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Research Article

Positive Solutions of Nonlinear Eigenvalue Problems for a Nonlocal Fractional Differential Equation

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By using the fixed point theorem, positive solutions of nonlinear eigenvalue problems for a nonlocal fractional differential equation $D_{0+}^{\alpha}u(t) + \lambda a(t)f(t,u(t)) = 0,\ 0 < t < 1,\ u(0) = 0,\ u(1) = \Sigma_{i=1}^{\infty}\alpha_i u(\xi_i)$ are considered, where $1 < \alpha \leq 2$ is a real number, λ is a positive parameter, D_{0+}^{α} is the standard Riemann-Liouville differentiation, and $\xi_i \in (0,1), \alpha_i \in [0,\infty)$ with $\Sigma_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1} < 1$, $a(t) \in C([0,1],[0,\infty)),\ f(t,u) \in C([0,\infty),[0,\infty))$.

1. Introduction

Fractional differential equations have been of great interest recently. This is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in various sciences such as physics, mechanics, chemistry, and engineering. For details, see [1–6] and references therein.

Recently, many results were obtained dealing with the existence and multiplicity of solutions of nonlinear fractional differential equations by the use of techniques of nonlinear analysis, see [7–21] and the reference therein. Bai and Lü [7] studied the existence of positive solutions of nonlinear fractional differential equation

$$\begin{gathered} D_{0+}^{\alpha}u(t) + f(t,u(t)) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0, \end{gathered} \tag{1.1}$$

where $1 < \alpha \leq 2$ is a real number, D_{0+}^{α} is the standard Riemann-Liouville differentiation, and $f : [0,1]\times[0,\infty) \to [0,\infty)$ is continuous. They derived the corresponding Green function and obtained some properties as follows.

Proposition 1.1. *Green's function $G(t,s)$ satisfies the following conditions:*

(R1) $G(t,s) \in C([0,1]\times[0,1])$, *and* $G(t,s) > 0$ *for* $t,s \in (0,1)$;

(R2) *there exists a positive function* $\gamma \in C(0,1)$ *such that*

$$\min_{1/4\le t\le 3/4} G(t,s) \ge \gamma(s) \max_{0\le t\le 1} G(t,s) \ge \gamma(s)G(s,s), \quad s\in(0,1), \tag{1.2}$$

where

$$G(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0\le s\le t\le 1, \\ \dfrac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0\le t\le s\le 1. \end{cases} \tag{1.3}$$

It is well known that the cone plays a very important role in applying Green's function in research area. In [7], the authors cannot acquire a positive constant taken instead of the role of positive function $\gamma(s)$ with $1<\alpha<2$ in (1.2). In [9], Jiang and Yuan obtained some new properties of the Green function and established a new cone. The results can be stated as follows.

Proposition 1.2. *Green's function $G(t,s)$ defined by* (1.3) *has the following properties:* $G(t,s) = G(1-s,1-t)$ *and*

$$\frac{\alpha-1}{\Gamma(\alpha)} t^{\alpha-1}(1-t)(1-s)^{\alpha-1}s \le G(t,s) \le \frac{1}{\Gamma(\alpha)} t^{\alpha-1}(1-t)(1-s)^{\alpha-2}, \quad \forall t,s\in(0,1). \tag{1.4}$$

Proposition 1.3. *The function $G^*(t,s) := t^{2-\alpha}G(t,s)$ has the following properties:*

$$q(t)\Phi(s) \le G^*(t,s) \le \Phi(s), \quad \forall t,s\in[0,1], \tag{1.5}$$

where $q(t) = (\alpha-1)t(1-t)$, $\Phi(s) = (1/\Gamma(\alpha))s(1-s)^{\alpha-1}$.

In this paper, we study the existence of positive solutions of nonlinear eigenvalue problems for a nonlocal fractional differential equation

$$\begin{gathered} D_{0+}^{\alpha}u(t) + \lambda a(t)f(t,u(t)) = 0, \quad 0<t<1, \\ u(0)=0, \qquad u(1) = \sum_{i=1}^{\infty}\alpha_i u(\xi_i), \end{gathered} \tag{1.6}$$

where $1<\alpha\le 2$ is a real number, λ is a positive parameter, D_{0+}^{α} is the standard Riemann-Liouville differentiation, and $\xi_i\in(0,1)$, $\alpha_i\in[0,\infty)$ with $\Sigma_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}<1$, $a(t)\in C([0,1],[0,\infty))$, $f(t,u)\in C([0,\infty),[0,\infty))$.

We assume the following conditions hold throughout the paper:

(H1) $\xi_i\in(0,1)$, $\alpha_i\in[0,\infty)$ are both constants with $\Sigma_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}<1$;

(H2) $a(t)\in C([0,1],\ [0,\infty))$, $a(t)\not\equiv 0$;

(H3) $f(t,u) \in C([0,1] \times [0,\infty), [0,\infty))$, and there exist $g \in C([0,+\infty), [0,+\infty)), q_1, q_2 \in C((0,1), (0,+\infty))$ such that

$$q_1(t)g(y) \le f\left(t, t^{\alpha-2}y\right) \le q_2(t)g(y), \quad t \in (0,1),\ y \in [0,+\infty), \tag{1.7}$$

where $\int_0^1 q_i(s)ds < +\infty, i = 1,2$.

2. The Preliminary Lemmas

For the convenience of the reader, we present here the necessary definitions from fractional calculus theory. These definitions can be found in the recent literature.

Definition 2.1. The fractional integral of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$I_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds \tag{2.1}$$

provided the right side is pointwise defined on $(0,\infty)$.

Definition 2.2. The fractional derivative of order $\alpha > 0$ of a function $y : (0,\infty) \to R$ is given by

$$D_{0+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{y(s)}{(t-s)^{\alpha-n+1}} ds, \tag{2.2}$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on $(0,\infty)$.

Lemma 2.3. *Let $\alpha > 0$. If one assumes $u \in C(0,1) \cap L(0,1)$, then the fractional differential equation*

$$D_{0+}^{\alpha} u(t) = 0 \tag{2.3}$$

has $u(t) = C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N},\ C_i \in R,\ i = 1,2,\ldots,N$, where N is the smallest integer greater than or equal to α, as unique solutions.

Lemma 2.4. *Assume that $u \in C(0,1) \cap L(0,1)$ with a fractional derivative of order $\alpha > 0$ that belongs to $u \in C(0,1) \cap L(0,1)$. Then*

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + C_1 t^{\alpha-1} + C_2 t^{\alpha-2} + \cdots + C_N t^{\alpha-N} \tag{2.4}$$

for some $C_i \in R, \quad i = 1,2,\ldots,N$.

Lemma 2.5 (see [7]). *Given $y \in C[0,1]$ and $1 < \alpha \le 2$, the unique solution of*

$$\begin{gathered} D_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \\ u(0) = u(1) = 0 \end{gathered} \tag{2.5}$$

is

$$u(t) = \int_0^1 G(t,s)y(s)ds, \tag{2.6}$$

where

$$G(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1. \end{cases} \tag{2.7}$$

Lemma 2.6. *Suppose (H1) holds. Given $y \in C[0,1]$ and $1 < \alpha \le 2$, the unique solution of*

$$\begin{aligned} &D_{0+}^{\alpha} u(t) + y(t) = 0, \quad 0 < t < 1, \\ &u(0) = 0, \qquad u(1) = \sum_{i=1}^{\infty} \alpha_i u(\xi_i) \end{aligned} \tag{2.8}$$

is

$$u(t) = \int_0^1 G(t,s)y(s)ds + A(y)t^{\alpha-1}, \tag{2.9}$$

where

$$G(t,s) = \begin{cases} \dfrac{[t(1-s)]^{\alpha-1} - (t-s)^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le s \le t \le 1, \\ \dfrac{[t(1-s)]^{\alpha-1}}{\Gamma(\alpha)}, & 0 \le t \le s \le 1, \end{cases} \tag{2.10}$$

$$A(y) = \frac{\sum_{i=1}^{\infty} \alpha_i \int_0^1 G(\xi_i, s)y(s)ds}{1 - \sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1}}.$$

Proof. By applying Lemmas 2.4 and 2.5, we have

$$u(t) = \int_0^1 G(t,s)y(s)ds + C_1 t^{\alpha-1} + C_2 t^{\alpha-2}. \tag{2.11}$$

Because

$$\sum_{i=1}^{\infty} \alpha_i \int_0^1 G(\xi_i, s)ds = \frac{\sum_{i=1}^{\infty} \alpha_i \xi_i^{\alpha-1}(1-\xi_i)}{\alpha\Gamma(\alpha)}, \quad \alpha_i \xi_i^{\alpha-1}(1-\xi_i) < \alpha_i \xi_i^{\alpha-1}, \tag{2.12}$$

by (H1), $\Sigma_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}(1-\xi_i)$ is convergent; therefore, $\Sigma_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)ds$ is convergent. Note that $y(t)$ is continuous function on $[0,1]$, so $\Sigma_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)y(s)ds$ is convergent.

From $u(0)=0, u(1)=\Sigma_{i=1}^{\infty}\alpha_i u(\xi_i)$, we have $C_2=0,\quad C_1=\Sigma_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)y(s)ds/(1-\Sigma_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1})$. Therefore,

$$u(t)=\int_0^1 G(t,s)y(s)ds+A(y)t^{\alpha-1},$$
$$A(y)=\frac{\Sigma_{i=1}^{\infty}\alpha_i\int_0^1 G(\xi_i,s)y(s)ds}{1-\Sigma_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}. \tag{2.13}$$

□

Lemma 2.7 (see [7]). *Let E be a Banach space, $P\subseteq E$ a cone, and Ω_1,Ω_2 two bounded open sets of E with $0\in\Omega_1\subset\overline{\Omega}_1\subset\Omega_2$. Suppose that $T:P\cap(\overline{\Omega}_2\backslash\Omega_1)\to P$ is a completely continuous operator such that either*

(i) $\|Tx\|\le\|x\|,\quad x\in P\cap\partial\Omega_1$, *and* $\|Tx\|\ge\|x\|,\quad x\in P\cap\partial\Omega_2$, *or*

(ii) $\|Tx\|\ge\|x\|,\quad x\in P\cap\partial\Omega_1$, *and* $\|Tx\|\le\|x\|,\quad x\in P\cap\partial\Omega_2$,

holds. Then T has a fixed point in $P\cap(\overline{\Omega}_2\setminus\Omega_1)$.

3. The Main Results

Let

$$G_0(t,s)=G(t,s)+\frac{t^{\alpha-1}\Sigma_{i=1}^{\infty}\alpha_i G(\xi_i,s)}{1-\Sigma_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1}}. \tag{3.1}$$

Then $u(t)$ is the solution of BVP (1.6) if and only if $Tu(t)=u(t)$, where T is the operator defined by

$$Tu(t):=\lambda\int_0^1 G_0(t,s)a(s)f(s,u(s))ds. \tag{3.2}$$

By similar arguments to Proposition 1.3, we obtain the following result.

Lemma 3.1. *Suppose (H1) holds. The function $\overline{G}(t,s):=t^{2-\alpha}G_0(t,s)$ has the following properties:*

$$q(t)\Psi(s)\le\overline{G}(t,s)\le\Psi(s),\quad\forall t,s\in[0,1], \tag{3.3}$$

where $q(t)=(\alpha-1)t(1-t),\Psi(s)=\Phi(s)+\Sigma_{i=1}^{\infty}\alpha_i G(\xi_i,s)/(1-\Sigma_{i=1}^{\infty}\alpha_i\xi_i^{\alpha-1})$.

Let $E=C[0,1]$ be endowed with the ordering $u\le v$ if $u(t)\le v(t)$ for all $t\in[0,1]$, and the maximum norm $\|u\|=\max_{0\le t\le 1}|u(t)|$. Define the cone $P\subset E$ by $P=\{u\in E\mid u(t)\ge 0\}$, and

$$K=\{u\in P\mid u(t)\ge q(t)\|u\|\}, \tag{3.4}$$

where $q(t)$ is defined by (3.3).

It is easy to see that P and K are cones in E. For any $0 < r < R < +\infty$, let $K_r = \{u \in K \|u\| < r\}$, $\partial K_r = \{u \in K \|u\| = r\}$, $\overline{K}_r = \{u \in K \|u\| \le r\}$, and $\overline{K}_R \setminus K_r = \{u \in K \mid r \le \|u\| \le R\}$.

For convenience, we introduce the following notations:

$$g_0 = \lim_{u \to 0^+} \frac{g(u)}{u}, \qquad g_\infty = \lim_{u \to +\infty} \frac{g(u)}{u}. \tag{3.5}$$

By similar arguments to Lemma 4.1 of [9], we obtain the following result.

Lemma 3.2. *Assume that (H1)–(H3) hold. Let $\overline{T} : K \to E$ be the operator defined by*

$$\overline{T}u(t) := \lambda \int_0^1 \overline{G}(t,s)a(s)f\left(s, s^{\alpha-2}u(s)\right)ds. \tag{3.6}$$

Then $\overline{T} : K \to K$ is completely continuous.

Theorem 3.3. *Assuming (H1)–(H3) hold, g_0, g_∞ exist. Then, for each λ satisfying*

$$\frac{1}{\left[((\alpha-1)/16)^2 \int_{1/4}^{3/4} \Psi(s)a(s)q_1(s)ds\right]g_0} < \lambda < \frac{1}{\left(\int_0^1 \Psi(s)a(s)q_2(s)ds\right)g_\infty}, \tag{3.7}$$

there exists at least one positive solution of BVP (1.6) *in P.*

Theorem 3.4. *Assuming (H1)–(H3) hold, g_0, g_∞ exist. Then, for each λ satisfying*

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there exists at least one positive solution of BVP (1.6) *in P.*

Proof of Theorem 3.3. Let λ be given as in (3.7), and choose $\varepsilon > 0$ such that

$$\frac{1}{\left[((\alpha-1)/16)^2 \int_{1/4}^{3/4} \Psi(s)a(s)q_1(s)ds\right](g_0-\varepsilon)} \le \lambda \le \frac{1}{\left(\int_0^1 \Psi(s)a(s)q_2(s)ds\right)(g_\infty+\varepsilon)}. \tag{3.9}$$

Beginning with g_0, there exists an $H_1 > 0$ such that $g(u) \ge (g_0 - \varepsilon)u$, for $0 < u \le H_1$. So $u \in K$ and $\|u\| = H_1$. For $t \in [1/4, 3/4]$, we have

$$\begin{aligned}
\overline{T}u(t) &= \lambda \int_0^1 \overline{G}(t,s)a(s)f\left(s, s^{\alpha-2}u(s)\right)ds \\
&\ge \lambda \int_0^1 q(t)\Psi(s)a(s)q_1(s)g(u(s))ds
\end{aligned}$$

$$\begin{aligned}
&\geq \lambda \int_0^1 q(t)\Psi(s)a(s)q_1(s)\left(g_0-\varepsilon\right)u(s)ds \\
&\geq \lambda\left(\frac{\alpha-1}{16}\right)^2 \int_{1/4}^{3/4} \Psi(s)a(s)q_1(s)\left(g_0-\varepsilon\right)ds\|u\| \\
&\geq \|u\| = H_1.
\end{aligned} \tag{3.10}$$

Thus, $\|\overline{T}u\| \geq \|u\|$. So, if we let

$$\Omega_1 = \{u \in K \mid \|u\| < H_1\}, \tag{3.11}$$

then

$$\left\|\overline{T}u\right\| \geq \|u\|, \quad u \in \partial\Omega_1. \tag{3.12}$$

It remains to consider g_∞. There exists an $\overline{H}_2$ such that $g(u) \leq (g_\infty+\varepsilon)u$, for all $u \geq \overline{H}_2$. There are the two cases, (a), where g is bounded, and (b), where g is unbounded.

Case a. Suppose $N > 0$ is such that $g(u) \leq N$, for all $0 < u < \infty$.

Let $H_2 = \max\{2H_1, N\lambda \int_0^1 \Psi(s)a(s)q_2(s)ds\}$. Then, for $u \in K$ with $\|u\| = H_2$, we have

$$\overline{T}u(t) \leq \lambda \int_0^1 \Psi(s)a(s)q_2(s)g(u(s))ds \leq \lambda N \int_0^1 \Psi(s)a(s)q_2(s)ds \leq H_2 = \|u\|. \tag{3.13}$$

So, if we let

$$\Omega_2 = \{u \in K \mid \|u\| < H_2\}, \tag{3.14}$$

then

$$\left\|\overline{T}u\right\| \leq \|u\|, \quad u \in \partial\Omega_2. \tag{3.15}$$

Case b. Let $H_2 > \max\{2H_1, \overline{H}_2\}$ be such that $g(u) \leq g(H_2),\ 0 < u < H_2$. Choosing $u \in K$ with $\|u\| = H_2$,

$$\overline{T}u(t) \leq \lambda \int_0^1 \Psi(s)a(s)q_2(s)g(H_2)ds \leq \lambda \int_0^1 \Psi(s)a(s)q_2(s)\left(g_\infty+\varepsilon\right)ds\, H_2 \leq H_2 = \|u\|, \tag{3.16}$$

and so $\|\overline{T}u\| \leq \|u\|$. For this case, if we let

$$\Omega_2 = \{u \in K \mid \|u\| < H_2\}, \tag{3.17}$$

then

$$\left\|\overline{T}u\right\| \leq \|u\|, \quad u \in \partial\Omega_2. \tag{3.18}$$

Therefore, by (ii) of Lemma 2.7, $\overline{T}$ has a fixed point u such that $H_1 \leq \|u\| \leq H_2$ and satisfies

$$u(t) = \int_0^1 \overline{G}(t,s)a(s)f\left(s, s^{\alpha-2}u(s)\right)ds. \tag{3.19}$$

It is obvious that $y(t) = t^{\alpha-2}u(t)$ is solution of (1.6) for $t \in (0,1]$, and

$$y(t) = \int_0^1 G_0(t,s)a(s)f\left(s, y(s)\right)ds, \quad t \in (0,1]. \tag{3.20}$$

Next, we will prove $y(0) = 0$. From $u \in C[0,1]$ and (H1)–(H3), we have

$$\begin{aligned}
\lim_{t\to 0^+} y(t) &= \lim_{t\to 0^+} \int_0^1 G_0(t,s)a(s)f\left(s, y(s)\right)ds \\
&= \lim_{t\to 0^+} \int_0^1 G_0(t,s)a(s)f\left(s, s^{\alpha-2}u(s)\right)ds \\
&\leq \lim_{t\to 0^+} \int_0^1 G_0(t,s)a(s)q_2(s)g(u(s))ds \\
&\leq \lim_{t\to 0^+} \int_0^1 G_0(t,s)a(s)q_2(s)ds \max_{\|u\|\leq H_2} g(u) \\
&\leq 0.
\end{aligned} \tag{3.21}$$

Thus, $y(0) = 0$. then $y(t) = t^{\alpha-2}u(t)$ is solution of (1.6) for $t \in [0,1]$. □

Proof of Theorem 3.4. Let λ be given as in (3.8), and choose $\varepsilon > 0$ such that

$$\frac{1}{\left[((\alpha-1)/16)^2 \int_{1/4}^{3/4} \Psi(s)a(s)q_1(s)ds\right](g_\infty - \varepsilon)} \leq \lambda \leq \frac{1}{\left(\int_0^1 \Psi(s)a(s)q_2(s)ds\right)(g_0 + \varepsilon)}. \tag{3.22}$$

Beginning with g_0, there exists an $H_1 > 0$ such that $g(u) \leq (g_0 + \varepsilon)u$, for $0 < u \leq H_1$. So, for $u \in K$ and $\|u\| = H_1$, we have

$$\begin{aligned}\overline{T}u(t) &= \lambda \int_0^1 \overline{G}(t,s)a(s)f\left(s, s^{\alpha-2}u(s)\right)ds \\ &\leq \lambda \int_0^1 \Psi(s)a(s)q_2(s)ds\left(g_0 + \varepsilon\right)\|u\| \\ &\leq \|u\|.\end{aligned} \tag{3.23}$$

Thus, $\|\overline{T}u\| \leq \|u\|$. So, if we let

$$\Omega_1 = \{u \in K \mid \|u\| < H_1\}, \tag{3.24}$$

then

$$\left\|\overline{T}u\right\| \leq \|u\|, \quad u \in \partial\Omega_1. \tag{3.25}$$

Next, considering g_∞, there exists an $\overline{H}_2$ such that $g(u) \geq (g_\infty - \varepsilon)u$, for all $u \geq \overline{H}_2$. Let $H_2 = \max\{2H_1, (16/(\alpha-1))\overline{H}_2\}$. Then, $u \in K\|u\| = H_2$. For $t \in [1/4, 3/4]$, we have

$$\begin{aligned}\overline{T}u(t) &= \lambda \int_0^1 \overline{G}(t,s)a(s)f\left(s, s^{\alpha-2}u(s)\right)ds \\ &\geq \lambda \int_0^1 q(t)\Psi(s)a(s)q_1(s)g(u(s))ds \\ &\geq \lambda \int_0^1 q(t)\Psi(s)a(s)q_1(s)\left(g_\infty - \varepsilon\right)u(s)ds \\ &\geq \lambda\left(\frac{\alpha-1}{16}\right)^2 \int_{1/4}^{3/4} \Psi(s)a(s)q_1(s)\left(g_\infty - \varepsilon\right)ds\|u\| \\ &\geq \|u\| = H_2,\end{aligned} \tag{3.26}$$

and so $\|\overline{T}u\| \geq \|u\|$. For this case, if we let

$$\Omega_2 = \{u \in K \mid \|u\| < H_2\}, \tag{3.27}$$

then

$$\left\|\overline{T}u\right\| \geq \|u\|, \quad u \in \partial\Omega_2. \tag{3.28}$$

Therefore, by (i) of Lemma 2.7, $\overline{T}$ has a fixed point u such that $H_1 \leq \|u\| \leq H_2$ and satisfy

$$u(t) = \int_0^1 \overline{G}(t,s)a(s)f\left(s, s^{\alpha-2}u(s)\right)ds. \tag{3.29}$$

By similar method to Theorem 3.3, we can get $y(0) = 0$, then $y(t) = t^{\alpha-2}u(t)$ is solution of (1.6) for $t \in [0,1]$. We complete the proof. □

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