Research Article

Trigonometric Function Periodic Wave Solutions and Their Limit Forms for the KdV-Like and the PC-Like Equations

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We use the bifurcation method of dynamical systems to study the periodic wave solutions and their limit forms for the KdV-like equation $u_t + a(1 + bu)uu_x + u_{xxx} = 0$, and PC-like equation $v_{tt} - v_{ttxx} - (a_1v + a_2v^2 + a_3v^3)_{xx} = 0$, respectively. Via some special phase orbits, we obtain some new explicit periodic wave solutions which are called trigonometric function periodic wave solutions because they are expressed in terms of trigonometric functions. We also show that the trigonometric function periodic wave solutions can be obtained from the limits of elliptic function periodic wave solutions. It is very interesting that the two equations have similar periodic wave solutions. Our work extend previous some results.

1. Introduction

Many authors have investigated the KdV-like equation

$$u_t + a(1 + bu)uu_x + u_{xxx} = 0, \quad (1.1)$$

and the PC-like equation

$$v_{tt} - v_{ttxx} - (a_1v + a_2v^2 + a_3v^3)_{xx} = 0. \quad (1.2)$$

For example, Dey [1, 2] studied the exact Hamiltonian density and the conservation laws, and gave two kink solutions for (1.1). Zhang et al. [3, 4] gave some solitary wave solutions and singular wave solutions for (1.1) by using two different methods. Yu [5] got an exact kink soliton for (1.1) by using homogeneous balance method. Grimshaw et al. [6] studied the

large-amplitude solitons for (1.1). Fan [7, 8] gave some bell-shaped soliton solutions, kink-shaped soliton, and Jacobi periodic solutions for (1.1) by using algebraic method. Tang et al. [9] investigated solitary waves and their bifurcations for (1.1) by employing bifurcation method of dynamical systems. Peng [10] used the modified mapping method to get some solitary wave solutions composed of hyperbolic functions, periodic wave solutions composed of Jacobi elliptic functions, and singular wave solution composed of triangle functions for (1.1). Chow et al. [11] described the interaction between a soliton and a breather for (1.1) by using the Hirota bilinear method. Kaya and Inan [12] studied solitary wave solutions for (1.1) by using Adomian decomposition method. Yomba [13] used Fan's subequation method to construct exact traveling wave solutions composed of hyperbolic functions or Jacobi elliptic functions for (1.1).

Zhang and Ma [14] gave some explicit solitary wave solutions composed of hyperbolic functions by using solving algebraic equations for (1.2). Li and Zhang [15] used bifurcation method of dynamical system to study the bifurcation of traveling wave solutions and construct solitary wave solutions for (1.2). Kaya [16] discussed the exact and numerical solitary wave solutions by using a decomposition method for (1.2). Rafei et al. [17] gave numerical solutions by using He's method for (1.2).

Recently, many authors have presented some useful methods to deal with the problems in equations, for instance [18–30].

In this paper, we use the bifurcation method mentioned above to study the periodic wave solutions for (1.1) and (1.2). Through some special phase orbits, we obtain new expressions of periodic wave solutions which are composed of trigonometric functions $\sin \xi$ or $\cos \xi$. These solutions are called trigonometric function periodic wave solutions. We also check the correctness by using the software Mathematica.

In Section 2, we will state our results for (1.1). In Section 3, we will state our results for (1.2). In Sections 4, and 5, we will give derivations for our main results. Some discussions and the orders for testing the correctness of the solutions will be given in Section 6.

2. Trigonometric Function Periodic Wave Solutions for (1.1)

In this section, we state our main results for (1.1). In order to state these results conveniently, we give some preparations. For given constant $c \neq 0$, on $a - b$ plane we define some lines and regions as follows.

(1) When $c < 0$, we define lines

$$\begin{aligned} l_1: b &= 0, \\ l_2: b &= -\frac{a}{6c}, \\ l_3: b &= -\frac{3a}{16c}, \\ l_4: a &= 0, \end{aligned} \tag{2.1}$$

and regions A_i ($i = 1-8$), as Figure 1(a).

Figure 1: The locations of the lines l_i , k_i ($i = 1, 2, 3, 4$) and the regions A_j , B_j ($j = 1, 2, \dots, 8$) for given constant $c \neq 0$.

Figure 2: The limiting process of $u_1(\xi)$ when $c < 0$, $(a, b) \in A_1$, and (a, b) tends to the line l_1 , where $a = 4$ and $c = -1$.

(2) When $c > 0$, we define lines

$$\begin{aligned}
 k_1: b &= 0, \\
 k_2: a &= 0, \\
 k_3: b &= -\frac{3a}{16c}, \\
 k_4: b &= -\frac{a}{6c},
 \end{aligned} \tag{2.2}$$

and regions B_i ($i = 1-8$), as Figure 1(b).

Using the lines and regions in Figure 1, we narrate our results as follows.

Proposition 2.1. For arbitrary given constant $c \neq 0$, let

$$\xi = x - ct. \tag{2.3}$$

Then, (1.1) has the following periodic wave solutions.

Figure 3: The limiting process of $u_1(\xi)$ when $c < 0$, $(a, b) \in A_5$, and (a, b) tends to the line l_1 , where $a = -4$ and $c = -1$.

(1) When $c < 0$ and $(a, b) \in A_1$ or A_5 , the expression of the periodic wave solution is

$$u_1(\xi) = \frac{6c}{a + \sqrt{a(a + 6bc)} \cos(\sqrt{-c}\xi)}, \quad (2.4)$$

which has the following limit forms.

(i) When $c < 0$, $(a, b) \in A_1$ and (a, b) tends to the line l_1 , $u_1(\xi)$ tends to the periodic blow-up solution

$$u_1^\circ(\xi) = \frac{6c}{a[1 + \cos(\sqrt{-c}\xi)]} \quad (2.5)$$

(see Figure 2).

(ii) When $c < 0$, $(a, b) \in A_5$ and (a, b) tends to the line l_1 , $u_1(\xi)$ tends to the periodic blow-up solution

$$u_1^*(\xi) = \frac{6c}{a[1 - \cos(\sqrt{-c}\xi)]} \quad (2.6)$$

(see Figure 3).

(iii) When $c < 0$, $(a, b) \in A_1$ or A_5 , and (a, b) tends to l_2 , $u_1(\xi)$ tends to the trivial solution $u(\xi) = 6c/a$.

(2) When $c < 0$ and $(a, b) \in A_2$, or when $c > 0$ and $(a, b) \in B_5$, the expression of the periodic wave solution is

$$u_2(\xi) = \frac{\alpha_0 \cos(w_0 \xi) + \beta_0}{p_0 \cos(w_0 \xi) + q_0}, \quad (2.7)$$

where

$$\Delta = 3a(3a + 16bc), \quad (2.8)$$

$$\alpha_0 = -\frac{(3a + \sqrt{\Delta})\sqrt{a(a - \sqrt{\Delta})}}{4a^2b^2}, \quad (2.9)$$

$$\beta_0 = -\frac{3a + 24bc + \sqrt{\Delta}}{2ab^2}, \quad (2.10)$$

$$p_0 = \frac{\sqrt{a(a - \sqrt{\Delta})}}{ab}, \quad (2.11)$$

$$q_0 = -\frac{1}{ab}(a + \sqrt{\Delta}), \quad (2.12)$$

$$w_0 = \sqrt{\frac{3a + 16bc + \sqrt{\Delta}}{8b}}. \quad (2.13)$$

The solution $u_2(\xi)$ has the following limit forms.

- (i) When $c < 0$, $(a, b) \in A_2$ and (a, b) tends to l_3 , the $u_2(\xi)$ tends to the peak-shaped solitary wave solution

$$u_2^\circ(\xi) = \frac{4c(3 + 2c\xi^2)}{a(-9 + 2c\xi^2)} \quad (2.14)$$

(see Figure 4).

- (ii) When $c < 0$, $(a, b) \in A_2$ and (a, b) tends to l_2 , $u_2(\xi)$ tends to the trivial solution $u(\xi) = 0$.
 (iii) When $c > 0$, $(a, b) \in B_5$ and (a, b) tends to k_1 , the $u_2(\xi)$ tends to the periodic blow-up solution

$$u_2^*(\xi) = \frac{c(2\sin^2(\sqrt{c}\xi/2) - 3)}{a\sin^2(\sqrt{c}\xi/2)} \quad (2.15)$$

(see Figure 5).

- (3) When $c < 0$ and $(a, b) \in A_6$, or when $c > 0$ and $(a, b) \in B_1$, the expressions of the solution is

$$u_3(\xi) = \frac{\alpha_1 \cos(w_1\xi) + \beta_1}{p_1 \cos(w_1\xi) + q_1}, \quad (2.16)$$

Figure 4: The limiting process of $u_2(\xi)$ when $c < 0$, $(a, b) \in A_2$, and (a, b) tends to the line l_3 , where $a = 4$ and $c = -1$.

Figure 5: The limiting process of $u_2(\xi)$ when $c > 0$, $(a, b) \in B_5$, and (a, b) tends to the line k_1 , where $a = -2$ and $c = 1$.

where

$$\begin{aligned}
 \alpha_1 &= \frac{(-3a + \sqrt{\Delta})\sqrt{a(a + \sqrt{\Delta})}}{4a^2b^2}, \\
 \beta_1 &= \frac{3a + 24bc - \sqrt{\Delta}}{2ab^2}, \\
 p_1 &= \frac{\sqrt{a(a + \sqrt{\Delta})}}{ab}, \\
 q_1 &= \frac{a - \sqrt{\Delta}}{ab}, \\
 w_1 &= \sqrt{\frac{3a + 16bc - \sqrt{\Delta}}{8b}}.
 \end{aligned} \tag{2.17}$$

The solution $u_3(\xi)$ has the following limit forms.

- (i) When $c < 0$, $(a, b) \in A_6$ and (a, b) tends to l_3 , the $u_3(\xi)$ tends to the canyon-shaped solitary wave (see Figure 6) solution $u_2^\circ(\xi)$.
- (ii) When $c < 0$, $(a, b) \in A_6$ and (a, b) tends to l_2 , $u_3(\xi)$ tends to the trivial solution $u(\xi) = 0$.
- (iii) When $c > 0$, $(a, b) \in B_1$ and (a, b) tends to k_1 , the $u_3(\xi)$ tends to the periodic blow-up wave solution $u_1^*(\xi)$ (see Figure 3).

Figure 6: The limiting process of $u_3(\xi)$ when $c < 0$, $(a, b) \in A_6$, and (a, b) tends to the line l_3 , where $a = -9$ and $c = -1$.

Remark 2.2. Note that if $u = \varphi(\xi)$ is a solution of (1.1), then $u = \varphi(\xi + r)$ also is solution of (1.1), where r is a arbitrary constant. According to this fact and the results listed in Proposition 2.1, the following nine functions also are periodic wave solutions of (1.1).

(1) When $c < 0$ and $(a, b) \in A_1$ or A_5 , the functions are

$$\begin{aligned} u_1^1(\xi) &= \frac{6c}{a - \sqrt{a(a + 6bc)} \cos(\sqrt{-c}\xi)}, \\ u_2^1(\xi) &= \frac{6c}{a + \sqrt{a(a + 6bc)} \sin(\sqrt{-c}\xi)}, \\ u_3^1(\xi) &= \frac{6c}{a - \sqrt{a(a + 6bc)} \sin(\sqrt{-c}\xi)}. \end{aligned} \quad (2.18)$$

(2) When $c < 0$ and $(a, b) \in A_2$ or when $c > 0$ and $(a, b) \in B_5$, the functions are

$$\begin{aligned} u_1^2(\xi) &= \frac{-\alpha_0 \cos(w_0\xi) + \beta_0}{-p_0 \cos(w_0\xi) + q_0}, \\ u_2^2(\xi) &= \frac{\alpha_0 \sin(w_0\xi) + \beta_0}{p_0 \sin(w_0\xi) + q_0}, \\ u_3^2(\xi) &= \frac{-\alpha_0 \sin(w_0\xi) + \beta_0}{-p_0 \sin(w_0\xi) + q_0}. \end{aligned} \quad (2.19)$$

(3) When $c < 0$ and $(a, b) \in A_6$, or when $c > 0$ and $(a, b) \in B_1$, the functions are

$$\begin{aligned} u_1^3(\xi) &= \frac{-\alpha_1 \cos(w_1\xi) + \beta_1}{-p_1 \cos(w_1\xi) + q_1}, \\ u_2^3(\xi) &= \frac{\alpha_1 \sin(w_1\xi) + \beta_1}{p_1 \sin(w_1\xi) + q_1}, \\ u_3^3(\xi) &= \frac{-\alpha_1 \sin(w_1\xi) + \beta_1}{-p_1 \sin(w_1\xi) + q_1}. \end{aligned} \quad (2.20)$$

Figure 7: The locations of the rays Γ_i , L_i and the regions W_i , Ω_i ($i = 1, 2, \dots, 8$) for given a_1 and c .

Remark 2.3. In the given parametric regions, the solutions $u_i(\xi)$, $u_i^1(\xi)$, $u_i^2(\xi)$, $u_i^3(\xi)$ ($i = 1, 2, 3$), and $u_2^\circ(\xi)$ are nonsingular. The solutions $u_1^\circ(\xi)$, $u_1^*(\xi)$, and $u_2^*(\xi)$ are singular. The relationships of singular solutions and nonsingular solutions are displayed in the Proposition 2.1.

3. Trigonometric Function Periodic Wave Solutions for (1.2)

In this section, we state our main results for (1.2). For given a_1 and c ($a_1 \neq c^2$), on $a_2 - a_3$ plane we define some rays and regions as follows.

(1) When $c^2 < a_1$, we define curves

$$\begin{aligned}
 \Gamma_1: a_2 &> 0, \quad a_3 = 0, \\
 \Gamma_2: a_2 &> 0, \quad a_3 = \frac{2a_2^2}{9(a_1 - c^2)}, \\
 \Gamma_3: a_2 &> 0, \quad a_3 = \frac{a_2^2}{4(a_1 - c^2)}, \\
 \Gamma_4: a_2 &= 0, \quad a_3 > 0, \\
 \Gamma_5: a_2 &< 0, \quad a_3 = \frac{a_2^2}{4(a_1 - c^2)}, \\
 \Gamma_6: a_2 &< 0, \quad a_3 = \frac{2a_2^2}{9(a_1 - c^2)}, \\
 \Gamma_7: a_2 &< 0, \quad a_3 = 0, \\
 \Gamma_8: a_2 &= 0, \quad a_3 < 0,
 \end{aligned} \tag{3.1}$$

and region W_i as the domain surrounded by Γ_i and Γ_{i+1} ($i = 1-7$), W_8 as the domain surrounded by Γ_8 and Γ_1 (see Figure 7(a)).

(2) When $c^2 > a_1$, we define curves

$$\begin{aligned}
 L_1: a_2 &> 0, & a_3 &= 0, \\
 L_2: a_2 &= 0, & a_3 &> 0, \\
 L_3: a_2 &< 0, & a_3 &= 0, \\
 L_4: a_2 &< 0, & a_3 &= \frac{2a_2^2}{9(a_1 - c^2)}, \\
 L_5: a_2 &< 0, & a_3 &= \frac{a_1^2}{4(a_1 - c^2)}, \\
 L_6: a_2 &= 0, & a_3 &< 0, \\
 L_7: a_2 &> 0, & a_3 &= \frac{a_2^2}{4(a_1 - c^2)}, \\
 L_8: a_2 &> 0, & a_3 &= \frac{2a_2^2}{9(a_1 - c^2)},
 \end{aligned} \tag{3.2}$$

and region Ω_i as the domain surrounded by L_i and L_{i+1} ($i = 1-7$), Ω_8 as the domain surrounded by L_8 and L_1 (see Figure 7(b)).

Using the rays and regions above, we state our results as follows.

Proposition 3.1. *For given parameter a_1 and constant c satisfying $c^2 \neq a_1$, let $\xi = x - ct$. Then, (1.2) has the following periodic wave solutions.*

(1) When $c^2 < a_1$ and $(a_2, a_3) \in W_1$ or W_6 , the expression of the periodic wave solution is

$$v_1(\xi) = \frac{R_0}{R_1 + R_2 \cos(R_3 \xi)}, \tag{3.3}$$

where

$$\begin{aligned}
 R_0 &= 2(c^2 - a_1), \\
 R_1 &= \frac{2a_2}{3}, \\
 R_2 &= \frac{1}{3} \sqrt{18a_3(c^2 - a_1) + 4a_2^2}, \\
 R_3 &= \sqrt{\frac{a_1 - c^2}{c^2}}.
 \end{aligned} \tag{3.4}$$

For $a_2 \neq 0$, the periodic wave solution $v_1(\xi)$ has the following limit forms.

- (i) When $c^2 < a_1$, $(a_2, a_3) \in W_1$ and (a_2, a_3) tends to the ray Γ_1 , $v_1(\xi)$ tends to the periodic blow-up solution

$$v_1^o(\xi) = \frac{3(c^2 - a_1)}{a_2 \left(1 + \cos \left(\left(\sqrt{a_1 - c^2} / |c| \right) \xi \right) \right)}. \quad (3.5)$$

The limiting process is similar to that in Figure 2.

- (ii) When $c^2 < a_1$, $(a_2, a_3) \in W_6$ and (a_2, a_3) tends to the ray Γ_7 , $v_1(\xi)$ tends to the periodic blow-up solution

$$v_1^*(\xi) = \frac{3(c^2 - a_1)}{a_2 \left(1 - \cos \left(\left(\sqrt{a_1 - c^2} / |c| \right) \xi \right) \right)}. \quad (3.6)$$

The limiting process is similar to that in Figure 3.

- (iii) When $c^2 < a_1$, $(a_2, a_3) \in W_1$ and (a_2, a_3) tends to the curve Γ_2 , or $(a_2, a_3) \in W_6$ and (a_2, a_3) tends to the curve Γ_6 , $v_1(\xi)$ tends to the trivial solution $v(\xi) = 3(c^2 - a_1) / a_2$.

(2) When $c^2 < a_1$ and $(a_2, a_3) \in W_5$, or when $c^2 > a_1$ and $(a_2, a_3) \in \Omega_1$, the expression of the periodic wave solution is

$$v_2(\xi) = S_0 - \frac{2S_1}{-S_2 + S_3 \cos(S_4 \xi)}, \quad (3.7)$$

where

$$\begin{aligned} S_0 &= \frac{-a_2 + \sqrt{\omega}}{2a_3}, \\ S_1 &= \frac{-a_2^2 + 4a_3(a_1 - c^2) + a_2\sqrt{\omega}}{a_3^2}, \\ S_2 &= \frac{2}{3a_3}(-a_2 + 3\sqrt{\omega}), \\ S_3 &= \frac{2}{3a_3}\sqrt{a_2(a_2 + 3\sqrt{\omega})}, \\ S_4 &= \sqrt{-\frac{S_1 a_3}{2c^2}}, \end{aligned} \quad (3.8)$$

$$\omega = a_2^2 - 4a_3(a_1 - c^2). \quad (3.9)$$

The periodic wave solution $v_2(\xi)$ has the following limit forms.

- (i) When $c^2 < a_1$, $(a_2, a_3) \in W_5$, and (a_2, a_3) tends to the curve Γ_6 , $v_2(\xi)$ tends to the trivial solution $v(\xi) = 0$.

- (ii) When $c^2 < a_1$, $(a_2, a_3) \in W_5$, and (a_2, a_3) tends to the curve Γ_5 , the $v_2(\xi)$ tends to the canyon-shaped solitary wave solution

$$v_2^o(\xi) = \frac{2(a_1 - c^2)[12c^2 - 9c^2 - 2(a_1 - c^2)\xi^2]}{a_2[9c^2 + 2(a_1 - c^2)\xi^2]}. \quad (3.10)$$

The limiting process is similar to that in Figure 6.

- (iii) When $c^2 > a_1$, $(a_2, a_3) \in \Omega_1$, and (a_2, a_3) tends to the ray L_1 , $v_2(\xi)$ tends to the periodic blow-up wave solution

$$v_2^*(\xi) = \frac{a_1 - c^2}{2a_2} \left[1 + 3 \tan^2 \left(\sqrt{\frac{c^2 - a_1}{4c^2}} \xi \right) \right]. \quad (3.11)$$

The limiting process is similar to that in Figure 2.

- (3) When $c^2 < a_1$ and $(a_2, a_3) \in W_2$, or when $c^2 > a_1$ and $(a_2, a_3) \in \Omega_2$, the expression of the periodic wave solution is

$$v_3(\xi) = T_0 + \frac{2T_1}{-T_2 + T_3 \cos(T_4 \xi)}, \quad (3.12)$$

where

$$\begin{aligned} T_0 &= \frac{-a_2 - \sqrt{\omega}}{2a_3}, \\ T_1 &= \frac{-a_2^2 + 4a_3(a_1 - c^2) - a_2\sqrt{\omega}}{a_3^2}, \\ T_2 &= \frac{2}{3a_3}(a_2 + 3\sqrt{\omega}), \\ T_3 &= \frac{2}{3a_3}\sqrt{a_2(a_2 - 3\sqrt{\omega})}, \\ T_4 &= \sqrt{-\frac{T_1 a_3}{2c^2}}. \end{aligned} \quad (3.13)$$

The periodic wave solution $v_3(\xi)$ has the following limit forms:

- (i) When $c^2 < a_1$, $(a_2, a_3) \in W_2$, and (a_2, a_3) tends to the curve Γ_2 , $v_3(\xi)$ tends to the trivial solution $v(\xi) = 0$.
- (ii) When $c^2 < a_1$, $(a_2, a_3) \in W_2$, and (a_2, a_3) tends to the curve Γ_3 , the $v_3(\xi)$ tends to the peak-shaped solitary wave solution $v_2^o(\xi)$. The limiting process is similar to that in Figure 4.
- (iii) When $c^2 > a_1$, $(a_2, a_3) \in \Omega_2$, and (a_2, a_3) tends to the ray L_3 , the $v_3(\xi)$ tends to the periodic blow-up wave solution $v_2^*(\xi)$. The limiting process is similar to that in Figure 5.

Remark 3.2. Similar to the reason in Remark 2.2, the following nine functions also are periodic wave solutions of (1.2).

(1) When $c^2 < a_1$ and $(a_2, a_3) \in W_1$ or W_6 , the functions are

$$\begin{aligned} v_1^1(\xi) &= \frac{R_0}{R_1 - R_2 \cos(R_3 \xi)}, \\ v_2^1(\xi) &= \frac{R_0}{R_1 + R_2 \sin(R_3 \xi)}, \\ v_3^1(\xi) &= \frac{R_0}{R_1 - R_2 \sin(R_3 \xi)}. \end{aligned} \quad (3.14)$$

(2) When $c^2 < a_1$ and $(a_2, a_3) \in W_5$ or when $c^2 > a_1$ and $(a_2, a_3) \in \Omega_1$, the functions are

$$\begin{aligned} v_1^2(\xi) &= S_0 + \frac{2S_1}{S_2 + S_3 \cos(S_4 \xi)}, \\ v_2^2(\xi) &= S_0 - \frac{2S_1}{-S_2 + S_3 \sin(S_4 \xi)}, \\ v_3^2(\xi) &= S_0 + \frac{2S_1}{S_2 + S_3 \sin(S_4 \xi)}. \end{aligned} \quad (3.15)$$

(3) When $c^2 < a_1$ and $(a_2, a_3) \in W_2$ or when $c^2 > a_1$ and $(a_2, a_3) \in \Omega_2$, the functions are

$$\begin{aligned} v_1^3(\xi) &= T_0 - \frac{2T_1}{T_2 + T_3 \cos(T_4 \xi)}, \\ v_2^3(\xi) &= T_0 + \frac{2T_1}{-T_2 + T_3 \sin(T_4 \xi)}, \\ v_3^3(\xi) &= T_0 - \frac{2T_1}{T_2 + T_3 \sin(T_4 \xi)}. \end{aligned} \quad (3.16)$$

Remark 3.3. In the given regions, the solutions $v_i(\xi)$, $v_i^1(\xi)$, $v_i^2(\xi)$, $v_i^3(\xi)$ ($i = 1, 2, 3$), and $v_2^0(\xi)$ are nonsingular. The solutions $v_1^0(\xi)$, $v_1^*(\xi)$, and $v_2^*(\xi)$ are singular. The relationships of nonsingular solutions and singular solutions are displayed in Proposition 3.1.

4. The Derivation on Proposition 2.1

In order to derive the Proposition 2.1, letting c be a constant and substituting $u = \varphi(\xi)$ with $\xi = x - ct$ into (1.1), we have

$$-c\varphi' + a\varphi\varphi' + ab\varphi^2\varphi' + \varphi''' = 0. \quad (4.1)$$

Integrating (4.1) once and letting the integral constant be zero, it follows that

$$-c\varphi + \frac{a}{2}\varphi^2 + \frac{ab}{3}\varphi^3 + \varphi'' = 0. \quad (4.2)$$

Letting $\varphi' = y$, yields the following planar system:

$$\varphi' = y, \quad y' = c\varphi - \frac{a}{2}\varphi^2 - \frac{ab}{3}\varphi^3. \quad (4.3)$$

Obviously, system (4.3) has the first integral

$$6y^2 - 6c\varphi^2 + 2a\varphi^3 + ab\varphi^4 = h. \quad (4.4)$$

Let

$$\begin{aligned} \varphi_1 &= \frac{-3a - \sqrt{\Delta}}{4ab}, \\ \varphi_2 &= \frac{-3a + \sqrt{\Delta}}{4ab}, \end{aligned} \quad (4.5)$$

where Δ is defined in (2.8). Then, it is easy to see that system (4.3) has three singular points $(\varphi_1, 0)$, $(0, 0)$ and $(\varphi_2, 0)$ when $\Delta > 0$, two singular points $((-3/4b), 0)$ and $(0, 0)$ when $\Delta = 0$, unique singular point $(0, 0)$ when $\Delta < 0$.

Let e_i and f_i ($i = 1, 2, 3$) be, respectively,

$$\begin{aligned} e_1 &= \frac{-a - \sqrt{a^2 + 6abc}}{ab}, \\ f_1 &= \frac{-a + \sqrt{a^2 + 6abc}}{ab}, \\ e_2 &= \frac{1}{4ab} \left(-a + \sqrt{\Delta} - 2\sqrt{a(a - \sqrt{\Delta})} \right), \end{aligned}$$

Figure 8: When $c < 0$, the bifurcation phase portraits of system (4.3) and the locations of e_i and f_i ($i = 1, 2, 3$).

$$\begin{aligned}
 f_2 &= \frac{1}{4ab} \left(-a + \sqrt{\Delta} + 2\sqrt{a(a - \sqrt{\Delta})} \right), \\
 e_3 &= -\frac{1}{4ab} \left(a + \sqrt{\Delta} + 2\sqrt{a(a + \sqrt{\Delta})} \right), \\
 f_3 &= -\frac{1}{4ab} \left(a + \sqrt{\Delta} - 2\sqrt{a(a + \sqrt{\Delta})} \right).
 \end{aligned} \tag{4.6}$$

Using the qualitative analysis of dynamical systems, we obtain the bifurcation phase portraits of system (4.3) and the locations of e_i and f_i ($i = 1, 2, 3$) as Figures 8 and 9.

It is easy to test that the closed orbit passing $(e_i, 0)$ passes $(f_i, 0)$ ($i = 1, 2, 3$). Thus, using the phase portraits in Figures 8 and 9, we derive $u_i(\xi)$ ($i = 1, 2, 3$) as follows.

(1) When $c < 0$ and $(a, b) \in A_1$ or A_5 , the closed orbit passing the points $(e_1, 0)$ and $(f_1, 0)$ has expression

$$y = \pm \sqrt{\frac{ab}{6}} \varphi \sqrt{-e_1 f_1 + (e_1 + f_1) \varphi - \varphi^2}, \quad \text{where } e_1 \leq \varphi \leq f_1. \tag{4.7}$$

Substituting (4.7) into $d\varphi/y = d\xi$, we have

$$\frac{d\varphi}{\sqrt{-e_1 f_1 + (e_1 + f_1) \varphi - \varphi^2}} = \sqrt{\frac{ab}{6}} d\xi. \tag{4.8}$$

Figure 9: When $c > 0$, the bifurcation phase portraits of system (4.3) and the locations of e_i and f_i ($i = 1, 2, 3$).

Integrating (4.8) along the closed orbit and noting that $u = \varphi(\xi)$, we obtain the solution $u_1(\xi)$ as (2.4).

(2) When $c < 0$ and $(a, b) \in A_2$ or when $c > 0$ and $(a, b) \in B_5$, the closed orbit passing the points $(e_2, 0)$ and $(f_2, 0)$ has expression

$$y = \pm \sqrt{\frac{ab}{6}} (\varphi - \varphi_1) \sqrt{-e_2 f_2 + (e_2 + f_2) \varphi - \varphi^2}, \quad \text{where } e_2 \leq \varphi \leq f_2. \quad (4.9)$$

Substituting (4.9) into $d\varphi/y = d\xi$, we get

$$\frac{d\varphi}{(\varphi - \varphi_1) \sqrt{-e_2 f_2 + (e_2 + f_2) \varphi - \varphi^2}} = \sqrt{\frac{ab}{6}} d\xi. \quad (4.10)$$

Along the closed orbit integrating (4.10) and noting that $u = \varphi(\xi)$, we get the solution $u_2(\xi)$ as (2.7).

(3) When $c < 0$ and $(a, b) \in A_6$ or when $c > 0$ and $(a, b) \in B_1$, the closed orbit passing the points $(e_3, 0)$ and $(f_3, 0)$ has expression

$$y = \pm \sqrt{\frac{ab}{6}} (\varphi_2 - \varphi) \sqrt{-e_3 f_3 + (e_3 + f_3) \varphi - \varphi^2}, \quad \text{where } e_3 \leq \varphi \leq f_3. \quad (4.11)$$

Substituting (4.11) into $d\varphi/y = d\xi$, it follows that

$$\frac{d\varphi}{(\varphi_2 - \varphi)\sqrt{-e_3 f_3 + (e_3 + f_3)\varphi - \varphi^2}} = \sqrt{\frac{ab}{6}} d\xi. \quad (4.12)$$

Similarly, along the closed orbit integrating (4.12), we obtain $u_3(\xi)$ as (2.16). From the expressions of these solutions, we get their limit forms. This completes the derivation on Proposition 2.1.

5. The Derivation on Proposition 3.1

In this section, we give derivation on Proposition 3.1. Let $v = \varphi(\xi)$ with $\xi = x - ct$, where c is a constant. Thus, (1.2) becomes

$$c^2 \psi'' - c^2 \psi'''' - (a_1 \psi + a_2 \psi^2 + a_3 \psi^3)'' = 0. \quad (5.1)$$

Integrating (5.1) twice and letting integral constant be zero, we get

$$c^2 (\psi - \psi'') = a_1 \psi + a_2 \psi^2 + a_3 \psi^3. \quad (5.2)$$

Letting $\psi' = y$, we have the planar system

$$\psi' = y, \quad c^2 y' = (c^2 - a_1) \psi - a_2 \psi^2 - a_3 \psi^3. \quad (5.3)$$

It is easy to see that system (5.3) has the first integral

$$c^2 y^2 + \psi^2 \left(\frac{a_3}{2} \psi^2 + \frac{2a_2}{3} \psi + a_1 - c^2 \right) = h, \quad (5.4)$$

and three singular points $(0, 0)$, $(\psi_1, 0)$, and $(\psi_2, 0)$, where

$$\begin{aligned} \psi_1 &= \frac{-a_2 - \sqrt{\omega}}{2a_3}, \\ \psi_2 &= \frac{-a_2 + \sqrt{\omega}}{2a_3} \end{aligned} \quad (5.5)$$

and ω is defined in (3.9).

Figure 10: When $c^2 < a_1$, the bifurcation phase portraits of system (5.3) and the locations of m_i and n_i ($i = 1, 2, 3$).

Let m_i and n_i ($i = 1, 2, 3$) be, respectively,

$$\begin{aligned}
 m_1 &= \frac{-2a_2 - \sqrt{2(a_2^2 - 9a_1a_3 + 9a_3c^2)}}{3a_3}, \\
 n_1 &= \frac{-2a_2 + \sqrt{2(a_2^2 - 9a_1a_3 + 9a_3c^2)}}{3a_3}, \\
 m_2 &= \frac{-a_2 - 3\sqrt{\omega} - 2\sqrt{a_2(a_2 + 3\sqrt{\omega})}}{6a_3}, \\
 n_2 &= \frac{-a_2 - 3\sqrt{\omega} + 2\sqrt{a_2(a_2 + 3\sqrt{\omega})}}{6a_3}, \\
 m_3 &= \frac{-a_2 + 3\sqrt{\omega} - 2\sqrt{a_2(a_2 - 3\sqrt{\omega})}}{6a_3}, \\
 n_3 &= \frac{-a_2 + 3\sqrt{\omega} + 2\sqrt{a_2(a_2 - 3\sqrt{\omega})}}{6a_3}.
 \end{aligned} \tag{5.6}$$

Similarly, using the qualitative analysis of dynamical systems, we get the bifurcation phase portraits of system (5.3) and the locations of m_i and n_i ($i = 1, 2, 3$) as Figures 10 and 11.

It is easy to test that the closed orbit passing $(m_i, 0)$ passes $(n_i, 0)$ ($i = 1, 2, 3$). Thus, using the phase portraits in Figures 10 and 11, we derive $v_i(\xi)$ ($i = 1, 2, 3$) as follows.

Figure 11: When $c^2 > a_1$, the bifurcation phase portraits of system (5.3) and the locations of m_i and n_i ($i = 1, 2, 3$).

(1) When $c^2 < a_1$ and $(a_2, a_3) \in W_1$ or W_6 , the closed orbit passing the points $(m_1, 0)$ and $(n_1, 0)$ has expression

$$y = \pm \sqrt{\frac{a_3}{2c^2}} \psi \sqrt{-m_1 n_1 + (m_1 + n_1) \psi - \psi^2}, \quad \text{where } m_1 \leq \psi \leq n_1. \quad (5.7)$$

Substituting (5.7) into $d\psi/y = d\xi$, we have

$$\frac{d\psi}{\psi \sqrt{-m_1 n_1 + (m_1 + n_1) \psi - \psi^2}} = \sqrt{\frac{a_3}{2c^2}} d\xi. \quad (5.8)$$

Integrating (5.8) along the closed orbit and noting that $v = \psi(\xi)$, we get the solution $v_1(\xi)$ as (3.3).

(2) When $c^2 < a_1$ and $(a_2, a_3) \in W_5$, or when $c^2 > a_1$ and $(a_2, a_3) \in \Omega_1$, the closed orbit passing the points $(m_2, 0)$ and $(n_2, 0)$ has expression

$$y = \pm \sqrt{\frac{a_3}{2c^2}} (\psi_2 - \psi) \sqrt{-m_2 n_2 + (m_2 + n_2) \psi - \psi^2}, \quad \text{where } m_2 \leq \psi \leq n_2. \quad (5.9)$$

From $d\psi/y = d\xi$ and (5.9), it follows that

$$\frac{d\psi}{(\psi_2 - \psi)\sqrt{-m_2n_2 + (m_2 + n_2)\psi - \psi^2}} = \sqrt{\frac{a_3}{2c^2}} d\xi. \quad (5.10)$$

Integrating (5.10) along the closed orbit, we get $v_2(\xi)$ as (3.7).

(3) When $c^2 < a_1$ and $(a_2, a_3) \in W_2$, or when $c^2 > a_1$ and $(a_2, a_3) \in \Omega_2$, the closed orbit passing the points $(m_3, 0)$ and $(n_3, 0)$ has expression

$$y = \pm \sqrt{\frac{a_3}{2c^2}} (\psi - \psi_1) \sqrt{-m_3n_3 + (m_3 + n_3)\psi - \psi^2}, \quad \text{where } m_3 \leq \psi \leq n_3. \quad (5.11)$$

Substituting (5.11) into $d\psi/y = d\xi$, we have

$$\frac{d\psi}{(\psi - \psi_1)\sqrt{-m_3n_3 + (m_3 + n_3)\psi - \psi^2}} = \sqrt{\frac{a_3}{2c^2}} d\xi. \quad (5.12)$$

Integrating (5.12) along the closed orbit, we obtain $v_3(\xi)$ as (3.12). From the expressions of these solutions, we get their limiting properties. This completes the derivation on Proposition 3.1.

6. Discussions and Testing Orders

In this paper, Using the special closed orbits, we have obtained trigonometric function periodic wave solutions for (1.1) and (1.2), respectively. Their limit forms have been given. From these expressions, an interesting phenomena has been seen, that is, (1.1) and (1.2) have similar periodic wave solutions. Our work has extended previous results on periodic wave solutions.

Now, we point out that the trigonometric function periodic wave solutions can be obtained from the limits of the elliptic function periodic wave solution. For given real number μ , let

$$\begin{aligned} \mu_1 &= \frac{1}{12ab} \left(-4a(2 + b\mu) + \frac{4(1 + i\sqrt{3})aF_{02}}{F} + 2i(i + \sqrt{3})F \right), \\ \mu_2 &= \frac{1}{12ab} \left(-4a(2 + b\mu) + \frac{4(1 - i\sqrt{3})aF_{02}}{F} - 2i(i + \sqrt{3})F \right), \\ \mu_3 &= \frac{1}{6ab} \left(-2a(2 + b\mu) - \frac{4aF_{02}}{F} + 2F \right), \end{aligned} \quad (6.1)$$

Figure 12: The locations of l_μ^1 and l_μ^2 when $c < 0$ and $(a, b) \in A_1$.

where

$$\begin{aligned}
 F_{01} &= (8 - 6b\mu + 15b^2\mu^2 + 10b^3\mu^3), \\
 F_{02} &= (-9bc + a(-2 + b\mu + b^2\mu^2)), \\
 F_{03} &= \sqrt{a^3(8F_{02}^3 + a(-54bc(-1 + b\mu) + aF_{01})^2)}, \\
 F &= (54a^2bc(-1 + b\mu) - a^3F_{01} + F_{03})^{1/3}.
 \end{aligned} \tag{6.2}$$

Assume that $c < 0$, $(a, b) \in (A_1)$, and $\varphi_1 < \mu < e_1$. It is easy to check that μ_i ($i = 1, 2, 3$) are real and satisfy

$$\mu < e_1 < \varphi_2 < f_1 < \mu_1 < \varphi_3 < \mu_2 < 0 < \mu_3 < \varphi_4. \tag{6.3}$$

There are two closed orbits l_μ^1 and l_μ^2 (see Figure 12). The closed orbit l_μ^1 passes the points $(\mu, 0)$ and $(\mu_1, 0)$. The closed orbit l_μ^2 passes the points $(\mu_2, 0)$ and $(\mu_3, 0)$.

On $\varphi - y$ plane, the expression of l_μ^1 is

$$y^2 = \frac{ab}{6}(\mu_3 - \varphi)(\mu_2 - \varphi)(\mu_1 - \varphi)(\varphi - \mu), \quad \text{where } \mu \leq \varphi \leq \mu_1. \tag{6.4}$$

Substituting (6.4) into $d\varphi/y = d\xi$ and integrating it along l_μ^1 , we have

$$g \operatorname{sn}^{-1}(\sin z, k) = \sqrt{\frac{ab}{6}}|\xi|, \tag{6.5}$$

where

$$\begin{aligned} g &= \frac{2}{\sqrt{(\mu_3 - \mu_1)(\mu_2 - \mu)}}, \\ k &= \sqrt{\frac{(\mu_3 - \mu_2)(\mu_1 - \mu)}{(\mu_3 - \mu_1)(\mu_2 - \mu)}}, \\ \sin z &= \sqrt{\frac{(\mu_3 - \mu_1)(\varphi - \mu)}{(\mu_1 - \mu)(\mu_3 - \varphi)}}. \end{aligned} \quad (6.6)$$

Solving (6.5) for φ and noting that $u = \varphi(\xi)$, we obtain an elliptic function periodic wave solution

$$u(\xi) = \frac{\mu(\mu_3 - \mu_1) + \mu_3(\mu_1 - \mu)\operatorname{sn}^2(\eta\xi, k)}{\mu_3 - \mu_1 + (\mu_1 - \mu)\operatorname{sn}^2(\eta\xi, k)}, \quad (6.7)$$

where

$$\eta = \sqrt{\frac{ab(\mu_3 - \mu_1)(\mu_2 - \mu)}{24}}. \quad (6.8)$$

Letting $\mu \rightarrow e_1 - 0$, it follows that $\mu_1 \rightarrow f_1$, $\mu_2 \rightarrow 0$, $\mu_3 \rightarrow 0$, $k \rightarrow 0$, $\eta \rightarrow \sqrt{(abe_1f_1)/24}$ and $\operatorname{sn}^2(\eta\xi, k) \rightarrow \operatorname{sn}^2(\sqrt{(abe_1f_1/24)}\xi, 0) = \sin^2(\sqrt{(abe_1f_1/24)}\xi)$.

Therefore, in (6.7) letting $\mu \rightarrow e_1 - 0$, we obtain the trigonometric function periodic wave solution

$$\begin{aligned} u(\xi) &= \frac{e_1f_1}{f_1 + (e_1 - f_1)\sin^2\left(\sqrt{(abe_1f_1/24)}\xi\right)} \\ &= \frac{-6c}{-a + \sqrt{a(a+6bc)} - 2\sqrt{a(a+6bc)}\sin^2\left(\left(\sqrt{|c|}/2\right)\xi\right)} \\ &= \frac{6c}{a - \sqrt{a(a+6bc)}\cos\left(\sqrt{|c|}\xi\right)} = u_1^1(\xi). \end{aligned} \quad (6.9)$$

Via Remark 2.2 and $u_1^1(\xi)$, further we get $u_2^1(\xi)$, $u_3^1(\xi)$ and $u_1(\xi)$. Similarly, we can derive others trigonometric function periodic wave solutions.

We also have tested the correctness of each solution by using the software Mathematica. Here, we list two testing orders. Others testing orders are similar.

(1) The orders for testing $u_1(\xi)$

$$u = \frac{6c}{a + \sqrt{a(a+6bc)} \cos[\sqrt{-c}(x-ct)]} \quad (6.10)$$

Simplify $[D[u, t] + a(1 + bu)D[u, x]u + D[u, \{x, 3\}]]$.

(2) The orders for testing $v_1(\xi)$

$$\begin{aligned} R_0 &= 2(-a_1 + c^2), \\ R_1 &= \frac{2a_2}{3}, \\ R_2 &= \sqrt{2a_3(-a_1 + c^2) + \frac{4a_2^2}{9}}, \\ R_3 &= \sqrt{\frac{a_1 - c^2}{c^2}}, \\ v &= \frac{R_0}{R_1 + R_2 \cos[R_3(x-ct)]}, \\ v_{tt} &= D[v, \{t, 2\}], \\ v_{ttxx} &= D[v_{tt}, \{x, 2\}] \end{aligned} \quad (6.11)$$

Simplify $[v_{tt} - v_{ttxx} - D[a_1v + a_2v^2 + a_3v^3, \{x, 2\}]]$.

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