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Research Article

Control and Synchronization of the Fractional-Order Lorenz Chaotic System via Fractional-Order Derivative

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The unstable equilibrium points of the fractional-order Lorenz chaotic system can be controlled via fractional-order derivative, and chaos synchronization for the fractional-order Lorenz chaotic system can be achieved via fractional-order derivative. The control and synchronization technique, based on stability theory of fractional-order systems, is simple and theoretically rigorous. The numerical simulations demonstrate the validity and feasibility of the proposed method.

1. Introduction

The theory of fractional-order derivatives can be dated back to the 17th century [1] and developed comprehensively in the last century due to its application in a wide variety of scientific and technological fields such as thermal, viscoelastic, acoustic, electrochemical, rheological, and polymeric disciplines [1, 2]. On the other hand, it has been shown that many fractional-order dynamical systems, as some well-known integer-order systems, can also display complex bifurcation and chaotic phenomena. For example, the fractional-order Lorenz system, the fractional-order Chen system, the fractional-order Lü system, and the fractional-order unified system also exhibit chaotic behavior. Due to its potential applications in secure communication and control processing, the fractional-order chaotic systems have been studied extensively in recent years in many aspects such as chaotic phenomena, chaotic control, chaotic synchronization, and other related studies [3–12].

It is verified that the fractional-order controllers can have better disturbance rejection ratios and less sensitivity to plant parameter variations compared to the traditional

controllers [13]. A fractional-order controller is presented to stabilize the unstable equilibrium points of integer orders chaos systems [13, 14]. But the previously presented in [13, 14] only discussed the control problem for integer orders chaos systems, not for fractional-order chaotic systems. Up to now, to the best of our knowledge, amongst all kinds of chaos control and chaos synchronization for the fractional-order chaotic systems, very few results on control and synchronization of fractional-order chaotic systems are presented via fractional-order derivative. Motivated by the above discussion, a novel control method for the fractional-order Lorenz chaotic system is investigated in this paper. A fractional-order controller is presented to stabilize the unstable equilibrium points of the fractional-order Lorenz chaotic system via fractional-order derivative, and a fractional-order controller is presented to synchronize the fractional-order Lorenz chaotic system via fractional-order derivative. The control and synchronization technique, based on stability theory of fractional-order systems, is simple and theoretically rigorous. The numerical simulations demonstrate the validity and feasibility of the proposed method.

2. The Fractional Derivatives and the Fractional-Order Lorenz Chaotic System

The Caputo definition of the fractional derivative, which sometimes is called smooth fractional derivative, is described as

$$D^q f(t) = \frac{1}{\Gamma(m-q)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{q+1-m}} d\tau, \quad m-1 < q < m, \tag{2.1}$$

where D^q denotes the Caputo definition of the fractional derivative. m is the smallest integer larger than q, and $f^{(m)}(t)$ is the m-order derivative in the usual sense. $\Gamma(\cdot)$ is gamma function.

The Lorenz system, as the first chaotic model, revealed the complex and fundamental behaviors of the nonlinear dynamical systems. In 2003, I. Grigorenko and E. Grigorenko [4] pointed out that the fractional-order Lorenz system exhibits chaotic behavior for fractional-order $q \geq 0.993$. The fractional-order Lorenz system is described as follows:

$$\begin{aligned} D^q x_1 &= 10(x_2 - x_1), \\ D^q x_2 &= 28x_1 - x_2 - x_1 x_3, \\ D^q x_3 &= x_1 x_2 - \frac{8x_3}{3}, \end{aligned} \tag{2.2}$$

where $0 < q < 1$. In this paper, we choose $q = 0.995$ for the fractional-order Lorenz chaotic system.

Now, we discuss the numerical solution of fractional differential equations. All the numerical simulation of fractional-order system in this paper is based on [3]. We can set

$h = T/N$, $t_n = nh$ $(n = 0,1,2\ldots,N)$, and initial condition $(x_1(0), x_2(0), x_3(0))$. So, the fractional-order Lorenz chaotic system (2.2) can be discretized as follows:

$$\begin{aligned}
x_1(n+1) &= x_1(0) + \frac{h^q}{\Gamma(q+2)}\left[10\left(x_2^p(n+1) - x_1^p(n+1)\right) + \sum_{j=0}^{n} \alpha_{1,j,n+1} \times 10\left(x_2(j) - x_1(j)\right)\right],\\
x_2(n+1) &= x_2(0) + \frac{h^q}{\Gamma(q+2)}\left[28x_1^p(n+1) - x_2^p(n+1) - x_1^p(n+1)x_3^p(n+1)\right.\\
&\qquad \left. + \sum_{j=0}^{n} \alpha_{2,j,n+1}\left(28x_1(j) - x_2(j) - x_1(j)x_3(j)\right)\right],\\
x_3(n+1) &= x_3(0) + \frac{h^q}{\Gamma(q+2)}\left[x_1^p(n+1)x_2^p(n+1) - \frac{8x_3^p(n+1)}{3}\right.\\
&\qquad \left. + \sum_{j=0}^{n} \alpha_{3,j,n+1}\left(x_1(j)x_2(j) - \frac{8x_3(j)}{3}\right)\right],
\end{aligned} \tag{2.3}$$

where

$$\begin{aligned}
x_1^p(n+1) &= x_1(0) + \frac{1}{\Gamma(q)}\sum_{j=0}^{n} b_{1,j,n+1} \times 10\left(x_2(j) - x_1(j)\right),\\
x_2^p(n+1) &= x_2(0) + \frac{1}{\Gamma(q)}\sum_{j=0}^{n} b_{2,j,n+1}\left(28x_1(j) - x_2(j) - x_1(j)x_3(j)\right),\\
x_3^p(n+1) &= x_3(0) + \frac{1}{\Gamma(q)}\sum_{j=0}^{n} b_{3,j,n+1}\left(x_1(j)x_2(j) - \frac{8x_3(j)}{3}\right),
\end{aligned} \tag{2.4}$$

and for $i = 1,2,3$,

$$\begin{aligned}
\alpha_{i,j,n+1} &= \begin{cases} n^{q+1} - (n-q)(n+1)^q, & j = 0,\\ (n-j+2)^{q+1} + (n-j)^{q+1} - 2(n-j+1)^{q+1}, & 1 \le j \le n,\\ 1, & j = n+1, \end{cases}\\
b_{i,j,n+1} &= \frac{h^q}{q}\left[(n-j+1)^q - (n-j)^q\right], \quad 0 \le j \le n.
\end{aligned} \tag{2.5}$$

The error of this approximation is described as follows:

$$|x_i(t_n) - x_i(n)| = o(h^p), \quad p = \min(2, 1+q). \tag{2.6}$$

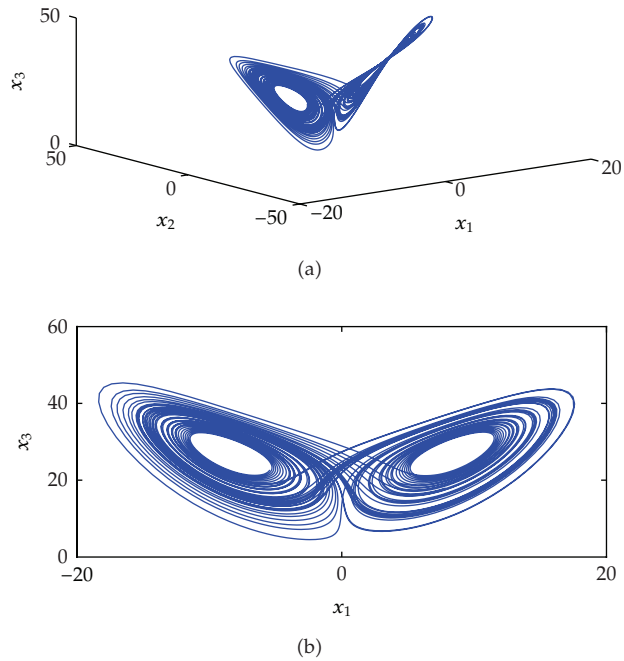

Figure 1: Chaotic attractors of the fractional-order Lorenz chaotic system (2.2) for $q = 0.995$.

Using the above numerical solution for fractional-order Lorenz chaotic system (2.2), the chaotic attractor of fractional-order Lorenz chaotic system (2.2) for $q = 0.995$ is shown in Figure 1.

3. Stabilizing the Unstable Equilibrium Points of the Fractional-Order Lorenz Chaotic System via Fractional-Order Derivative

It is obvious that the fractional-order Lorenz chaotic system (2.2) has three unstable equilibrium points. The unstable equilibrium points are $p_0 = (0,0,0)$ and $p_\pm = (\pm\sqrt{72}, \pm\sqrt{72}, 27)$. In this section, we will discuss how to stabilize the unstable equilibrium points of the fractional-order Lorenz chaotic system (2.2) via fractional-order derivative. First, let us present the stability theorem for linear commensurate fractional-order systems and nonlinear commensurate fractional-order systems.

Lemma 3.1 (see [13, 15]). *The following linear commensurate fractional-order autonomous system*

$$D^q x = Ax, \quad x(0) = x_0 \tag{3.1}$$

is asymptotically stable if and only if $|\arg \lambda| > 0.5\pi q$ *is satisfied for all eigenvalues* (λ) *of matrix* A. *Also, this system is stable if and only if* $|\arg \lambda| \geq 0.5\pi q$ *is satisfied for all eigenvalues of matrix* A, *and those critical eigenvalues which satisfy* $|\arg \lambda| = 0.5\pi q$ *have geometric multiplicity one, where* $0 < q < 1$, $x \in R^n$ *and* $A \in R^{n\times n}$.

Lemma 3.2 (see [16, 17]). *The fixed points of the following nonlinear commensurate fractional-order autonomous system:*

$$D^q x = f(x) \tag{3.2}$$

is asymptotically stable if all eigenvalues (λ) of the Jacobian matrix $A = \partial f / \partial x$ evaluated at the fixed points satisfy $|\arg \lambda| > 0.5\pi q$, where $0 < q < 1$, $x \in R^n$, $f : R^n \to R^n$ are continuous nonlinear vector functions, and the fixed points of this nonlinear commensurate fractional-order system are calculated by solving equation $f(x) = 0$.

3.1. Stabilizing the Unstable Equilibrium Point $p_0 = (0,0,0)$ via Fractional-Order Derivative

Now, let us design a controller for fractional-order Lorenz chaotic system (2.2) via fractional-order derivative, and we can obtain the following results.

Theorem 3.3. *Let the controlled fractional-order Lorenz chaotic system be*

$$\begin{aligned} D^q x_1 &= 10(x_2 - x_1), \\ D^q x_2 &= 28x_1 - x_2 - x_1 x_3 + u_1(x_1), \\ D^q x_3 &= x_1 x_2 - \frac{8x_3}{3}, \end{aligned} \tag{3.3}$$

where $u_1(x_1) = -k_{11} D^q x_1 - k_{12} x_1$ is the fractional-order controller, and k_{1i} $(i = 1,2)$ is the feedback coefficient. If $k_{12} = 38 + 10k_{11}$ and $-10k_{11} - 11 \le 0$, then system (3.3) will asymptotically converge to the unstable equilibrium point $p_0 = (0,0,0)$.

Proof. The Jacobi matrix of the controlled fractional-order Lorenz chaotic system (3.3) at equilibrium $p_0 = (0,0,0)$ is

$$J = \begin{bmatrix} -10 & 10 & 0 \\ 28 + 10k_{11} - k_{12} & -10k_{11} - 1 & 0 \\ 0 & 0 & -\dfrac{8}{3} \end{bmatrix}. \tag{3.4}$$

Because $k_{12} = 38 + 10k_{11}$, so

$$J = \begin{bmatrix} -10 & 10 & 0 \\ -10 & -10k_{11} - 1 & 0 \\ 0 & 0 & -\dfrac{8}{3} \end{bmatrix}. \tag{3.5}$$

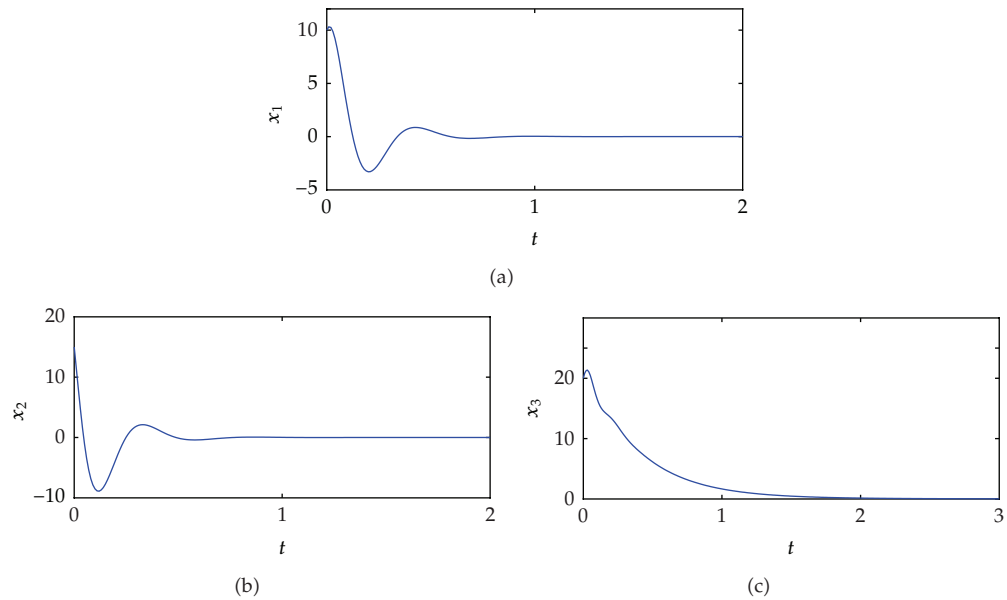

Figure 2: Time evolution of the equilibrium point $p_0 = (0, 0, 0)$.

Its characteristic equation is $(\lambda + 8/3)[\lambda^2 + (10k_{11} + 11)\lambda + 10(10k_{11} + 1) + 100] = 0$, and the eigenvalues are

$$\lambda_{1,2} = \frac{-(10k_{11} + 11) \pm \sqrt{(10k_{11} + 11)^2 - 40(10k_{11} + 11)}}{2}, \quad \lambda_3 = -\frac{8}{3}. \tag{3.6}$$

Because $-10k_{11} - 11 \leq 0$, so

$$|\arg \lambda_i| > \frac{\pi}{2} > 0.5\pi q, \quad (i = 1, 2, 3). \tag{3.7}$$

According to Lemma 3.2, it implies that the equilibrium point $p_0 = (0, 0, 0)$ of system (3.3) is asymptotically stable, that is, the unstable equilibrium point $p_0 = (0, 0, 0)$ in fractional-order Lorenz system (2.2) can be stabilized via fractional-order derivative. The proof is completed. □

For example, choose $k_{11} = 0.1$, then $k_{12} = 39$. The corresponding numerical result is shown in Figure 2, in which the initial conditions are $(10, 15, 20)^T$ in this paper.

Theorem 3.4. *Consider the controlled fractional-order Lorenz chaotic system as follows:*

$$\begin{aligned} D^q x_1 &= 10(x_2 - x_1) + u_2(x_2), \\ D^q x_2 &= 28x_1 - x_2 - x_1 x_3, \\ D^q x_3 &= x_1 x_2 - \frac{8x_3}{3}, \end{aligned} \tag{3.8}$$

where $u_2(x_2) = -k_{21}D^q x_2 - k_{22}x_2$ *is the fractional-order controller, and* k_{2i} $(i = 1, 2)$ *is the feedback coefficient. If* $k_{22} = 38 - k_{21}$ *and* $-28k_{21} - 11 \leq 0$, *then system* (3.8) *will gradually converge to the unstable equilibrium point* $p_0 = (0, 0, 0)$.

Proof. The Jacobi matrix of the controlled fractional-order Lorenz chaotic system (3.8) at equilibrium $p_0 = (0, 0, 0)$ is

$$J = \begin{bmatrix} -10 - 28k_{21} & 10 - k_{21} - k_{22} & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\dfrac{8}{3} \end{bmatrix}. \tag{3.9}$$

If $k_{22} = 38 - k_{21}$, then the Jacobi matrix is

$$J = \begin{bmatrix} -10 - 28k_{21} & -28 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -\dfrac{8}{3} \end{bmatrix}. \tag{3.10}$$

Its characteristic equation is $(\lambda + 8/3)[\lambda^2 + (28k_{21} + 11)\lambda + (28k_{21} + 10) + 784] = 0$, and the eigenvalues are

$$\lambda_{1,2} = \frac{-(28k_{21} + 11) \pm \sqrt{(28k_{21} + 11)^2 - 4[(28k_{21} + 10) + 784]}}{2}, \quad \lambda_3 = -\frac{8}{3}. \tag{3.11}$$

Because $-28k_{21} - 11 \leq 0$, so

$$\left|\arg \lambda_i\right| > \frac{\pi}{2} > 0.5\pi q, \quad (i = 1, 2, 3). \tag{3.12}$$

According to Lemma 3.2, it implies that the equilibrium point $p_0 = (0, 0, 0)$ of system (3.8) is asymptotically stable, that is, the unstable equilibrium point $p_0 = (0, 0, 0)$ in fractional-order Lorenz system (2.2) can be stabilized via fractional-order derivative. The proof is completed. □

For example, choose $k_{21} = 0.1$, then $k_{22} = 37.9$. The corresponding numerical result is shown in Figure 3.

3.2. Stabilizing the Unstable Equilibrium Point $p_\pm = (\pm\sqrt{72}, \pm\sqrt{72}, 27)$ via Fractional-Order Derivative

Let us design a fractional-order controller for fractional-order Lorenz chaotic system (2.2), and we can yield the following results.

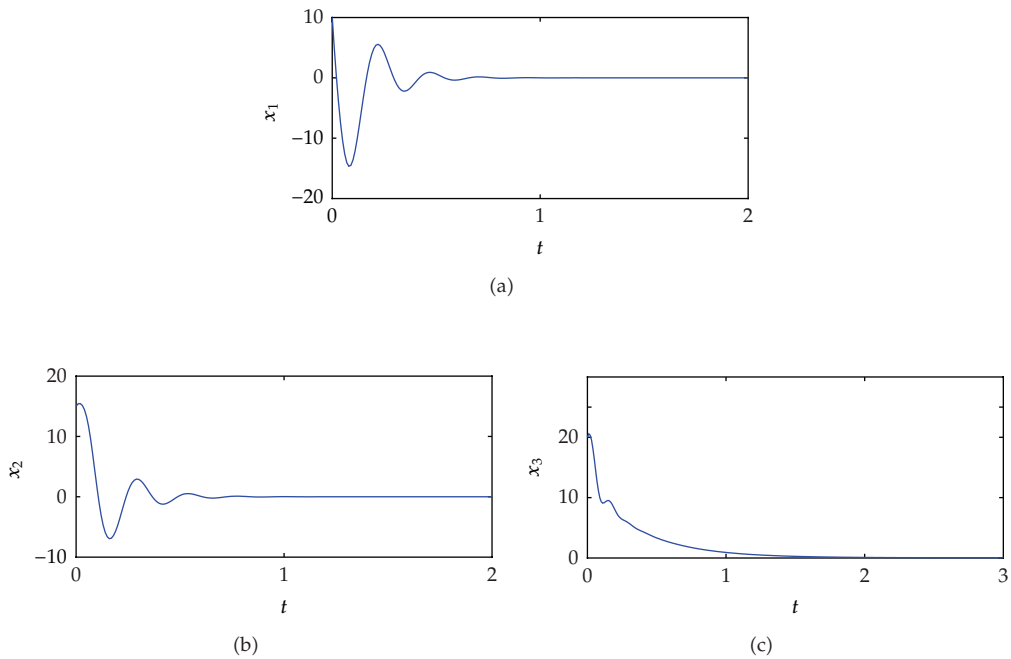

Figure 3: Time evolution of the equilibrium point $p_0 = (0, 0, 0)$.

Theorem 3.5. *Consider that the controlled fractional-order Lorenz chaotic system is*

$$\begin{aligned} D^q x_1 &= 10(x_2 - x_1), \\ D^q x_2 &= 28x_1 - x_2 - x_1 x_3 + w_1(x_1), \\ D^q x_3 &= x_1 x_2 - \frac{8x_3}{3}, \end{aligned} \tag{3.13}$$

where $w_1(x_1) = -k_1 D^q x_1$ is the fractional-order controller, and k_1 is the feedback coefficient. If $k_1 > (-155 + \sqrt{24769})/60$, then system (3.13) *will gradually converge to the unstable equilibrium point $p_+ = (\sqrt{72}, \sqrt{72}, 27)$.*

Proof. Let $x = x_1 - \sqrt{72}$, $y = x_2 - \sqrt{72}$, $z = x_3 - 27$, and the controlled fractional-order Lorenz chaotic system (3.13) can be

$$\begin{aligned} D^q x &= 10(y - x), \\ D^q y &= x - y - xz - \sqrt{72}z - k_1 D^q x, \\ D^q z &= xy + \sqrt{72}x + \sqrt{72}y - \frac{8z}{3}. \end{aligned} \tag{3.14}$$

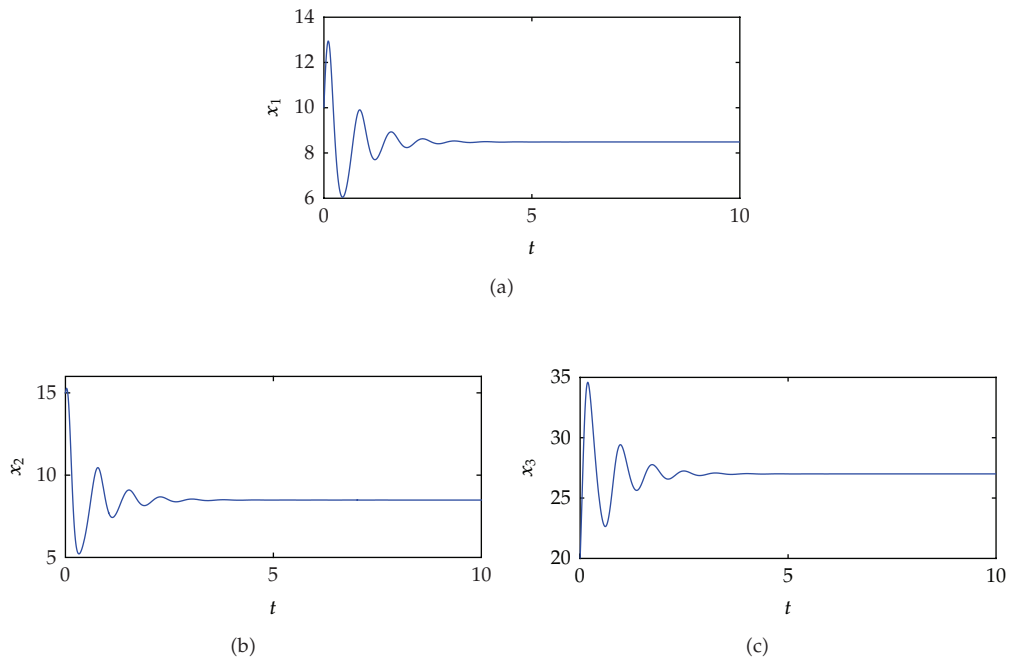

Figure 4: Time evolution of the equilibrium point $p_+ = (\sqrt{72}, \sqrt{72}, 27)$.

The Jacobi matrix of the fractional-order system (3.14) at equilibrium $(0, 0, 0)$ is

$$J = \begin{bmatrix} -10 & 10 & 0 \\ 10k_1 + 1 & -10k_1 - 1 & -\sqrt{72} \\ \sqrt{72} & \sqrt{72} & -\dfrac{8}{3} \end{bmatrix}. \tag{3.15}$$

Its characteristic equation is $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$, where $a_1 = 10k_1 + 41/3$, $a_2 = 8(10k_1 + 11)/3 + 72$, and $a_3 = 1440$. If $k_1 > (-155 + \sqrt{24769})/60$, then $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, and $a_1a_2 - a_3 > 0$. So,

$$|\arg \lambda_i| > \frac{\pi}{2} > 0.5\pi q, \quad (i = 1, 2, 3). \tag{3.16}$$

According to Lemma 3.2, it implies that the equilibrium point $p_+ = (\sqrt{72}, \sqrt{72}, 27)$ of system (3.13) is asymptotically stable, that is, the unstable equilibrium point $p_+ = (\sqrt{72}, \sqrt{72}, 27)$ in fractional-order Lorenz system (2.2) can be stabilized via fractional-order derivative. The proof is completed. □

For example, choose $k_1 = 1$. The corresponding numerical result is shown in Figure 4.

Theorem 3.6. *Consider that the controlled fractional-order Lorenz chaotic system is*

$$\begin{aligned} D^q x_1 &= 10(x_2 - x_1) + w_2(x_2), \\ D^q x_2 &= 28x_1 - x_2 - x_1 x_3, \\ D^q x_3 &= x_1 x_2 - \frac{8x_3}{3}, \end{aligned} \tag{3.17}$$

where $w_2(x_2) = -k_2 D^q x_2$ *is the fractional-order controller, and* k_2 *is the feedback coefficient. If* $(-7616 - \sqrt{56765440})/1248 < k_2 < (-7616 + \sqrt{56765440})/1248$, *then system* (3.17) *will gradually converge to the unstable equilibrium point* $p_+ = (\sqrt{72}, \sqrt{72}, 27)$.

Proof. Let $x = x_1 - \sqrt{72}$, $y = x_2 - \sqrt{72}$, $z = x_3 - 27$, and the controlled fractional-order Lorenz chaotic system (3.17) can be

$$\begin{aligned} D^q x &= 10(y - x) - k_2 D^q y, \\ D^q y &= x - y - xz - \sqrt{72}z, \\ D^q z &= xy + \sqrt{72}x + \sqrt{72}y - \frac{8z}{3}. \end{aligned} \tag{3.18}$$

The Jacobi matrix of the fractional-order system (3.18) at equilibrium $(0, 0, 0)$ is

$$J = \begin{bmatrix} -10 - k_2 & 10 + k_2 & k_2\sqrt{72} \\ 1 & -1 & -\sqrt{72} \\ \sqrt{72} & \sqrt{72} & -\frac{8}{3} \end{bmatrix}. \tag{3.19}$$

Its characteristic equation is $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$, where $a_1 = k_2 + 41/3$, $a_2 = 8(k_2 + 11)/3 + 72(1 - k_2)$, and $a_3 = 1440$. If $(-7616 - \sqrt{56765440})/1248 < k_2 < (-7616 + \sqrt{56765440})/1248$, then $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, and $a_1 a_2 - a_3 > 0$. So,

$$|\arg \lambda_i| > \frac{\pi}{2} > 0.5\pi q, \quad (i = 1, 2, 3). \tag{3.20}$$

According to Lemma 3.2, it implies that the equilibrium point $p_+ = (\sqrt{72}, \sqrt{72}, 27)$ of system (3.17) is asymptotically stable, that is, the unstable equilibrium point $p_+ = (\sqrt{72}, \sqrt{72}, 27)$ in fractional-order Lorenz system (2.2) can be stabilized via fractional-order derivative. The proof is completed. □

For example, choose $k_2 = -2$. The corresponding numerical result is shown in Figure 5.

Similarly, the fractional-order Lorenz chaotic system can be easily controlled to another unstable equilibrium point $p_- = (-\sqrt{72}, -\sqrt{72}, 27)$.

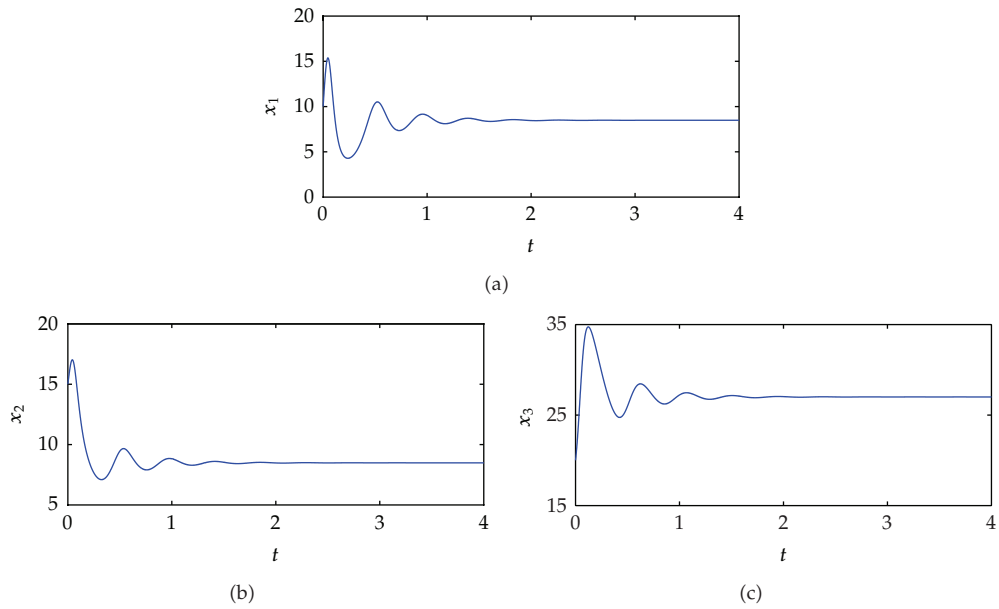

Figure 5: Time evolution of the equilibrium point $p_+ = (\sqrt{72}, \sqrt{72}, 27)$.

Remark 3.7. In general, there is no universal method to select the controller, and these particular controllers in our paper depend on the structure of the fractional-order chaotic system.

Remark 3.8. The differences of the present control strategy in our paper are compared to the result reported by Tavazoei and Haeri [13] as follows. First, we use the scalar controller in our paper, but they used the vector controller. Second, we discuss the control problem for fractional-order chaotic systems via fractional-order derivative, but they discussed the control problem for integer orders chaos systems via fractional-order derivative.

Remark 3.9. The differences of the present control strategy in our paper are compared to the result reported by Razminia et al. [18] as follows. We discuss the control problem for fractional-order chaotic systems via fractional-order derivative, but they discussed the control problem for fractional-order chaotic systems via state feedback, and they did not use fractional-order derivative.

4. Synchronizing the Fractional-Order Lorenz Chaotic System via Fractional-Order Derivative

Now, we design a feedback controller for the fractional-order Lorenz chaotic system (2.2) via fractional-order derivative and obtain the controlled response system (4.1)

$$\begin{aligned} D^q y_1 &= 10(y_2 - y_1), \\ D^q y_2 &= 28y_1 - y_2 - y_1 y_3 + V, \\ D^q y_3 &= y_1 y_2 - \frac{8y_3}{3}, \end{aligned} \tag{4.1}$$

where $V = k_1(D^q y_1 - D^q x_1) + k_2(y_1 - x_1) + y_1 y_3 - x_1 x_3$ is the fractional-order controller, and k_i $(i = 1, 2)$ is the feedback coefficient. Now, we can yield the following theorem.

Theorem 4.1. *If the feedback coefficients k_i $(i = 1, 2)$ satisfy*

$$28 - 10k_1 + k_2 = -10, \qquad -1 + 10(k_1 - 1) < 0, \tag{4.2}$$

then the fractional-order Lorenz chaotic system (2.2) *and the controlled fractional-order Lorenz chaotic system* (4.1) *achieved synchronization via fractional-order derivative.*

Proof. Define the synchronization error variables as follows:

$$e_i = y_i - x_i, \quad (i = 1, 2, 3). \tag{4.3}$$

By subtracting (2.2) from (4.1), we obtain

$$\begin{pmatrix} D^q e_1 \\ D^q e_2 \\ D^q e_3 \end{pmatrix} = A \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{4.4}$$

where

$$A = \begin{bmatrix} -10 & 10 & 0 \\ 28 - 10k_1 + k_2 & -1 + 10k_1 & 0 \\ x_2 & y_1 & -\dfrac{8}{3} \end{bmatrix}. \tag{4.5}$$

Because $28 - 10k_1 + k_2 = -10$, so

$$A = \begin{bmatrix} -10 & 10 & 0 \\ -10 & -1 + 10k_1 & 0 \\ x_2 & y_1 & -\dfrac{8}{3} \end{bmatrix}. \tag{4.6}$$

Therefore, the eigenvalues are

$$\lambda_{1,2} = \frac{\beta \pm \sqrt{\beta^2 + 40\beta}}{2}, \qquad \lambda_3 = -\frac{8}{3} < 0, \tag{4.7}$$

where $\beta = -1 + 10(k_1 - 1)$.

Because $\beta = -1 + 10(k_1 - 1) < 0$, then all the eigenvalues of matrix A have negative real part. Therefore,

$$\left|\arg \lambda_i(A)\right| > \frac{\pi}{2} > 0.5\pi q, \quad (i = 1, 2, 3). \tag{4.8}$$

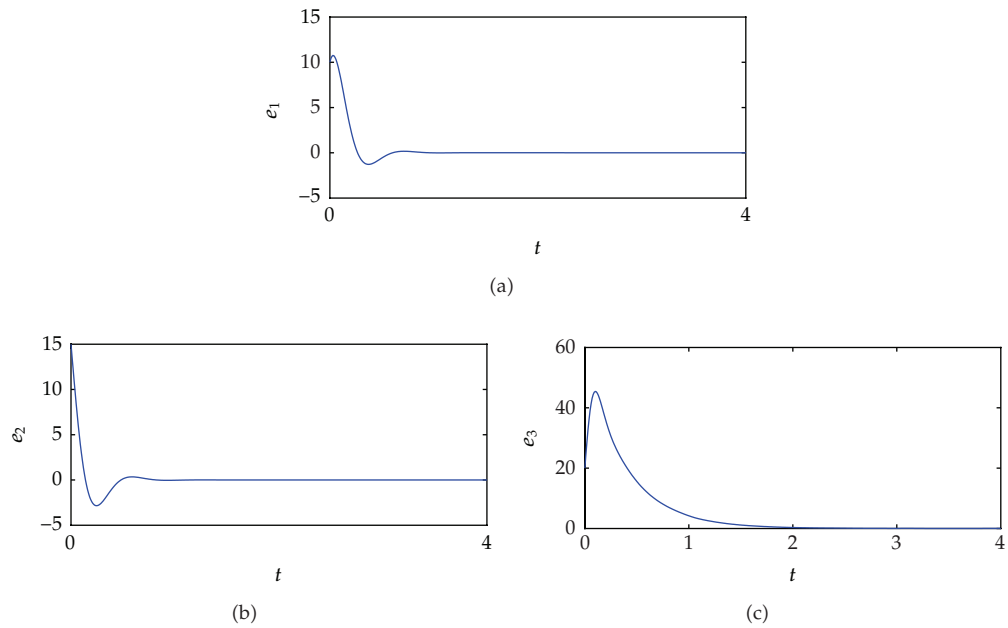

Figure 6: The time variation of synchronization error.

According to Lemma 3.2, it implies that the equilibrium point $(0, 0, 0)$ of error system (4.4) is asymptotically stable, that is, $\lim_{t\to+\infty} e_i = 0$ $(i = 1, 2, 3)$. So, the fractional-order Lorenz chaotic system (2.2) and the controlled fractional-order Lorenz chaotic system (4.1) achieved synchronization via fractional-order derivative. The proof is completed. □

Next, in order to verify the effectiveness and feasibility of the proposed synchronization scheme, the corresponding numerical simulations are given. For example, choose $k_1 = -0.1$ and $k_2 = -39$; the time variation of synchronization error is shown in Figure 6. The initial conditions are $\mathbf{x}(0) = (10, 20, 30)^{\mathbf{T}}$, and $\mathbf{y}(0) = (20, 35, 50)^{\mathbf{T}}$ respectively.

5. Conclusion

Using fractional-order derivative, we can stabilize the unstable equilibrium points of the fractional-order Lorenz chaotic system and realize chaos synchronization for the fractional-order Lorenz chaotic system. The control technique in our paper is simple and theoretically rigorous. Some examples are also given to illustrate the effectiveness of the theoretical result. This proposed control method is different from the previous works and can be applied to other fractional-order chaotic systems.

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