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Research Article

On the Perturbation Bounds of Projected Generalized Continuous-Time Sylvester Equations

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This paper is devoted to the perturbation analysis for a projected generalized continuous-time Sylvester equation. Perturbation bounds of the solution based on the Euclidean norm are presented.

1. Introduction

In this paper we study the sensitivity of and derive perturbation bounds for the projected generalized continuous-time Sylvester equation

$$\begin{gathered} AXC + DXB + P_{l,1}EP_{r,2} = 0, \\ X = P_{r,1}XP_{l,2}, \end{gathered} \tag{1.1}$$

where A, D are $n \times n$ matrices, B, C are $m \times m$ matrices, E is an $n \times m$ matrix, and X is the unknown $n \times m$ matrix, respectively, with real entries. Here, $P_{r,1}$ and $P_{r,2}$ are the spectral projectors onto the right deflating subspaces corresponding to the finite eigenvalues of the pencils $\lambda D - A$ and $\lambda C - B$, respectively, while $P_{l,1}$ and $P_{l,2}$ are the spectral projectors onto the left deflating subspaces corresponding to the finite eigenvalues of $\lambda D - A$ and $\lambda C - B$, respectively. We assume that the pencils $\lambda D - A$ and $\lambda C - B$ are regular, that is, $\det(\lambda D - A)$ and $\det(\lambda C - B)$ are not identically zero. Under the assumption, the pencils $\lambda D - A$ and $\lambda C - B$

have the Weierstrass canonical forms [1]: there exist nonsingular $n \times n$ matrices W_1, T_1 and $m \times m$ matrices W_2, T_2 such that

$$D = W_1 \begin{bmatrix} I & 0 \\ 0 & N^{(A)} \end{bmatrix} T_1, \qquad A = W_1 \begin{bmatrix} J^{(A)} & 0 \\ 0 & I \end{bmatrix} T_1,$$
$$C = W_2 \begin{bmatrix} I & 0 \\ 0 & N^{(B)} \end{bmatrix} T_2, \qquad B = W_2 \begin{bmatrix} J^{(B)} & 0 \\ 0 & I \end{bmatrix} T_2, \tag{1.2}$$

where $J^{(A)}$, $J^{(B)}$, $N^{(A)}$, and $N^{(B)}$ are block diagonal matrices with each diagonal block being the Jordan block. The eigenvalues of $J^{(A)}$ and $J^{(B)}$ are the finite eigenvalues of the pencils $\lambda D - A$ and $\lambda C - B$, respectively. $N^{(A)}$ and $N^{(B)}$ correspond to the eigenvalue at infinity. Using (1.2), $P_{l,1}$, $P_{l,2}$, $P_{r,1}$, and $P_{r,2}$ can be expressed as

$$P_{l,1} = W_1 \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} W_1^{-1}, \qquad P_{l,2} = W_2 \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} W_2^{-1},$$
$$P_{r,1} = T_1^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_1, \qquad P_{r,2} = T_2^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_2. \tag{1.3}$$

If D and C are nonsingular, then $P_{r,1} = P_{l,1} = I_n$, $P_{r,2} = P_{l,2} = I_m$, and (1.1) reduces to the generalized Sylvester equation $AXC + DXB + E = 0$. As $m = n$, $B = A^T$, and $C = D^T$, (1.1) is referred to as the projected generalized continuous-time Lyapunov equation. The projected generalized continuous-time Lyapunov equation plays an important role in stability analysis and control design problems for descriptor systems including the characterization of controllability and observability properties, computing $\mathcal{H}_2$ and Hankel norms, determining the minimal and balanced realizations as well as balanced truncation model order reduction; see [2–6] and the references therein. If the pencil $\lambda D - A$ is c-stable, that is, all its finite eigenvalues have negative real parts, then the projected generalized Lyapunov equation has a unique solution for each E, and if, additionally, E is symmetric and positive semidefinite, then the solution X is symmetric and positive semidefinite see, for example, [4] for details. In [7], the generalized Bartels-Stewart method and the generalized Hammarling method are presented for solving the projected generalized Lyapunov equation. The generalized Hammarling method is designed to obtain the Cholesky factor of the solution. These two methods are based on the generalized real Schur factorization (generalized Schur factorization if the matrix entries are complex) of the pencil $\lambda D - A$.

Zhou et al. [8] considered the projected generalized continuous-time Sylvester equation (1.1). They firstly presented one sufficient condition for the existence and uniqueness of the solution of this equation. Then, several numerical methods were proposed for solving (1.1). Finally, they shew that the solution of this equation is useful for computing the $\mathcal{HL}_2$ inner product of two descriptor systems.

The perturbation analysis for Sylvester-type equations has been considered by several authors. Higham [9] presented a perturbation analysis of the standard Sylvetser equation $AX - XB = C$. By taking into account its specific structure, he derived expressions for the backward error and a normwise condition number which measures the worst-case sensitivity of a solution to small perturbations in the data A, B, and C. In [10], a complete perturbation analysis of the nonsingular general Lyapunov equation was presented. Stykel [7] discussed the perturbation theory for the projected generalized continuous-time algebraic Lyapunov

equation. Konstantinov et al. considered the perturbation analysis for several types of matrix equations in their monograph [11]. They presented the framework of the perturbation analysis and derived condition numbers, first-order homogeneous bounds, componentwise bounds, and nonlocal normwise and componentwise bounds for the general Sylvester equation and the general Lyapunov equation.

In this paper, we study the perturbation theory for the projected generalized continuous-time Sylvester equation (1.1) and derive a perturbation bound for its solution.

Throughout this paper, we adopt the following notation. I_l denotes the $l \times l$ identity matrix and 0 denotes the zero vector or zero matrix. If the dimension of I_l is apparent from the context, we drop the index and simply use I. The space of $m \times n$ real matrices is denoted by $\mathbb{R}^{n\times m}$. The Euclidean norm for a vector or its associated induced matrix norm is denoted by $\|\cdot\|_2$. The superscript T denotes the transpose of a vector or a matrix and A^{-1} is the inverse of nonsingular A.

The remainder of the paper is organized as follows. In Section 2, we present the perturbation results for the projected generalized continuous-time Sylvester equation and the generalized continuous-time Sylvester equation. Conclusions are given in Section 3.

2. Perturbation Results for the Projected Generalized Continuous-Time Sylvester Equation

In this section, we firstly review and introduce one important theorem, for example, [12], which gives sufficient conditions for the existence, uniqueness, and analytic formula of the solution of the projected generalized continuous-time Sylvester equation (1.1).

Theorem 2.1. *Let $\lambda D - A$ and $\lambda C - B$ be regular pencils with finite eigenvalues $\{\mu_1, \mu_2, \ldots, \mu_{n_1}\}$ and $\{\nu_1, \nu_2, \ldots, \nu_{m_1}\}$ counted according to their multiplicities, respectively. Let $P_{r,1}$ and $P_{r,2}$ be the spectral projectors onto the right deflating subspaces corresponding to the finite eigenvalues of the pencils $\lambda D - A$ and $\lambda C - B$, respectively, and let $P_{l,1}$ and $P_{l,2}$ be the spectral projectors onto the left deflating subspaces corresponding to the finite eigenvalues of $\lambda D - A$ and $\lambda C - B$, respectively. Then, the projected generalized continuous-time Sylvester equation* (1.1) *has a unique solution for every E if $\mu_i + \nu_j \neq 0$ for any $i = 1, 2, \ldots, n_1$ and $j = 1, 2, \ldots, m_1$.*

Moreover, if $\lambda D - A$ and $\lambda C - B$ are c-stable, that is, all their finite eigenvalues have negative real part, then X can be expressed as

$$X = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-i\omega D - A)^{-1} P_{l,1} E P_{r,2} (i\omega C - B)^{-1} d\omega. \tag{2.1}$$

Now let us define a linear operator

$$\mathcal{S}^- : \mathbb{R}^{n\times m} \to \mathbb{R}^{n\times m}, \tag{2.2}$$

which satisfies the following: for a matrix $E \in \mathbb{R}^{n\times m}$, $X = \mathcal{S}^-(E)$ is the unique solution of the projected generalized continuous-time Sylvester equation (1.1), that is,

$$X = \mathcal{S}^-(E) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-i\omega D - A)^{-1} P_{l,1} E P_{r,2} (i\omega C - B)^{-1} d\omega. \tag{2.3}$$

The following result shows that the linear operator $\mathcal{S}^-$ is bounded and is very useful for the perturbation analysis of the projected generalized continuous-time Sylvester equation. Although the proof is similar to that of Lemma 3.6 in [7], we include it in this paper for completeness.

Lemma 2.2. *Assume that the pencils $\lambda D - A$ and $\lambda C - B$ are c-stable. Then*

$$\|\mathcal{S}^-\|_2 \leq \sqrt{\|H_1\|_2 \|H_2\|_2}, \tag{2.4}$$

where H_1 and H_2 are the solutions of the projected generalized continuous-time Lyapunov equations

$$\begin{gathered} AH_1D^T + DH_1A^T + P_{l,1}P_{l,1}^T = 0, \\ H_1 = P_{r,1}H_1P_{r,1}^T, \end{gathered} \tag{2.5}$$

$$\begin{gathered} B^TH_2C + C^TH_2B + P_{r,2}^TP_{r,2} = 0, \\ H_2 = P_{l,2}^TH_2P_{l,2}, \end{gathered} \tag{2.6}$$

respectively.

Proof. Let $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^m$ be the left and right singular vectors of unit length corresponding to the largest singular value of the solution X. Then, for any $E \in \mathbb{R}^{n\times m}$,

$$\begin{aligned} \|\mathcal{S}^-(E)\|_2 &= \|X\|_2 = u^TXv \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} u^T(-i\omega D - A)^{-1}P_{l,1}EP_{r,2}(i\omega C - B)^{-1}v\,d\omega \\ &\leq \frac{1}{2\pi}\|E\|_2\int_{-\infty}^{\infty}\left\|u^T(-i\omega D - A)^{-1}P_{l,1}\right\|_2\left\|P_{r,2}(i\omega C - B)^{-1}v\right\|_2 d\omega \\ &\leq \frac{1}{2\pi}\|E\|_2\left(\int_{-\infty}^{\infty}\left\|u^T(-i\omega D - A)^{-1}P_{l,1}\right\|_2^2 d\omega\right)^{1/2} \\ &\quad\cdot\left(\int_{-\infty}^{\infty}\left\|P_{r,2}(i\omega C - B)^{-1}v\right\|_2^2 d\omega\right)^{1/2}. \end{aligned} \tag{2.7}$$

Here, we have used the Cauchy-Schwarz inequality. It holds that

$$\begin{aligned} \int_{-\infty}^{\infty}\left\|u^T(-i\omega D - A)^{-1}P_{l,1}\right\|_2^2 d\omega &= \int_{-\infty}^{\infty} u^T(-i\omega D - A)^{-1}P_{l,1}P_{l,1}^T\left(i\omega D^T - A^T\right)^{-1}u\,d\omega \\ &\leq \left\|\int_{-\infty}^{\infty}(-i\omega D - A)^{-1}P_{l,1}P_{l,1}^T\left(i\omega D^T - A^T\right)^{-1}d\omega\right\|_2 \\ &= 2\pi\|H_1\|_2, \end{aligned} \tag{2.8}$$

where

$$H_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\omega D - A)^{-1}P_{l,1}P_{l,1}^T\left(i\omega D^T - A^T\right)^{-1}d\omega \tag{2.9}$$

is the unique solution of (2.5).

Similarly, we obtain

$$\int_{-\infty}^{\infty}\left\|P_{r,2}(i\omega C - B)^{-1}v\right\|_2^2 d\omega \le 2\pi\|H_2\|_2, \tag{2.10}$$

where

$$H_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(-i\omega C^T - B^T\right)^{-1}P_{l,1}P_{l,1}^T(i\omega C - B)^{-1}d\omega \tag{2.11}$$

is the unique solution of (2.6).

From (2.7), (2.8), and (2.10), it follows that for any $E \in \mathbb{R}^{n\times m}$,

$$\left\|\mathcal{S}^-(E)\right\|_2 \le \|E\|_2\sqrt{\|H_1\|_2\|H_2\|_2}. \tag{2.12}$$

Hence,

$$\left\|\mathcal{S}^-\right\|_2 = \sup_{E\in\mathbb{R}^{n\times m}, E\neq 0}\frac{\|\mathcal{S}^-(E)\|_2}{\|E\|_2} \le \sqrt{\|H_1\|_2\|H_2\|_2}. \tag{2.13}$$

□

Let A, B, C, D, and E be slightly perturbed to

$$\tilde{A} = A + \Delta A, \qquad \tilde{B} = B + \Delta B, \qquad \tilde{C} = C + \Delta C, \qquad \tilde{D} = D + \Delta D, \qquad \tilde{E} = E + \Delta E, \tag{2.14}$$

respectively, where $\Delta A, \Delta D \in \mathbb{R}^{n\times n}$, $\Delta B, \Delta C \in \mathbb{R}^{m\times m}$, $\Delta E \in \mathbb{R}^{n\times m}$ with

$$\begin{aligned} &\|\Delta A\|_2 \le \varepsilon\|A\|_2, \qquad \|\Delta B\|_2 \le \varepsilon\|B\|_2, \qquad \|\Delta C\|_2 \le \varepsilon\|C\|_2, \\ &\|\Delta D\|_2 \le \varepsilon\|D\|_2, \qquad \|\Delta E\|_2 \le \varepsilon\|E\|_2. \end{aligned} \tag{2.15}$$

Then the projected generalized continuous-time Sylvester equation (1.1) is perturbed to:

$$\begin{aligned} \tilde{A}\tilde{X}\tilde{C} + \tilde{D}\tilde{X}\tilde{B} + \tilde{P}_{l,1}\tilde{E}\tilde{P}_{r,2} &= 0, \\ \tilde{X} &= \tilde{P}_{r,1}\tilde{X}\tilde{P}_{l,2}, \end{aligned} \tag{2.16}$$

where $\tilde{X} = X + \Delta X$ with $\Delta X \in \mathbb{R}^{n\times m}$, $\tilde{P}_{r,1}$ and $\tilde{P}_{r,2}$ are the spectral projectors onto the right deflating subspaces corresponding to the finite eigenvalues of the pencils $\lambda\tilde{D} - \tilde{A}$

and $\lambda\widetilde{C} - \widetilde{B}$, respectively, and $\widetilde{P}_{l,1}$ and $\widetilde{P}_{l,2}$ are the spectral projectors onto the left deflating subspaces corresponding to the finite eigenvalues of $\lambda\widetilde{D} - \widetilde{A}$ and $\lambda\widetilde{C} - \widetilde{B}$, respectively.

We assume in this paper that the spectral projectors of the pencils $\lambda D - A$, $\lambda C - B$ and the perturbed pencils $\lambda\widetilde{D} - \widetilde{A}$, $\lambda\widetilde{C} - \widetilde{B}$ satisfy

$$\operatorname{Ker}(P_{r,i}) = \operatorname{Ker}\left(\widetilde{P}_{r,i}\right), \qquad \operatorname{Ker}\left(P_{l,i}^T\right) = \operatorname{Ker}\left(\widetilde{P}_{l,i}^T\right), \quad i = 1, 2. \tag{2.17}$$

Such an assumption is reasonable in some applications; see, for example, [13]. We further assume that

$$\left\|\widetilde{P}_{l,i} - P_{l,i}\right\|_2 \le \varepsilon K, \qquad \left\|\widetilde{P}_{r,i} - P_{r,i}\right\|_2 \le \varepsilon K, \quad i = 1, 2, \tag{2.18}$$

where K is a constant.

From (2.17), it follows that for $i = 1, 2$,

$$P_{r,i}\widetilde{P}_{r,i} = P_{r,i}, \qquad \widetilde{P}_{r,i}P_{r,i} = \widetilde{P}_{r,i}, \qquad \widetilde{P}_{l,i}P_{l,i} = P_{l,i}, \qquad P_{l,i}\widetilde{P}_{l,i} = \widetilde{P}_{l,i}. \tag{2.19}$$

Moreover, it is not difficult to verify that

$$P_{l,1}D = P_{l,1}DP_{r,1} = DP_{r,1}, \qquad P_{l,1}A = P_{l,1}AP_{r,1} = AP_{r,1}, \tag{2.20}$$

$$P_{l,2}C = P_{l,2}CP_{r,2} = CP_{r,2}, \qquad P_{l,2}B = P_{l,2}BP_{r,2} = BP_{r,2}, \tag{2.21}$$

$$\widetilde{P}_{l,1}\widetilde{D} = \widetilde{P}_{l,1}\widetilde{D}\widetilde{P}_{r,1} = \widetilde{D}\widetilde{P}_{r,1}, \qquad \widetilde{P}_{l,1}\widetilde{A} = \widetilde{P}_{l,1}\widetilde{A}\widetilde{P}_{r,1} = \widetilde{A}\widetilde{P}_{r,1}, \tag{2.22}$$

$$\widetilde{P}_{l,2}\widetilde{C} = \widetilde{P}_{l,2}\widetilde{C}\widetilde{P}_{r,2} = \widetilde{C}\widetilde{P}_{r,2}, \qquad \widetilde{P}_{l,2}\widetilde{B} = \widetilde{P}_{l,2}\widetilde{B}\widetilde{P}_{r,2} = \widetilde{B}\widetilde{P}_{r,2}. \tag{2.23}$$

We reformulate the first equation of the perturbed equation (2.16) as

$$A\widetilde{X}C + D\widetilde{X}B + \widetilde{P}_{l,1}\widetilde{E}\widetilde{P}_{r,2} + F\left(\widetilde{X}\right) = 0, \tag{2.24}$$

where $F(\widetilde{X}) \in \mathbb{R}^{n\times m}$ is defined by

$$F\left(\widetilde{X}\right) = \widetilde{A}\widetilde{X}\Delta C + \Delta A\widetilde{X}C + \widetilde{D}\widetilde{X}\Delta B + \Delta D\widetilde{X}B. \tag{2.25}$$

Then we have the following lemma.

Lemma 2.3. *The following relation holds*

$$\widetilde{P}_{l,1}\widetilde{E}\widetilde{P}_{r,2} + F\left(\widetilde{X}\right) = P_{l,1}\left(\widetilde{P}_{l,1}\widetilde{E}\widetilde{P}_{r,2} + F\left(\widetilde{X}\right)\right)P_{r,2}. \tag{2.26}$$

Proof. The equality $\widetilde{P}_{l,1}\widetilde{E}\widetilde{P}_{r,2} = P_{l,1}\widetilde{P}_{l,1}\widetilde{E}\widetilde{P}_{r,2}P_{r,2}$ follows directly from (2.19).

Since $\tilde{X} = \tilde{P}_{r,1}\tilde{X}\tilde{P}_{l,2}$, we have

$$\tilde{P}_{r,1}\tilde{X} = \tilde{P}_{r,1}\tilde{P}_{r,1}\tilde{X}\tilde{P}_{l,2} = \tilde{P}_{r,1}\tilde{X}\tilde{P}_{l,2} = \tilde{X}. \tag{2.27}$$

Similarly, $\tilde{X}\tilde{P}_{l,2} = \tilde{X}$.

By using (2.19), (2.20), (2.21), and (2.23), we obtain

$$\begin{aligned}
\tilde{X}\Delta C P_{r,2} &= \tilde{X}\tilde{P}_{l,2}\left(\tilde{C} - C\right)P_{r,2} = \tilde{X}\tilde{P}_{l,2}\tilde{C}P_{r,2} - \tilde{X}P_{l,2}C \\
&= \tilde{X}\tilde{C}\tilde{P}_{r,2}P_{r,2} - \tilde{X}P_{l,2}C = \tilde{X}\tilde{C}\tilde{P}_{r,2} - \tilde{X}P_{l,2}C \\
&= \tilde{X}\tilde{P}_{l,2}\tilde{C} - \tilde{X}P_{l,2}C = \tilde{X}\Delta C, \\
P_{l,1}\tilde{A}\tilde{X} &= P_{l,1}\tilde{A}\tilde{P}_{r,1}\tilde{X} = P_{l,1}\tilde{P}_{l,1}\tilde{A}\tilde{X} = \tilde{P}_{l,1}\tilde{A}\tilde{X} = \tilde{A}\tilde{P}_{r,1}\tilde{X} = \tilde{A}\tilde{X}.
\end{aligned} \tag{2.28}$$

Hence,

$$P_{l,1}\tilde{A}\tilde{X}\Delta C P_{r,2} = \tilde{A}\tilde{X}\Delta C P_{r,2} = \tilde{A}\tilde{X}\Delta C. \tag{2.29}$$

Using the similar manipulation, we can show that

$$P_{l,1}\tilde{D}\tilde{X}\Delta B P_{r,2} = \tilde{D}\tilde{X}\Delta B, \quad P_{l,1}\Delta A\tilde{X}C P_{r,2} = \Delta A\tilde{X}C, \quad P_{l,1}\Delta D\tilde{X}B P_{r,2} = \Delta D\tilde{X}B. \tag{2.30}$$

Then, the equality (2.26) follows. □

The following theorem provides the result on the relative error bound for the solution of the projected generalized continuous-time Sylvester equation (1.1).

Theorem 2.4. *Assume that the pencils $\lambda D - A$ and $\lambda C - B$ are c-stable. Let X be the unique solution of the projected generalized continuous-time Sylvester equation* (1.1). *Define*

$$\alpha = (\|A\|_2\|C\|_2 + \|D\|_2\|B\|_2)\sqrt{\|H_1\|_2\|H_2\|_2}, \qquad \beta = \|E\|_2\sqrt{\|H_1\|_2\|H_2\|_2}. \tag{2.31}$$

If $\varepsilon(2+\varepsilon)\alpha < 1$, then the perturbed equation (2.16) *has a unique solution $\tilde{X}$. Moreover, the relative error satisfies the following inequality:*

$$\frac{\left\|\tilde{X} - X\right\|_2}{\|X\|_2} \le \frac{\varepsilon(2+\varepsilon)\alpha}{1-\varepsilon(2+\varepsilon)\alpha} + \frac{\varepsilon\beta\left[(1+\varepsilon)K\left(\varepsilon K + \|P_{r,2}\|_2\right) + (1+\varepsilon)K\|P_{l,1}\|_2 + \|P_{l,1}\|_2\|P_{r,2}\|_2\right]}{(1-\varepsilon(2+\varepsilon)\alpha)\|X\|_2}. \tag{2.32}$$

Proof. By (2.26), the perturbed equation can be rewritten as

$$\begin{gathered}
A\tilde{X}C + D\tilde{X}B + P_{l,1}\left(\tilde{P}_{l,1}\tilde{E}\tilde{P}_{r,2} + F\left(\tilde{X}\right)\right)P_{r,2} = 0, \\
\tilde{X} = \tilde{P}_{r,1}\tilde{X}\tilde{P}_{l,2}.
\end{gathered} \tag{2.33}$$

It follows from (2.33) and Theorem 2.1 that the perturbed solution $\widetilde{X}$ can be expressed as

$$\widetilde{X} = \frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\omega D - A)^{-1}P_{l,1}\left(\widetilde{P}_{l,1}\widetilde{E}\widetilde{P}_{r,2} + F\left(\widetilde{X}\right)\right)P_{r,2}(i\omega C - B)^{-1}d\omega \equiv \Phi\left(\widetilde{X}\right). \tag{2.34}$$

For any $\widetilde{X} \in \mathbb{R}^{n\times m}$, we have

$$\begin{aligned}\left\|F\left(\widetilde{X}\right)\right\|_2 &\le \left(\left\|\widetilde{A}\right\|_2\|\Delta C\|_2 + \|\Delta A\|_2\|C\|_2 + \left\|\widetilde{D}\right\|_2\|\Delta B\|_2 + \|\Delta D\|_2\|B\|_2\right)\left\|\widetilde{X}\right\|_2 \\ &\le \varepsilon(2+\varepsilon)(\|A\|_2\|C\|_2 + \|D\|_2\|B\|_2)\left\|\widetilde{X}\right\|_2.\end{aligned} \tag{2.35}$$

According to the definition of the linear operator Φ, by making use of Lemma 2.2 and (2.35), we get, for any $\widetilde{X}_1, \widetilde{X}_2 \in \mathbb{R}^{n\times m}$,

$$\begin{aligned}\left\|\Phi\left(\widetilde{X}_1\right) - \Phi\left(\widetilde{X}_2\right)\right\|_2 &\le \left\|\frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\omega D - A)^{-1}P_{l,1}F\left(\widetilde{X}_1 - \widetilde{X}_2\right)P_{r,2}(i\omega C - B)^{-1}d\omega\right\|_2 \\ &\le \left\|F\left(\widetilde{X}_1 - \widetilde{X}_2\right)\right\|_2\sqrt{\|H_1\|_2\|H_2\|_2} \\ &\le \varepsilon(2+\varepsilon)(\|A\|_2\|C\|_2 + \|D\|_2\|B\|_2)\sqrt{\|H_1\|_2\|H_2\|_2}\left\|\widetilde{X}_1 - \widetilde{X}_2\right\|_2.\end{aligned} \tag{2.36}$$

It shows that if $\varepsilon(2+\varepsilon)(\|A\|_2\|C\|_2+\|D\|_2\|B\|_2)\sqrt{\|H_1\|_2\|H_2\|_2} < 1$, then Φ is a contractive linear operator. At this moment, by the fixed point theorem [14], $\Phi(\widetilde{X}) = \widetilde{X}$ has a unique solution.

It holds that

$$\begin{aligned}&\left\|\widetilde{P}_{l,1}\widetilde{E}\widetilde{P}_{r,2} - P_{l,1}EP_{r,2}\right\|_2 \\ &\quad= \left\|\widetilde{P}_{l,1}\widetilde{E}\widetilde{P}_{r,2} - P_{l,1}\widetilde{E}\widetilde{P}_{r,2} + P_{l,1}\widetilde{E}\widetilde{P}_{r,2} - P_{l,1}\widetilde{E}P_{r,2} + P_{l,1}\widetilde{E}P_{r,2} - P_{l,1}EP_{r,2}\right\|_2 \\ &\quad\le \left\|\widetilde{P}_{l,1} - P_{l,1}\right\|_2\left\|\widetilde{E}\right\|_2\left\|\widetilde{P}_{r,2}\right\|_2 + \|P_{l,1}\|_2\left\|\widetilde{E}\right\|_2\left\|\widetilde{P}_{r,2} - P_{r,2}\right\|_2 \\ &\qquad+ \|P_{l,1}\|_2\|\Delta E\|_2\|P_{r,2}\|_2 \\ &\quad\le \varepsilon K(1+\varepsilon)\|E\|_2\left(\varepsilon K + \|P_{r,2}\|_2\right) \\ &\qquad+ \varepsilon(1+\varepsilon)K\|P_{l,1}\|_2\|E\|_2 + \varepsilon\|P_{l,1}\|_2\|E\|_2\|P_{r,2}\|_2 \\ &\quad= \varepsilon\left[(1+\varepsilon)K\left(\varepsilon K + \|P_{r,2}\|_2\right) + (1+\varepsilon)K\|P_{l,1}\|_2 + \|P_{l,1}\|_2\|P_{r,2}\|_2\right]\|E\|_2.\end{aligned} \tag{2.37}$$

It follows from (2.35) that

$$\left\|F\left(\widetilde{X}\right)\right\|_2 \le \varepsilon(2+\varepsilon)(\|A\|_2\|C\|_2 + \|D\|_2\|B\|_2)\left(\|X\|_2 + \left\|\widetilde{X} - X\right\|_2\right). \tag{2.38}$$

By using (2.1), (2.34), (2.37), (2.38), and Lemma 2.2, we obtain

$$\begin{aligned}
\left\|\widetilde{X}-X\right\|_2 &= \left\|\frac{1}{2\pi}\int_{-\infty}^{\infty}(-i\omega D-A)^{-1}P_{l,1}\left(F\left(\widetilde{X}\right)+\widetilde{P}_{l,1}\widetilde{E}\widetilde{P}_{r,2}-P_{l,1}EP_{r,2}\right)P_{r,2}(i\omega C-B)^{-1}1d\omega\right\|_2 \\
&\le \left\|F\left(\widetilde{X}\right)+\widetilde{P}_{l,1}\widetilde{E}\widetilde{P}_{r,2}-P_{l,1}EP_{r,2}\right\|_2\sqrt{\|H_1\|_2\|H_2\|_2} \\
&\le \varepsilon(2+\varepsilon)\alpha\|X\|_2+\varepsilon(2+\varepsilon)\alpha\left\|\widetilde{X}-X\right\|_2 \\
&\quad +\varepsilon\beta\left[(1+\varepsilon)K\left(\varepsilon K+\|P_{r,2}\|_2\right)+(1+\varepsilon)K\|P_{l,1}\|_2+\|P_{l,1}\|_2\|P_{r,2}\|_2\right],
\end{aligned} \tag{2.39}$$

where α and β are defined in (2.31). Then, the inequality (2.32) of the relative error bound results from the above inequality. □

If D and C are also nonsingular, then $P_{r,1}=P_{l,1}=I_n$ and $P_{r,2}=P_{l,2}=I_m$. In this case, the projected generalized continuous-time Sylvester equation (1.1) reduces to the generalized continuous-time Sylvester equation

$$AXC+DXB+E=0. \tag{2.40}$$

The corresponding perturbed equation is

$$\widetilde{A}\widetilde{X}\widetilde{C}+\widetilde{D}\widetilde{X}\widetilde{B}+\widetilde{E}=0. \tag{2.41}$$

Let H_1 and H_2 be the solutions of the generalized continuous-time Lyapunov equations

$$\begin{aligned}
AH_1D^T+DH_1A^T+I_n&=0, \\
B^TH_2C+C^TH_2B+I_m&=0,
\end{aligned} \tag{2.42}$$

respectively.

Note that (2.37) reduces to

$$\left\|\widetilde{E}-E\right\|_2=\|\Delta E\|_2\le\varepsilon\|E\|_2. \tag{2.43}$$

Hence, we have

$$\begin{aligned}
\left\|\widetilde{X}-X\right\|_2 &\le \left\|F\left(\widetilde{X}\right)+\widetilde{E}-E\right\|_2\sqrt{\|H_1\|_2\|H_2\|_2} \\
&\le \varepsilon(2+\varepsilon)\alpha\|X\|_2+\varepsilon(2+\varepsilon)\alpha\left\|\widetilde{X}-X\right\|_2+\varepsilon\|E\|_2\sqrt{\|H_1\|_2\|H_2\|_2}
\end{aligned}$$

$$\begin{aligned} &\leq \varepsilon(2+\varepsilon)\alpha\|X\|_2 + \varepsilon(2+\varepsilon)\alpha\left\|\widetilde{X}-X\right\|_2 + \varepsilon\alpha\|X\|_2 \\ &= \varepsilon(3+\varepsilon)\alpha\|X\|_2 + \varepsilon(2+\varepsilon)\alpha\left\|\widetilde{X}-X\right\|_2. \end{aligned} \tag{2.44}$$

From the above inequality, we obtain the relative error bound for the generalized continuous-time Sylvester equation (2.40).

Theorem 2.5. *Assume that D and C are nonsingular, and the pencils $\lambda D - A$ and $\lambda C - B$ are c-stable. Let X be the unique solution of the generalized continuous-time Sylvester equation* (2.40)*. Let $\alpha = (\|A\|_2\|C\|_2 + \|D\|_2\|B\|_2)\sqrt{\|H_1\|_2\|H_2\|_2}$, where H_1 and H_2 are the solutions of the generalized continuous-time Lyapunov equation* (2.42)*, respectively. If $\varepsilon(2+\varepsilon)\alpha < 1$, then the perturbed equation* (2.41) *has a unique solution $\widetilde{X}$. Moreover, the relative error satisfies the following inequality:*

$$\frac{\left\|\widetilde{X}-X\right\|_2}{\|X\|_2} \leq \frac{\varepsilon(3+\varepsilon)\alpha}{1-\varepsilon(2+\varepsilon)\alpha}. \tag{2.45}$$

This result shows that $(\|A\|_2\|C\|_2 + \|D\|_2\|B\|_2)\sqrt{\|H_1\|_2\|H_2\|_2}$ may be used to measure the sensitivity of the solution of the generalized continuous-time Sylvester equation (2.40).

3. Conclusions

In this paper, we have studied the perturbation analysis for the projected generalized continuous-time Sylvester equation and the generalized continuous-time Sylvester equation. By making use of solutions of two special projected generalized continuous-time Sylvester equations, we obtain the perturbation bounds based on the Euclidean norm for their solutions.

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