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Letter to the Editor

Comment on "Highly Efficient Sigma Point Filter for Spacecraft Attitude and Rate Estimation"

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In light of the intuition that a better symmetrical structure can further increase the numerical accuracy, the paper by Fan and Zeng (2009) developed a new sigma point construction strategy for the unscented Kalman filter (UKF), namely, geometric simplex sigma points (GSSP). This comment presents a different perspective from the standpoint of the numerical integration. In this respect, the GSSP constitutes an integration formula of degree 2 with equal weights. Then, we demonstrate that the GSSP can be derived through the orthogonal transformation from the basic points set of degree 2. Moreover, the method presented in this comment can be used to construct more accurate sigma points set for certain dynamic problems.

With the intuition that a better symmetry property provides a better numerical behavior [1], addressed the construction strategies to make the best symmetric structure in simplex sigma point set and derived the so-called geometric simplex sigma points (GSSP) for Euclidean geometric space. As compared with the previously exiting simplex sigma points set, the GSSP has a symmetric structure and a lower computational expense, is numerically more accurate, and can be used in a variety of 3-dimensional modeled dynamic problems.

In this comment we will show that the GSSP can also be derived from the integration rule of degree 2. Embedding the Gaussian assumption in the Bayesian filter we can reach the idea that the functional recursion of the Bayesian filter reduces to an algebraic recursion operating only on conditional means and covariances which share the same structure of Gaussian weighted integrals whose integrands are all of the form *nonlinear function* $\times$ *Gaussian density*. The multidimensional integrals are usually intractable for systems involving nonlinearity, so the recursive estimation problem boils down to how to compute the integrals using approximate methods. There are many well-known numerical integration methods such as Gauss-Hermite quadrature, cubature rules, fully symmetric integration rule, and central-difference-based methods that can be used to handle such integrals [2–4]. The unscented transformation (UT) used in the traditional unscented Kalman filter (UKF) can be interpreted as either fully symmetric integration rule or cubature rule of degree 3. The

simplex UT can also be interpreted as a numerical integration formula of degree 2 [3]. Next we will focus on the numerical integration formula of degree 2 in order to derive the GSSP.

Before getting involved in further details, we first introduce some definitions when constructing the exact monomials rule as follows [3, 4].

Definition 1. Consider the monomials of the form $\prod_{i=1}^{d} x_i^{\alpha_i}$, where the powers α_i are nonnegative integers and $\sum_{i=1}^{d} \alpha_i \leq p$, a rule said to have precision p if it can integrate such monomials accurately and it is not exact for monomials of degree $p+1$.

The numerical integration formulas are conducted by approximating the integrals with the weighted sum of an elaborately chosen set of points as follows [5–7]:

$$\int_{R^n} g(x) \cdot W(x) \approx \sum_k \alpha_k g(\chi_k), \tag{1}$$

where R^n is a region in an n-dimensional, real, Euclidean space, $x = (x_1, x_2, x_3, \ldots, x_n)$ is the state variable, α_k are constants, and χ_k are points in the space. The integral weight is a Gaussian distribution as discussed above. Since an arbitrary Gaussian distribution with mean μ and covariance Σ can always be transformed into the unit Gaussian distribution as (see prove in [4])

$$\int g(x) \cdot N(x; \mu, \Sigma) dx = \int g(A\xi + \mu) \cdot N(\xi; 0, I) d\xi, \tag{2}$$

where $AA^T = \Sigma$ and I is the identity matrix, we can start by considering the multidimensional unit Gaussian integral. Based on Definition 1, we can construct a rule, which is exact up to degree 2 by determining the weighted points set χ_k such that it is exact for selections $g_i(\xi) = 1$, $g_i(\xi) = \xi_i$, $g_{i,j}(\xi) = \xi_i \xi_j$, $i \neq j$, and $g_{i,i}(\xi) = \xi_i^2$. The true values of the integrals are

$$I_0 = \int 1 \cdot N(\xi; 0, I) d\xi = 1, \tag{3}$$

$$\begin{aligned} I_1 &= \int \xi_i \cdot N(\xi; 0, I) d\xi = 0, \quad i = 1, 2, \ldots n, \\ I_2 &= \int \xi_i^2 \cdot N(\xi; 0, I) d\xi = 1, \quad i = 1, 2, \ldots n, \\ I_{1 \times 1} &= \int \xi_i \xi_j \cdot N(\xi; 0, I) d\xi = 0, \quad i \neq j = 1, 2, \ldots n. \end{aligned} \tag{4}$$

In [5] Stroud had proved that $n+1$ is the minimum number of points for equally weighted degree 2 formulas. Let us define $n+1$ equally weighted points

$$\chi = (\chi_1, \chi_2, \ldots, \chi_{n+1}), \tag{5}$$

where $\chi_k = (\chi_{k,1}, \chi_{k,2}, \ldots, \chi_{k,n})^T$, $k = 1, 2, \ldots, n+1$. In order to calculate (4) accurately using these points through (1), we can get the following equations:

$$\begin{aligned} \frac{1}{n+1}\sum_{k=1}^{n+1}\chi_k &= 0, \\ \frac{1}{n+1}\sum_{k=1}^{n+1}\chi_k \cdot \chi_k^T &= I_n, \end{aligned} \tag{6}$$

where I_n is the n-dimensional identity matrix. Any equally weighted points set that fulfills (6) can approximate the unit Gaussian integral accurately up to degree 2. References [6, 7] have presented a basic points set that fulfills such conditions with the form as

$$\begin{aligned} \chi_{k,2r-1} &= \sqrt{2}\cos\frac{2rk\pi}{n+1}, \\ \chi_{k,2r} &= \sqrt{2}\sin\frac{2rk\pi}{n+1}, \end{aligned} \tag{7}$$

$r = 1, 2, \ldots, [n/2]$, and if n is odd, $\chi_{k,n} = (-1)^k$. $[n/2]$ is the greatest integer not exceeding $n/2$. When $n = 3$, the basic points set is

$$S_1 = [\chi_1 \mid \chi_2 \mid \chi_3 \mid \chi_4] = \begin{bmatrix} 0 & -\sqrt{2} & 0 & \sqrt{2} \\ \sqrt{2} & 0 & -\sqrt{2} & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}. \tag{8}$$

Next we will give a theorem through which we can get the GSSP from the basic points set of degree 2.

Theorem 1. *Assume that $n+1$ equally weighted points set as that in (5) constitutes an integration formula of degree 2. A is an $n \times n$ orthogonal matrix. Then, $A\chi$ also constitutes an integration formula of degree 2.*

Proof. By defining a matrix

$$M = \begin{bmatrix} \chi_{1,1} & \chi_{2,1} & \chi_{3,1} & \cdots & \chi_{n+1,1} \\ \chi_{1,2} & \chi_{2,2} & \chi_{3,2} & \cdots & \chi_{n+1,2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \chi_{1,n} & \chi_{2,n} & \chi_{3,n} & \cdots & \chi_{n+1,n} \end{bmatrix}, \tag{9}$$

we can rewrite (6) as

$$MM^T = (n+1)I_n, \tag{10}$$

where I_n is the n-dimensional identity matrix. A is an orthogonal matrix, so

$$AM(AM)^T = AMM^TA^T = (n+1)I_nAA^T = (n+1)I_n. \tag{11}$$

Hence, $A\chi$ also fulfills (6) which completes the proof. □

For the 3-dimensional Euclidean space, there are many orthogonal matrixes. Here, we use the direction cosine matrix (DCM) which is widely used in the practical systems such as guidance and navigation [8]. The DCM that rotates an angle ϕ about $u = [0\ \ 0\ \ 1]^T$ is

$$C(\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{12}$$

By a simple computation, it is obvious that

$$S_2 = C\left(\frac{3\pi}{4}\right)S_1 = \begin{bmatrix} 1 & 1 & -1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}. \tag{13}$$

Since all the points share equal weight, S_2 is virtually just the GSSP derived in [1].

Up to this point we have derived the GSSP through the numerical integration formulas method. Compared with the intuitionistic method in [1], our method is more principled in mathematical terms. Although Theorem 1 is proposed for integration formula of degree 2, it can be generalized for different degrees, that is, the orthogonal transformation on the numerical integration formula will not change its accurate degree. Reference [7] also presented the points set of degree 3, that is,

$$\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_{2n}), \tag{14}$$

where $\gamma_k = (\gamma_{k,1}, \gamma_{k,2}, \ldots, \gamma_{k,n})^T$, $k = 1, 2, \ldots, 2n$ with

$$\begin{aligned} \gamma_{k,2r-1} &= \sqrt{2}\cos\left(\frac{(2r-1)k\pi}{n}\right), \\ \gamma_{k,2r} &= \sqrt{2}\sin\left(\frac{(2r-1)k\pi}{n}\right), \end{aligned} \tag{15}$$

$r = 1, 2, \ldots, [n/2]$, and if n is odd, $\gamma_{k,n} = (-1)^k$. $[n/2]$ is the greatest integer not exceeding $n/2$. It can be proven that the points set (14) can be derived through orthogonal transformation on the cubature points set with the form [4]

$$\lambda = \left[\sqrt{n}e_k \ \ -\sqrt{n}e_k\right], \quad k = 1, \ldots, n, \tag{16}$$

where e_i denotes a unit vector to the direction of coordinate axis i.

As can be seen from (16) the distance of the cubature point from the mean is proportional to $\sqrt{n}$. So, for high-dimensional problems, the cubature points set bears the nonlocal sampling problem [9–11]. For many kinds of nonlinearities (such as exponents or trigonometric functions), this can lead to significant difficulties. In contrast, the points set (14) does not bear such nonlocal sampling problem. Under this condition, the points set (14) is more accurate than the cubature points set (16). Therefore, to this respect, we can derive different sigma points set which may be more accurate for certain dynamic problems by the presented method.

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