t, x, P_2(G(t, x, u - M\omega_1)) - M\omega_2)\|_{L^1}. \end{aligned} \quad (3.15)$$

So, we get

$$Tu \geq \frac{G_1^2 \|a_1\|_{L^1}}{\overline{G}_1} \|Tu\| \geq \delta_1 \|Tu\|, \quad (3.16)$$

namely, $T(K) \subseteq K$.

Let

$$\Omega_r = \{u \in E : \|u\| < r\}, \quad \Omega_R = \{u \in E : \|u\| < R\}. \quad (3.17)$$

Since $r \leq \|u\| \leq R$ for any $u \in K \cap (\overline{\Omega_R} \setminus \Omega_r)$, we have $0 < \delta_1 r - M\|\omega\| \leq u - M\omega_1 \leq R$.

First, we show

$$\|Tu\| \leq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_r. \quad (3.18)$$

In fact, if $u \in K \cap \partial\Omega_r$, then $\|u\| = r$ and $u \geq \delta_1 r > M\|\omega_1\|$ for $(t, x) \in \mathbb{T}^2$. By (H3) and (H4), we have

$$\begin{aligned} P_2(G(t, x, u - M\omega_1)) &\leq \frac{\overline{G_2}}{\underline{G_2}\|a_2\|_{L^1}} \|G(t, x, u - M\omega_1)\|_{L^1} \\ &\leq \frac{\overline{G_2}}{\underline{G_2}\|a_2\|_{L^1}} \left\| j_2(u - M\omega_1) \left(1 + \frac{h_2(u - M\omega_1)}{j_2(u - M\omega_1)} \right) \right\|_{L^1} \\ &\leq \frac{\overline{G_2}}{\underline{G_2}\|a_2\|_{L^1}} j_2(\delta_1 r - M\|\omega_1\|) \left\{ 1 + \frac{h_2(r)}{j_2(r)} \right\} 4\pi^2, \end{aligned} \quad (3.19)$$

$$\begin{aligned} P_2(G(t, x, u - M\omega_1)) &\geq \underline{G_2} \|G(t, x, u - M\omega_1)\|_{L^1} \\ &\geq \underline{G_2} \left\| j_4(u - M\omega_1) \left(1 + \frac{h_4(u - M\omega_1)}{j_4(u - M\omega_1)} \right) \right\|_{L^1} \\ &\geq \underline{G_2} j_4(r) \left\{ 1 + \frac{h_4(\delta_1 r - M\|\omega_1\|)}{j_4(\delta_1 r - M\|\omega_1\|)} \right\} 4\pi^2. \end{aligned} \quad (3.20)$$

In addition, we also have

$$\begin{aligned} P_2(G(t, x, u - M\omega_1)) &\geq \underline{G_2} j_4(r) \left\{ 1 + \frac{h_4(\delta_1 r - M\|\omega_1\|)}{j_4(\delta_1 r - M\|\omega_1\|)} \right\} 4\pi^2 \\ &\geq \underline{G_2} j_4(R) \left\{ 1 + \frac{h_4(\delta_1 r - M\|\omega_1\|)}{j_4(\delta_1 r - M\|\omega_1\|)} \right\} 4\pi^2 \\ &> \frac{\overline{G_2}}{\underline{G_2}\|a_2\|_{L^1}} M 4\pi^2 \\ &\geq M\omega_2, \end{aligned} \quad (3.21)$$

by (H5), (H6), and (3.20).

So, we have

$$\begin{aligned} Tu &= P_1(F(t, x, v - M\omega_2)) \\ &\leq \frac{\overline{G_1}}{\underline{G_1}\|a_1\|_{L^1}} \|F(t, x, v - M\omega_2)\|_{L^1} \\ &\leq \frac{\overline{G_1}}{\underline{G_1}\|a_1\|_{L^1}} \left\| j_1(v - M\omega_2) \left\{ 1 + \frac{h_1(v - M\omega_2)}{j_1(v - M\omega_2)} \right\} \right\|_{L^1} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\overline{G_1}}{\underline{G_1}\|a_1\|_{L^1}} \|j_1(P_2(G(t, x, u - M\omega_1)) - M\omega_2) \\
&\quad \times \left\{ 1 + \frac{h_1(P_2(G(t, x, u - M\omega_1)) - M\omega_2)}{j_1(P_2(G(t, x, u - M\omega_1)) - M\omega_2)} \right\} \|_{L^1} \\
&\leq \frac{\overline{G_1}}{\underline{G_1}\|a_1\|_{L^1}} j_1 \left(\frac{\overline{G_2}}{\underline{G_2}\|a_2\|_{L^1}} j_2(\delta_1 r - M\|\omega_1\|) \left\{ 1 + \frac{h_4(\delta_1 r - M\|\omega_1\|)}{j_4(\delta_1 r - M\|\omega_1\|)} \right\} 4\pi^2 - M\|\omega_2\| \right) \\
&\quad \times \left\{ 1 + \frac{h_1 \left(\left(\frac{\overline{G_2}}{\underline{G_2}} \|a_2\|_{L^1} \right) j_2(\delta_1 r - M\|\omega_1\|) \{1 + h_2(r)/j_2(r)\} 4\pi^2 \right)}{j_1 \left(\left(\frac{\overline{G_2}}{\underline{G_2}} \|a_2\|_{L^1} \right) j_2(\delta_1 r - M\|\omega_1\|) \{1 + h_2(r)/j_2(r)\} 4\pi^2 \right)} \right\} 4\pi^2 \\
&\leq r = \|u\|
\end{aligned} \tag{3.22}$$

for $(t, x) \in \mathbb{T}^2$, since $\delta_1 r - M\|\omega_1\| \leq u - M\omega_1 \leq r$.

This implies that $\|Tu\| \leq \|u\|$; that is, (3.18) holds.

Next, we show

$$\|Tu\| \geq \|u\|, \quad \text{for } u \in K \cap \partial\Omega_R. \tag{3.23}$$

If $u \in K \cap \partial\Omega_R$, then $\|u\| = R$ and $u \geq \delta R > M\|\omega_1\|$ for $(t, x) \in \mathbb{T}^2$. From (H4) and (H6), we have

$$\begin{aligned}
Tu &= P_1(F(t, x, v - M\omega_1)) \\
&\geq \underline{G_1} \left\| j_3(v - M\omega_2) \left\{ 1 + \frac{h_3(v - M\omega_2)}{j_3(v - M\omega_2)} \right\} \right\|_{L^1} \\
&\geq \underline{G_1} \left\| j_3(P_2(G(t, x, u - M\omega_1)) - M\omega_2) \times \left\{ 1 + \frac{h_3(P_2(G(t, x, u - M\omega_1)) - M\omega_2)}{j_3(P_2(G(t, x, u - M\omega_1)) - M\omega_2)} \right\} \right\|_{L^1} \\
&\geq \underline{G_1} \left\| j_3 \left(\frac{\overline{G_2}}{\underline{G_2}\|a_2\|_{L^1}} j_2(\delta_1 R - M\|\omega_1\|) \left\{ 1 + \frac{h_2(R)}{j_2(R)} \right\} 4\pi^2 \right) \right. \\
&\quad \times \left. \left\{ 1 + \frac{h_3 \left(\frac{\overline{G_2}}{\underline{G_2}\|a_2\|_{L^1}} j_2(\delta_1 R - M\|\omega_1\|) \left\{ 1 + \frac{h_2(R)}{j_2(R)} \right\} 4\pi^2 - M\|\omega_2\| \right)}{j_3 \left(\frac{\overline{G_2}}{\underline{G_2}\|a_2\|_{L^1}} j_2(\delta_1 R - M\|\omega_1\|) \left\{ 1 + \frac{h_2(R)}{j_2(R)} \right\} 4\pi^2 - M\|\omega_2\| \right)} \right\} \right\|_{L^1} \\
&\geq R = \|u\|
\end{aligned} \tag{3.24}$$

for $(t, x) \in \mathbb{T}^2$, since $\delta_1 R - M\|\omega_1\| \leq u - M\omega_1 \leq R$.

This implies that $Tu \geq \|u\|$; that is, (3.23) holds.

Finally, (3.18), (3.23), and Lemma 1.1 guarantee that T has a fixed point $u \in K \cap \overline{\Omega_R} \setminus \Omega_r$ with $r \leq \|u\| \leq R$. Clearly, $u > M\omega_1$.

Since

$$\begin{aligned}
 P_2(G(t, x, u - M\omega_1)) &\geq \underline{G}_2 \|G(t, x, M\omega_1)\|_{L^1} \\
 &\geq \underline{G}_2 \left\| j_4(u - M\omega_1) \left(1 + \frac{h_4(u - M\omega_1)}{j_4(u - M\omega_1)} \right) \right\|_{L^1} \\
 &\geq \underline{G}_2 j_4(R) \left\{ 1 + \frac{h_4(\delta_1 r - M\|\omega_1\|)}{j_4(\delta_1 r - M\|\omega_1\|)} \right\} 4\pi^2 \\
 &> \frac{\overline{G}_2}{\underline{G}_2 \|a_2\|_{L^1}} M 4\pi^2 \\
 &\geq M\omega_2,
 \end{aligned} \tag{3.25}$$

then we have a doubly periodic solution (u, v) of (3.9) with $u > M\omega_1$, $v > M\omega_2$, namely, $(u - M\omega_1, v - M\omega_2) > (0, 0)$ is a positive solution of (1.1) with (1.2). $\square$

Similarly, we also obtain the following result.

Theorem 3.2. Assume that (H1)–(H4) hold. In addition, we assume the following.

(H7) There exists

$$r > \frac{M\|\omega_2\|}{\delta_2}, \tag{3.26}$$

such that

$$r \geq \frac{4\pi^2 \overline{G}_2}{\underline{G}_2 \|a_2\|_{L^1}} I_5 \cdot I_6, \tag{3.27}$$

here

$$\begin{aligned}
 I_5 &= j_2 \left(4\pi^2 \underline{G}_1 j_3(r) \left\{ 1 + \frac{h_3(\delta_2 r - M\|\omega_2\|)}{j_3(\delta_2 r - M\|\omega_2\|)} \right\} - M\|\omega_1\| \right), \\
 I_6 &= 1 + \frac{h_2 \left(\left(4\pi^2 \overline{G}_1 / \underline{G}_1 \|a_1\|_{L^1} \right) j_1(\delta_2 r - M\|\omega_2\|) \{ 1 + h_1(r) / j_1(r) \} \right)}{j_2 \left(\left(4\pi^2 \overline{G}_1 / \underline{G}_1 \|a_1\|_{L^1} \right) j_1(\delta_2 r - M\|\omega_2\|) \{ 1 + h_1(r) / j_1(r) \} \right)}.
 \end{aligned} \tag{3.28}$$

(H8) There exists $R > r$, such that

$$\begin{aligned}
 4\pi^2 \underline{G}_2 I_7 \cdot I_8 &\geq R, \\
 \delta_1 j_3(R) \left\{ 1 + \frac{h_3(\delta_2 r - M\|\omega_2\|)}{j_3(\delta_2 r - M\|\omega_2\|)} \right\} &> M,
 \end{aligned} \tag{3.29}$$

where

$$\begin{aligned}
 I_7 &= j_4 \left(\frac{4\pi^2 \overline{G_1}}{\underline{G_1} \|a_1\|_{L^1}} j_1(\delta_2 R - M\|\omega_2\|) \left\{ 1 + \frac{h_1(R)}{j_1(R)} \right\} \right), \\
 I_8 &= 1 + \frac{h_4 \left(4\pi^2 \underline{G_1} j_3(R) \{ 1 + h_3(\delta_2 R - M\|\omega_2\|) / j_3(\delta_2 R - M\|\omega_2\|) \} - M\|\omega_1\| \right)}{j_4 \left(4\pi^2 \underline{G_1} j_3(R) \{ 1 + h_3(\delta_2 R - M\|\omega_2\|) / j_3(\delta_2 R - M\|\omega_2\|) \} - M\|\omega_1\| \right)}. \tag{3.30}
 \end{aligned}$$

Then, problem (1.1)-(1.2) has a positive periodic solution.

4. An Example

Consider the following system:

$$\begin{aligned}
 u_{tt} - u_{xx} + 2u_t + \sin^2(t+x)u &= \mu \left(v^{-\alpha} + v^\beta + k_1(t,x) \right), \\
 v_{tt} - v_{xx} + 2v_t + \cos^2(t+x)v &= \lambda \left(u^{-\tau} + u^\sigma + k_2(t,x) \right), \\
 u(t+2\pi, x) &= u(t, x+2\pi) = u(t, x), \quad (t, x) \in \mathbb{R}^2, \\
 v(t+2\pi, x) &= v(t, x+2\pi) = v(t, x), \quad (t, x) \in \mathbb{R}^2,
 \end{aligned} \tag{4.1}$$

where $c_1 = c_2 = 2$, $\mu, \lambda > 0$, $\alpha, \tau > 0$, $\beta, \sigma > 1$, $a_1(t, x) = \sin^2(t+x)$, $a_2(t, x) = \cos^2(t+x) \in C(\mathbb{T}^2, \mathbb{R}^+)$, $k_i : \mathbb{T}^2 \rightarrow \mathbb{R}$ is continuous. When μ is chosen such that

$$\mu < \sup_{u \in ((M\|\omega_1\|)/\delta_1, \infty)} \frac{\underline{G} \|a_1\|_{L^1}}{\overline{G} 4\pi^2} \frac{I^1}{I^2}, \tag{4.2}$$

here we denote

$$\begin{aligned}
 I^1 &= u \left(\underline{G} \lambda u^{-\tau} \{ 1 + (\delta_1 u - M\|\omega_1\|)^{\sigma+\tau} \} 4\pi^2 - M\|\omega_2\| \right)^\alpha, \\
 I^2 &= 1 + \left(\frac{\overline{G}}{\underline{G} \|a_2\|_{L^1}} \lambda (\delta_1 u - M\|\omega_1\|)^{-\tau} (1 + u^{\sigma+\tau} + 2Hu^\tau) 4\pi^2 \right)^{\beta+\alpha} \\
 &\quad + 2H \left(\frac{\overline{G}}{\underline{G} \|a_2\|_{L^1}} \lambda (\delta_1 u - M\|\omega_1\|)^{-\tau} (1 + u^{\sigma+\tau} + 2Hu^\tau) 4\pi^2 \right),
 \end{aligned} \tag{4.3}$$

where $H = \max\{\|k_1\|, \|k_2\|\}$ and the Green function $G_1 = G_2 = G$. Then, problem (4.1) has a positive solution.

To verify the result, we will apply Theorem 3.1 with $M = \max\{\mu H, \lambda H\}$ and

$$\begin{aligned} j_1(v) &= j_3(v) = \mu v^{-\alpha}, \quad h_1(v) = \mu(v^\beta + 2H), \quad h_3(v) = \mu v^\beta, \\ j_2(u) &= j_4(u) = \lambda u^{-\tau}, \quad h_2(u) = \mu(u^\sigma + 2H), \quad h_4(u) = \mu u^\sigma. \end{aligned} \quad (4.4)$$

Clearly, (H1)–(H4) are satisfied.

Set

$$T(u) = \frac{\underline{G}\|a_1\|_{L^1}}{\overline{G}4\pi^2} \frac{I^1}{I^2}, \quad u \in \left(\frac{(M\|\omega_1\|)}{\delta_1}, +\infty \right). \quad (4.5)$$

Obviously, $T((M\|\omega_1\|)/\delta_1) = 0$, $T(\infty) = 0$, then there exists $r \in ((M\|\omega_1\|)/\delta_1, +\infty)$ such that

$$T(r) = \sup_{u \in ((M\|\omega_1\|)/\delta_1, \infty)} \frac{\underline{G}\|a_1\|_{L^1}}{\overline{G}4\pi^2} \frac{I^1}{I^2}. \quad (4.6)$$

This implies that there exists

$$r \in \left(\frac{(M\|\omega_1\|)}{\delta_1}, +\infty \right), \quad (4.7)$$

such that

$$\mu < \sup_{u \in ((M\|\omega_1\|)/\delta_1, \infty)} \frac{\underline{G}\|a_1\|_{L^1}}{\overline{G}4\pi^2} \frac{I^1}{I^2}. \quad (4.8)$$

So, (H5) is satisfied.

Finally, since

$$\frac{R \left(\left(\overline{G}/\underline{G}\|a_2\|_{L^1} \right) \lambda (\delta_1 R - M\|\omega_1\|)^{-\tau} (1 + R^{\sigma+\tau} + 2HR^\tau) 4\pi^2 \right)^\alpha}{\mu \underline{G} \left[1 + (\underline{G}\lambda R^{-\tau} \{1 + (\delta_1 R - M\|\omega_1\|)^{\sigma+\tau}\} 4\pi^2 - M\|\omega_2\|)^{\alpha+\beta} \right]} \longrightarrow 0 \quad \text{as } R \longrightarrow \infty, \quad (4.9)$$

this implies that there exists R . In addition, for fixed r, R , choosing λ sufficiently large, we have

$$\delta_2 \lambda R^{-\tau} \{1 + (\delta_1 r - M\|\omega_1\|)^{\sigma+\tau}\} > M. \quad (4.10)$$

Thus, (H6) is satisfied. So, all the conditions of Theorem 3.1 are satisfied.

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