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Research Article

Adaptive State-Feedback Stabilization for High-Order Stochastic Nonlinear Systems Driven by Noise of Unknown Covariance

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This paper further considers more general high-order stochastic nonlinear system driven by noise of unknown covariance and its adaptive state-feedback stabilization problem. A smooth state-feedback controller is designed to guarantee that the origin of the closed-loop system is globally stable in probability.

1. Introduction

In this paper, we consider the following high-order stochastic nonlinear system:

$$\begin{aligned} dx_1 &= x_2^p dt + f_1(x_1)dt + g_1(x_1)\Sigma d\omega, \\ dx_2 &= x_3^p dt + f_2(\overline{x}_2)dt + g_2(\overline{x}_2)\Sigma d\omega, \\ &\vdots \\ dx_n &= u^p dt + f_n(\overline{x}_n)dt + g_n(\overline{x}_n)\Sigma d\omega, \end{aligned} \tag{1.1}$$

where $x = (x_1, \ldots, x_n) \in R^n$, $u \in R$, are the state and control input, respectively. $\overline{x}_i = (x_1, \ldots, x_i)$, $i = 1, \ldots, n$. $p \geq 1$ is odd integer. w is an r-dimensional standard Wiener process defined in a complete probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ with Ω being a sample space, $\mathcal{F}$ being a σ-field, $\{\mathcal{F}_t\}_{t\geq 0}$ being a filtration, and P being the probability measure. $\Sigma : R_+ \to R^{r\times r}$ is an

unknown bounded nonnegative definite Borel measurable matrix function and $\Sigma\Sigma^T$ denotes the infinitesimal covariance function of the driving noise $\Sigma d\omega$. $f_i : R^i \to R$ and $g_i : R^i \to R^r$, $i = 1, \ldots, n$, are assumed to be smooth with $f_i(0) = 0$ and $g_i(0) = 0$.

When $p = 1$, system (1.1) reduced to the well-known normal form whose study on feedback control problem has achieved great development in recent years. According to the difference of selected Lyapunov functions, the existing literature on controller design can be mainly divided into two types. One type is based on quadratic Lyapunov functions which are multiplied by different weighting functions, see, for example, [1–5] and the references therein. Another essential improvement belongs to Krstić and Deng. By introducing the quartic Lyapunov function, [6, 7] presented asymptotical stabilization control in the large under the assumption that the nonlinearities equal to zero at the equilibrium point of the open-loop system. Subsequently, for several classes of stochastic nonlinear systems with unmodeled dynamics and uncertain nonlinear functions, by combining Krstić and Deng's method with stochastic small-gain theorem [8], and with dynamic signal and changing supply function [9, 10], different adaptive output-feedback control schemes are studied.

When $p > 1$, some intrinsic features of (1.1), such as that its Jacobian linearization is neither controllable nor feedback linearizable, lead to the existing design tools hardly applicable to this kind of systems. Motivated by the fruitful deterministic results in [11, 12] and the related papers and based on stochastic stability theory in [13–15], and so forth, [16] firstly considered, this class of systems with stochastic inverse dynamics. Subsequently, [17–21] considered respectively, the state-feedback stabilization problem for more general systems with different structures. [22, 23] solved the output-feedback stabilization, and [24] addressed the inverse optimal stabilization.

All the papers mentioned above, however, only consider the case of $\Sigma\Sigma^T = I$. In this paper, we will further consider more general high-order stochastic nonlinear system driven by noise of unknown covariance and its adaptive state-feedback stabilization problem. A smooth state-feedback controller is designed to guarantee that the origin of the closed-loop system is globally stable in probability. A simulation example verifies the effectiveness of the control scheme.

The paper is organized as follows. Section 2 provides some preliminary results. Section 3 gives the state-feedback controller design and stability analysis, following a simulation example in Section 4. In Section 5, we conclude the paper.

2. Preliminary Results

The following notations definitions and lemmas are to be used throughout the paper.

R_+ stands for the set of all nonnegative real numbers, R^n is the n-dimensional Euclidean space, $R^{n\times m}$ is the space of real $n \times m$-matrixes. $\mathcal{C}^2$ denotes the family of all the functions with continuous second partial derivatives. $|x|$ is the usual Euclidean norm of a vector x. $\|X\| = (\mathrm{Tr}\{XX^T\})^{1/2}$, where $\mathrm{Tr}\{X\}$ is its trace when X is a square matrix and X^T denotes the transpose of X. $\mathcal{K}$ denotes the set of all functions: $R_+ \to R_+$, which are continuous, strictly increasing and vanishing at zero; $\mathcal{K}_\infty$ is the set of all functions which are of class $\mathcal{K}$ and unbounded; $\mathcal{KL}$ denotes the set of all functions $\beta(s,t)$: $R_+ \times R_+ \to R_+$, which are of class $\mathcal{K}$ for each fixed t and decrease to zero as $t \to \infty$ for each fixed s. To simplify the procedure, we sometimes denote $\chi(t)$ by χ for any variable $\chi(t)$.

Consider the nonlinear stochastic system

$$dx = f(x)dt + g(x)d\omega, \tag{2.1}$$

where $x \in R^n$ is the state, w is an r-dimensional independent Wiener process with incremental covariance $\Sigma\Sigma^T dt$, that is, $E\{d\omega d\omega^T\} = \Sigma\Sigma^T dt$, where Σ is a bounded function taking values in the set of nonnegative definite matrices, $f : R^n \to R^n$ and $g : R^n \to R^{n\times r}$ are locally Lipschitz functions.

Definition 2.1 (see [13]). For any given $V(x) \in \mathcal{C}^2$ associated with stochastic system (2.1), the differential operator $\mathcal{L}$ is defined as

$$\mathcal{L}V(x) \triangleq \frac{\partial V(x)}{\partial x} f(x) + \frac{1}{2}\operatorname{Tr}\left\{ g^T(x) \frac{\partial^2 V(x)}{\partial x^2} g(x) \right\}. \tag{2.2}$$

Definition 2.2 (see [25]). For the stochastic system (2.1) with $f(0) = 0$, $g(0) = 0$, the equilibrium $x(t) = 0$ is globally asymptotically stable (GAS) in probability if for any $\xi > 0$, there exists a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that

$$P\{|x(t)| < \beta(|x_0|, t)\} \geq 1 - \xi, \qquad t \geq 0, \quad \forall x_0 \in R^n \setminus \{0\}. \tag{2.3}$$

Lemma 2.3 (see [14]). *Consider the stochastic system* (2.1). *If there exist a $\mathcal{C}^2$ function $V(x)$, class $\mathcal{K}_\infty$ function α_1 and α_2, constants $c_1 > 0$ and $c_2 \geq 0$, and a nonnegative function $W(x)$ such that for all $x \in R^n$, $t \geq 0$*

$$\alpha_1(|x|) \leq V(x) \leq \alpha_2(|x|), \qquad \mathcal{L}V \leq -c_1 W(x) + c_2, \tag{2.4}$$

then,

(a) *there exists an almost surely unique solution on $[0,\infty)$ for each $x_0 \in R^n$,*

(b) *when $c_2 = 0$, $f(0) = 0$, $g(0) = 0$, and $W(x)$ is continuous, then the equilibrium $x = 0$ is globally stable in probability and the solution $x(t)$ satisfies $P\{\lim_{t\to\infty} W(x(t)) = 0\} = 1$.*

Lemma 2.4 (see [12]). *Let x, y be real variables, for any positive integers m, n, positive real number b and nonnegative continuous function $a(\cdot)$, then*

$$a(\cdot)x^m y^n \leq b|x|^{m+n} + \frac{n}{m+n}\left(\frac{m+n}{m}\right)^{-m/n} (a(\cdot))^{(m+n)/n} b^{-m/n} |y|^{m+n}, \tag{2.5}$$

when $a(\cdot) = 1$, $b = (m/(m+n))d$, d is a positive constant, then the above inequality is

$$x^m y^n \leq \frac{m}{m+n} d|x|^{m+n} + \frac{n}{m+n} d^{-m/n} |y|^{m+n}. \tag{2.6}$$

Lemma 2.5 (see [12]). *Let x, y and z_i, $i = 1, \ldots, p$, be real variables and let $l_1(\cdot)$ and $l_2(\cdot)$ be smooth mappings. Then for any positive integers m, n and real number $N > 0$, there exist two nonnegative smooth functions $h_1(\cdot)$ and $h_2(\cdot)$ such that the following inequalities hold:*

(i) $|x^m[(y + xl_1(x))^n - (xl_1(x))^n]| \leq |x|^{m+n}/N + |y|^{m+n} h_1(x, y)$,

(ii) $|y^m(z_1^n + \cdots + z_p^n + y^n) l_2(z_1, \ldots, z_p, y)| \leq (1/N)\sum_{k=1}^{p} |z_k|^{m+n} + |y|^{m+n} h_2(z_1, \ldots, z_p, y)$.

Lemma 2.6 (see [12]). *Let $x_1, \ldots, x_n$, p, be positive real variables, then*

$$(x_1 + \cdots + x_n)^p \leq \max\left\{n^{p-1}, 1\right\}\left(x_1^p + \cdots + x_n^p\right). \tag{2.7}$$

3. Controller Design and Stability Analysis

The objective of this paper is to design a smooth state-feedback controller for system (1.1), such that the solution of the closed-loop system is GAS in probability. To achieve the control objective, we need the following assumption.

Assumption 3.1. There are nonnegative smooth functions f_{i1}, g_{i1}, $i = 1, \ldots, n$, such that

$$\left|f_i(\overline{x}_i)\right| \leq \left(\sum_{j=1}^{i}\left|x_j\right|^p\right) f_{i1}(\overline{x}_i), \qquad \left|g_i(\overline{x}_i)\right| \leq \left(\sum_{j=1}^{i}\left|x_j\right|^p\right) g_{i1}(\overline{x}_i). \tag{3.1}$$

3.1. Controller Design

Now, we give the controller design procedure by using the backstepping method.

First, we introduce the following coordinate change:

$$z_1 = x_1, \qquad z_i = x_i - \alpha_{i-1}\left(\overline{x}_{i-1}, \widehat{\theta}\right), \quad i = 2, \ldots, n, \tag{3.2}$$

where $\alpha_{i-1}(\overline{x}_{i-1}, \widehat{\theta})$, $i = 2, \ldots, n$, are smooth virtual controllers which will be designed later, $\widehat{\theta}$ is the estimation of θ, and

$$\theta \triangleq \max_{t\geq 0}\left\{\left\|\Sigma\Sigma^T\right\|^{(p+3)/2}, \left\|\Sigma\Sigma^T\right\|^{(p+3)/3}, \left\|\Sigma\Sigma^T\right\|\right\}. \tag{3.3}$$

Then, by Itô's differentiation rule, one has

$$\begin{aligned} dz_i &= d\left(x_i - \alpha_{i-1}\left(\overline{x}_{i-1}, \widehat{\theta}\right)\right) \\ &= \left(x_{i+1}^p + F_i(\overline{x}_i) - \sum_{l=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_l} x_{l+1}^p - \frac{1}{2}\sum_{k,m=1}^{i-1}\frac{\partial^2 \alpha_{i-1}}{\partial x_k \partial x_m} g_k(\overline{x}_k)\Sigma\Sigma^T g_m^T(\overline{x}_m) - \frac{\partial \alpha_{i-1}}{\partial \widehat{\theta}}\dot{\widehat{\theta}}\right)dt \\ &\quad + G_i(\overline{x}_i)\Sigma d\omega, \end{aligned} \tag{3.4}$$

where

$$F_i(\overline{x}_i) = f_i(\overline{x}_i) - \sum_{l=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_l} f_l(\overline{x}_l), \qquad G_i(\overline{x}_i) = g_i(\overline{x}_i) - \sum_{l=1}^{i-1}\frac{\partial \alpha_{i-1}}{\partial x_l} g_l(\overline{x}_l). \tag{3.5}$$

Step 1. Define the first Lyapunov function

$$V_1\left(z_1,\widehat{\theta}\right)=\frac{1}{4}z_1^4+\frac{1}{2\Gamma}\widetilde{\theta}^2, \tag{3.6}$$

where Γ is a positive constant, $\widetilde{\theta}=\theta-\widehat{\theta}$ is the parameter estimation error. By (3.2)–(3.4) and Assumption 3.1, there exist nonnegative smooth functions μ_{11} and μ_{15} such that

$$\begin{aligned}\mathcal{L}V_1&=z_1^3x_2^p+z_1^3f_1(x_1)+\frac{3}{2}z_1^2g_1(x_1)\Sigma\Sigma^Tg_1^T(x_1)-\frac{\widetilde{\theta}}{\Gamma}\dot{\widehat{\theta}}\\&\le z_1^3\left(x_2^p-\alpha_1^p\right)+z_1^3\alpha_1^p+z_1^{p+3}\mu_{11}(z_1)+z_1^{p+3}\mu_{15}(z_1)\theta-\frac{\widetilde{\theta}}{\Gamma}\dot{\widehat{\theta}}\\&\le z_1^3\left(x_2^p-\alpha_1^p\right)+z_1^3\alpha_1^p+z_1^{p+3}\mu_{11}(z_1)+z_1^{p+3}\mu_{15}(z_1)\sqrt{1+\widehat{\theta}^2}-\frac{\widetilde{\theta}}{\Gamma}\left(\dot{\widehat{\theta}}-\Gamma z_1^{p+3}\mu_{15}(z_1)\right).\end{aligned} \tag{3.7}$$

Choosing the first smooth virtual controller

$$\alpha_1\left(x_1,\widehat{\theta}\right)=-z_1\beta_1\left(z_1,\widehat{\theta}\right),\qquad \beta_1\left(z_1,\widehat{\theta}\right)=\left(c_{11}+\mu_{11}(z_1)+\mu_{15}(z_1)\sqrt{1+\widehat{\theta}^2}\right)^{1/p}, \tag{3.8}$$

and the tuning function

$$\tau_1(z_1)=\Gamma z_1^{p+3}\mu_{15}(z_1), \tag{3.9}$$

one has

$$\mathcal{L}V_1\le -c_{11}z_1^{p+3}+z_1^3\left(x_2^p-\alpha_1^p\right)-\frac{\widetilde{\theta}}{\Gamma}\left(\dot{\widehat{\theta}}-\tau_1\right), \tag{3.10}$$

where $c_{11}>0$ is a design parameter.

Step i $(2\le i\le n)$. For notational coherence, denote $u=x_{n+1}$. Assuming that at step $i-1$, one has

$$\mathcal{L}V_{i-1}\le -\sum_{j=1}^{i-1}c_{j,i-1}z_j^{p+3}-\left(\frac{\widetilde{\theta}}{\Gamma}+\sum_{j=2}^{i-1}z_j^3\frac{\partial\alpha_{j-1}}{\partial\widehat{\theta}}\right)\left(\dot{\widehat{\theta}}-\tau_{i-1}\right)+z_{i-1}^3\left(x_i^p-\alpha_{i-1}^p\right), \tag{3.11}$$

where $\tau_{i-1} = \tau_{i-2} + \Gamma z_{i-1}^{p+3}(\mu_{i-1,4} + \mu_{i-1,5})$. In the sequel, we will prove that (3.11) still holds for the ith Lyapunov function $V_i(\overline{z}_i, \widehat{\theta}) = V_{i-1}(\overline{z}_{i-1}, \widehat{\theta}) + (1/4)z_i^4$. By (3.4) and (3.11), one has

$$\begin{aligned}
\mathcal{L}V_i \le{}& \mathcal{L}V_{i-1} + z_i^3\left(x_{i+1}^p + F_i(\overline{x}_i) - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}x_{l+1}^p - \frac{1}{2}\sum_{k,m=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_k\partial x_m}g_k(\overline{x}_k)\Sigma\Sigma^T g_m^T(\overline{x}_m)\right)\\
&- z_i^3\frac{\partial\alpha_{i-1}}{\partial\widehat{\theta}}\dot{\widehat{\theta}} + \frac{3}{2}z_i^2\operatorname{Tr}\left\{\Sigma^T G_i^T(\overline{x}_i)G_i(\overline{x}_i)\Sigma\right\}\\
\le{}& -\sum_{j=1}^{i-1}c_{j,i-1}z_j^{p+3} - \left(\frac{\widetilde{\theta}}{\Gamma} + \sum_{j=2}^{i-1}z_j^3\frac{\partial\alpha_{j-1}}{\partial\widehat{\theta}}\right)\left(\dot{\widehat{\theta}} - \tau_{i-1}\right) + z_{i-1}^3\left(x_i^p - \alpha_{i-1}^p\right)\\
&+ z_i^3\left(x_{i+1}^p + F_i(\overline{x}_i) - \sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}x_{l+1}^p - \frac{1}{2}\sum_{k,m=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_k\partial x_m}g_k(\overline{x}_k)\Sigma\Sigma^T g_m^T(\overline{x}_m)\right)\\
&- z_i^3\frac{\partial\alpha_{i-1}}{\partial\widehat{\theta}}\dot{\widehat{\theta}} + \frac{3}{2}z_i^2\operatorname{Tr}\left\{\Sigma^T G_i^T(\overline{x}_i)G_i(\overline{x}_i)\Sigma\right\} + z_i^3\frac{\partial\alpha_{i-1}}{\partial\widehat{\theta}}\tau_{i-1} - z_i^3\frac{\partial\alpha_{i-1}}{\partial\widehat{\theta}}\tau_{i-1}.
\end{aligned} \tag{3.12}$$

To proceed further, an estimate for each term in the right-hand side of (3.12) is needed. Using Itô's differentiation rule, Lemmas 2.4–2.6, (3.2), and (3.3), it follows that

$$\begin{aligned}
z_{i-1}^3\left(x_i^p - \alpha_{i-1}^p\right) &= z_{i-1}^3\left((z_i + \alpha_{i-1})^p - \alpha_{i-1}^p\right)\\
&\le \xi_{i1}z_{i-1}^{p+3} + \mu_{i1}\left(\overline{z}_i, \widehat{\theta}\right)z_i^{p+3},\\
z_i^3F_i(\overline{x}_i) &\le |z_i|^3\sum_{j=1}^{i}|z_j|^p\rho_{i2j}\left(\overline{z}_i, \widehat{\theta}\right)\\
&\le \sum_{j=1}^{i-1}\xi_{i2j}z_j^{p+3} + \mu_{i2}\left(\overline{z}_i, \widehat{\theta}\right)z_i^{p+3},\\
-z_i^3\sum_{l=1}^{i-1}\frac{\partial\alpha_{i-1}}{\partial x_l}x_{l+1}^p &\le |z_i|^3\sum_{j=1}^{i-1}|z_j|^p\rho_{i3j}\left(\overline{z}_i, \widehat{\theta}\right)\\
&\le \sum_{j=1}^{i-1}\xi_{i3j}z_j^{p+3} + \mu_{i3}\left(\overline{z}_i, \widehat{\theta}\right)z_i^{p+3},\\
-\frac{1}{2}z_i^3\sum_{k,m=1}^{i-1}\frac{\partial^2\alpha_{i-1}}{\partial x_k\partial x_m}g_k(\overline{x}_k)\Sigma\Sigma^T g_m^T(\overline{x}_m) &\le |z_i|^3\sum_{j=1}^{i-1}|z_j|^{2p}\rho_{i4j}\left(\overline{z}_i, \widehat{\theta}\right)\left\|\Sigma\Sigma^T\right\|\\
&\le \sum_{j=1}^{i-1}\xi_{i4j}z_j^{p+3} + \mu_{i4}\left(\overline{z}_i, \widehat{\theta}\right)z_i^{p+3}\theta,\\
\frac{3}{2}z_i^2\operatorname{Tr}\left\{\Sigma^T G_i^T(\overline{x}_i)G_i(\overline{x}_i)\Sigma\right\} &\le z_i^2\sum_{j=1}^{i}z_j^{p+1}\rho_{i5j}\left(\overline{z}_i, \widehat{\theta}\right)\left\|\Sigma\Sigma^T\right\|
\end{aligned}$$

$$\begin{aligned}
&\leq \sum_{j=1}^{i-1} \xi_{i5j} z_j^{p+3} + \mu_{i5}\left(\overline{z}_i, \widehat{\theta}\right) z_i^{p+3}\theta, \\
-z_i^3 \frac{\partial \alpha_{i-1}}{\partial \widehat{\theta}} \tau_{i-1} &\leq |z_i|^3 \sum_{j=1}^{i-1} |z_j|^p \rho_{i6j}\left(\overline{z}_i, \widehat{\theta}\right) \\
&\leq \sum_{j=1}^{i-1} \xi_{i6j} z_j^{p+3} + \mu_{i6}\left(\overline{z}_i, \widehat{\theta}\right) z_i^{p+3},
\end{aligned} \tag{3.13}$$

where $\xi_{i1}, \xi_{i2j}, \xi_{i3j}, \xi_{i4j}, \xi_{i5j}, \xi_{i6j}, j = 1, \ldots, i-1$, are positive constants and $\mu_{i1}, \mu_{i2}, \mu_{i3}, \mu_{i4}, \mu_{i5}, \mu_{i6}, \rho_{i2j}, \rho_{i3j}, \rho_{i4j}, \rho_{i5j}, \rho_{i6j}, j = 1, \ldots, i$, are nonnegative smooth functions. Substituting (3.13) into (3.12), one has

$$\begin{aligned}
\mathcal{L}V_i \leq{}& -\sum_{j=1}^{i-1} c_{j,i-1} z_j^{p+3} + \xi_{i1} z_{i-1}^{p+3} - \left(\frac{\widetilde{\theta}}{\Gamma} + \sum_{j=2}^{i-1} z_j^3 \frac{\partial \alpha_{j-1}}{\partial \widehat{\theta}}\right)\left(\dot{\widehat{\theta}} - \tau_{i-1}\right) + z_i^3 \alpha_i^p \\
&+ z_i^{p+3}\left(\mu_{i1} + \mu_{i2} + \mu_{i3} + \mu_{i6} + (\mu_{i4} + \mu_{i5})\sqrt{1 + \widehat{\theta}^2}\right) + \frac{\widetilde{\theta}}{\Gamma}(\mu_{i4} + \mu_{i5})\Gamma z_i^{p+3} \\
&+ \sum_{j=1}^{i-1} (\xi_{i2j} + \xi_{i3j} + \xi_{i4j} + \xi_{i5j} + \xi_{i6j}) z_j^{p+3} + z_i^3\left(x_{i+1}^p - \alpha_i^p\right).
\end{aligned} \tag{3.14}$$

Choosing the ith smooth virtual controller α_i

$$\begin{aligned}
\alpha_i\left(\overline{x}_i, \widehat{\theta}\right) &= -z_i \beta_i\left(\overline{z}_i, \widehat{\theta}\right), \\
\beta_i\left(\overline{z}_i, \widehat{\theta}\right) &= \left(c_{ii} + \mu_{i1} + \mu_{i2} + \mu_{i3} + \mu_{i6} + (\mu_{i4} + \mu_{i5})\left(\sqrt{1 + \widehat{\theta}^2} + \sum_{j=2}^{i} z_j^3 \frac{\partial \alpha_{j-1}}{\partial \widehat{\theta}}\Gamma\right)\right)^{1/p},
\end{aligned} \tag{3.15}$$

and tuning function τ_i

$$\tau_i(\overline{z}_i) = \tau_{i-1}(\overline{z}_{i-1}) + \Gamma z_i^{p+3}(\mu_{i4} + \mu_{i5}), \tag{3.16}$$

and substituting (3.15) and (3.16) into (3.14), it follows that

$$\mathcal{L}V_i\left(\overline{z}_i, \widehat{\theta}\right) \leq -\sum_{j=1}^{i} c_{ji} z_j^{p+3} - \left(\frac{\widetilde{\theta}}{\Gamma} + \sum_{j=2}^{i} z_j^3 \frac{\partial \alpha_{j-1}}{\partial \widehat{\theta}}\right)\left(\dot{\widehat{\theta}} - \tau_i\right) + z_i^3\left(x_{i+1}^p - \alpha_i^p\right), \tag{3.17}$$

where $c_{ji} = c_{jj} - \xi_{j+1,1} - \sum_{k=2}^{6} \xi_{ikj}$, $j = 1, \ldots, i-1$.

Hence at step n, the smooth adaptive state-feedback controller

$$u = \alpha_n\left(\overline{x}_n, \widehat{\theta}\right) = -z_n\beta_n\left(\overline{z}_n, \widehat{\theta}\right), \qquad \dot{\widehat{\theta}} = \tau_n(\overline{z}_n),$$

$$\beta_n\left(\overline{z}_n, \widehat{\theta}\right) = \left(c_{nn} + \mu_{n1} + \mu_{n2} + \mu_{n3} + \mu_{n6} + (\mu_{n4} + \mu_{n5})\left(\sqrt{1+\widehat{\theta}^2} + \sum_{j=2}^{n} z_j^3 \frac{\partial \alpha_{j-1}}{\partial \widehat{\theta}}\Gamma\right)\right)^{1/p},$$

$$\tau_n(\overline{z}_n) = \Gamma z_1^{p+3}\mu_{15}(z_1) + \sum_{j=2}^{n}\Gamma z_j^{p+3}(\mu_{j4} + \mu_{j5}), \tag{3.18}$$

such that the nth Lyapunov function

$$V_n\left(\overline{z}_n, \widehat{\theta}\right) = \frac{1}{4}\sum_{j=1}^{n} z_j^4 + \frac{1}{2\Gamma}\widetilde{\theta}^2 \tag{3.19}$$

satisfies

$$\mathcal{L}V_n \le -\sum_{j=1}^{n} c_{jn} z_j^{p+3}, \tag{3.20}$$

where μ_{nl}, $l = 1, \ldots, 6$, are nonnegative smooth functions, c_{jn}, $j = 1, \ldots, n$, are constants, and

$$c_{jn} = c_{jj} - \xi_{j+1,1} - \sum_{k=2}^{6}\xi_{nkj}, \qquad j = 1, \ldots, n-1. \tag{3.21}$$

3.2. Stability Analysis

Theorem 3.2. *If Assumption 3.1 holds for the high-order stochastic nonlinear system* (1.1), *under the state-feedback controller* (3.18), *then*

(i) *the closed-loop system consisting of* (1.1), (3.2), (3.8), (3.9), (3.15), (3.16), *and* (3.18) *has an almost surely unique solution on* $[0, \infty)$ *for each* $(x_0, \widetilde{\theta}(0)) \in R^{n+1}$,

(ii) *the origin of the closed-loop system is globally stable in probability,*

(iii) $P\{\lim_{t\to\infty}|x(t)| = 0\} = 1$ *and* $P\{\lim_{t\to\infty}\widehat{\theta}(t)$ *exists and is finite*$\} = 1$.

Proof. It is easy to verify that $V_n(\overline{z}_n, \widehat{\theta})$ is $\mathcal{C}^2$ on $\overline{z}_n$ and $\widehat{\theta}$. For $j = 1, \ldots, n-1$, choose the design parameter $c_{jj} > \xi_{j+1,1} + \sum_{k=2}^{6}\xi_{nkj}$, $c_{nn} > 0$, then by (3.21), $c_{jn} > 0$, $j = 1, \ldots, n-1$. Since $V_n(\overline{z}_n, \widehat{\theta})$ is continuous, positive, and radially bounded, by (3.20), (3.21), and Lemma 4.3 in [25], there exist two class $\mathcal{K}_\infty$ functions α_1 and α_2 such that $\alpha_1(|x|, |\widetilde{\theta}|) \le V_n(\overline{z}_n, \widehat{\theta}) \le \alpha_2(|x|, |\widetilde{\theta}|)$. Hence, the condition of Lemma 2.3 holds. □

By Lemma 2.3, it follows that conclusion (i), (ii) hold, and $P\{\lim_{t\to\infty}|z(t)| = 0\} = 1$. In view of $\alpha_i(0, \widehat{\theta}) = 0$ and $x_i = z_i + \alpha_{i-1}(\overline{x}_{i-1}, \widehat{\theta})$, one has $P\{\lim_{t\to\infty}|x(t)| = 0\} = 1$. By (3.20)

and the definition of $V_n(\overline{z}_n, \widehat{\theta})$ in (3.19), it holds that $\widetilde{\theta}(t)$ converges a.s. to a finite limit $\widetilde{\theta}_\infty$ as $t \to \infty$, therefore $\widehat{\theta}(t)$ converges a.s. to a finite limit as $t \to \infty$.

4. A Simulation Example

Consider a two-order nonlinear stochastic system

$$\begin{aligned} dx_1 &= x_2^3 dt + f_1(x_1)dt + x_1^3 \Sigma d\omega, \\ dx_2 &= u^3 dt + f_2(\overline{x}_2)dt + x_2^3 \Sigma d\omega, \end{aligned} \tag{4.1}$$

where $f_1(x_1) = x_1^3$, $f_2(\overline{x}_2) = x_1 x_2^2$. By Lemma 2.4, one gets $|f_1(x_1)| \le |x_1|^3$, $|g_1(x_1)| \le |x_1|^3$, $|f_2(\overline{x}_2)| \le (1/3)|x_1|^3 + (2/3)|x_2|^3$, $g_2(\overline{x}_2) \le |x_2|^3$. We choose $f_{11}(x_1) = 1$, $g_{11}(x_1) = 1$, $f_{21}(\overline{x}_2) = 2/3$, $g_{21}(\overline{x}_2) = 1$, Assumption 3.1 is satisfied.

We now give the design of state-feedback controller for system (4.1).

Step 1. Define $z_1 = x_1$, $V_1(z_1, \widehat{\theta}) = (1/4)z_1^4 + (1/2\Gamma)\widetilde{\theta}^2$. A smooth virtual controller

$$\begin{aligned} \alpha_1\left(x_1, \widehat{\theta}\right) &= -z_1 \beta_1\left(z_1, \widehat{\theta}\right), \\ \beta_1\left(z_1, \widehat{\theta}\right) &= \left(c_{11} + 1 + \mu_{15}(z_1)\sqrt{1 + \widehat{\theta}^2}\right)^{1/3}, \end{aligned} \tag{4.2}$$

and the tuning function

$$\tau_1(z_1) = \Gamma z_1^6 \mu_{15}(z_1) \tag{4.3}$$

yield $\mathcal{L}V_1(z_1, \widehat{\theta}) \le -c_{11} z_1^6 + z_1^3(x_2^3 - \alpha_1^3) + (\widetilde{\theta}/\Gamma)(\dot{\widehat{\theta}} - \tau_1)$, where

$$\mu_{15}(z_1) = \frac{3}{2} z_1^2, \qquad \theta(t) = \max_{t \ge 0}\left\{ \left\|\Sigma(t)\Sigma(t)^T\right\|^3, \left\|\Sigma(t)\Sigma(t)^T\right\|^2, \left\|\Sigma(t)\Sigma(t)^T\right\| \right\}. \tag{4.4}$$

Step 2. Defining $z_2 = x_2 - \alpha_1(x_1, \widehat{\theta})$, $V_2(\overline{z}_2, \widehat{\theta}) = V_1(z_1, \widehat{\theta}) + (1/4)z_2^4$, by (3.12), one has

$$\begin{aligned} \mathcal{L}V_2\left(z_1, \widehat{\theta}\right) \le{}& -c_{11} z_1^6 + z_1^3\left(x_2^3 - \alpha_1^3\right) + \frac{\widetilde{\theta}}{\Gamma}\left(\dot{\widehat{\theta}} - \tau_1\right) \\ &+ z_2^3\left(u^3 + F_2(\overline{x}_2) - \frac{\partial \alpha_1}{\partial x_1} x_2^3 - \frac{1}{2}\frac{\partial^2 \alpha_1}{\partial x_1^2} g_1(\overline{x}_1)\Sigma\Sigma^T g_1^T(\overline{x}_1)\right) \\ &- z_2^3 \frac{\partial \alpha_1}{\partial \widehat{\theta}}\dot{\widehat{\theta}} + \frac{3}{2} z_2^2 \operatorname{Tr}\left\{\Sigma^T G_2^T(\overline{x}_2) G_2(\overline{x}_2)\Sigma\right\}, \end{aligned} \tag{4.5}$$

where $F_2(\overline{x}_2) = f_2(\overline{x}_2) - (\partial\alpha_1/\partial x_1)f_1(x_1)$, $G_2(\overline{x}_2) = g_2(\overline{x}_2) - (\partial\alpha_1/\partial x_1)g_1(x_1)$. By Lemma 2.4, the definition of z_2, and (4.2), one can obtain

$$\begin{aligned}
z_1^3\left(x_2^3 - \alpha_1^3\right) &\le \frac{1}{2}d_{11}z_1^6 + \frac{1}{2}d_{11}^{-1}z_2^6 + 3\left(\frac{2}{3}d_{12}z_1^6 + \frac{1}{3}d_{12}^{-1}\beta_1^3 z_2^6 + \frac{5}{6}d_{13}z_1^6 + \frac{1}{6}d_{13}^{-1}\beta_1^{12}z_2^6\right) \\
&= \xi_{21}z_1^6 + \mu_{21}\left(z_1, \widehat{\theta}\right)z_2^6, \\
z_2^3F_2(\overline{x}_2) &\le 2|z_2|^3\left(\frac{1}{3}|z_1|^3 + \frac{2}{3}|z_2|^3 + |z_1|^3\beta_1^2 - \frac{\partial\alpha_1}{\partial x_1}z_1^3\right) \\
&= \xi_{221}z_1^6 + \mu_{22}\left(\overline{z}_2, \widehat{\theta}\right), \\
-z_2^3\frac{\partial\alpha_1}{\partial x_1}x_2^3 &\le |z_2|^3\left|\frac{\partial\alpha_1}{\partial x_1}\right|\left(z_2^3 - 3z_2^2z_1\beta_1 + 3z_2z_1\beta_1 - z_1^3\beta_1^3\right) \\
&\le \xi_{231}z_1^6 + \mu_{23}\left(\overline{z}_2, \widehat{\theta}\right)z_2^6, \\
-\frac{1}{2}z_2^3\frac{\partial^2\alpha_1}{\partial x_1^2}g_1^3\Sigma\Sigma^T g_1^3 &\le |z_2|^3\frac{\partial^2\alpha_1}{\partial x_1^2}z_1^6\left\|\Sigma\Sigma^T\right\| \\
&\le \xi_{241}z_1^6 + \mu_{24}\left(\overline{z}_2, \widehat{\theta}\right)z_2^6\theta;
\end{aligned} \tag{4.6}$$

by (4.3), Lemmas 2.4, 2.6, and the definitions of z_2 and $G_2(\overline{x}_2)$, one has

$$\begin{aligned}
-z_2^3\frac{\partial\alpha_1}{\partial\widehat{\theta}}\tau_1 &\le |z_2|^3|z_1|^3\left(\frac{3}{2}\frac{\partial\alpha_1}{\partial\widehat{\theta}}\Gamma|z_1|^5\right) \\
&\le \xi_{261}z_1^6 + \mu_{26}\left(\overline{z}_2, \widehat{\theta}\right)z_2^6, \\
\frac{3}{2}z_2^2\operatorname{Tr}\left\{\Sigma^T G_2^T(\overline{x}_2)G_2(\overline{x}_2)\Sigma\right\} &\le \frac{3}{2}z_2^2\left(\left(z_2 - z_1\beta_1\right)^3 - \frac{\partial\alpha_1}{\partial x_1}z_1^3\right)^2\left\|\Sigma\Sigma^T\right\| \\
&\le z_2^2\left(\left(3\cdot 2^5\beta_1^6 + 3\cdot\left(\frac{\partial\alpha_1}{\partial x_1}\right)^2\right)z_1^6 + 2^5z_2^6\right)\left\|\Sigma\Sigma^T\right\| \\
&\le \xi_{251}z_1^6 + \mu_{25}\left(\overline{z}_2, \widehat{\theta}\right)z_2^6\theta,
\end{aligned} \tag{4.7}$$

where $\xi_{21} = (1/2)d_{11} + 2d_{12} + (5/2)d_{13}$, $\xi_{221} = (1/3)d_{21} + d_{22} + d_{23}$, $\xi_{231} = (1/2)d_{31} + (1/2)d_{32}$, $\xi_{241} = (1/2)d_{41}$, $\xi_{251} = (2/3)d_{51}$, $\xi_{261} = (1/2)d_{61}$, $\mu_{21}(z_1, \widehat{\theta}) = (1/2)d_{11}^{-1} + d_{12}^{-1}\beta_1^3(z_1, \widehat{\theta}) + (1/2)d_{13}^{-5}\beta_1^{12}(z_1, \widehat{\theta})$, $\mu_{22}(\overline{z}_2, \widehat{\theta}) = (1/3)d_{21}^{-1} + (4/3) + d_{22}^{-1}\beta_1^4 + d_{23}^{-1}(\partial\alpha_1/\partial x_1)^2$, $\mu_{23}(\overline{z}_2, \widehat{\theta}) = (\partial\alpha_1/\partial x_1)(1 - 2\beta_1^{3/2} + \beta_1^6) + (\partial\alpha_1/\partial x_1)^2((1/2)d_{31}^{-1} + (1/2)d_{32}^{-1})$, $\mu_{24}(\overline{z}_2, \widehat{\theta}) = (1/2)d_{41}^{-1}(\partial^2\alpha_1/\partial x_1^2)^2 z_1^6$, $\mu_{25}(\overline{z}_2, \widehat{\theta}) = 32z_2^2 + d_{51}^{-1}(128\beta_1^{18} + 4(\partial\alpha_1/\partial x_1)^6)z_1^6$, $\mu_{26}(\overline{z}_2, \widehat{\theta}) = (9/8)d_{61}^{-1}(\partial\alpha_1/\partial\widehat{\theta})^2\Gamma^2 z_1^{10}$, d_{11}, d_{12}, d_{13}, d_{21}, d_{22}, d_{23}, d_{31}, d_{32}, d_{41}, d_{51}, d_{61} are positive constants.

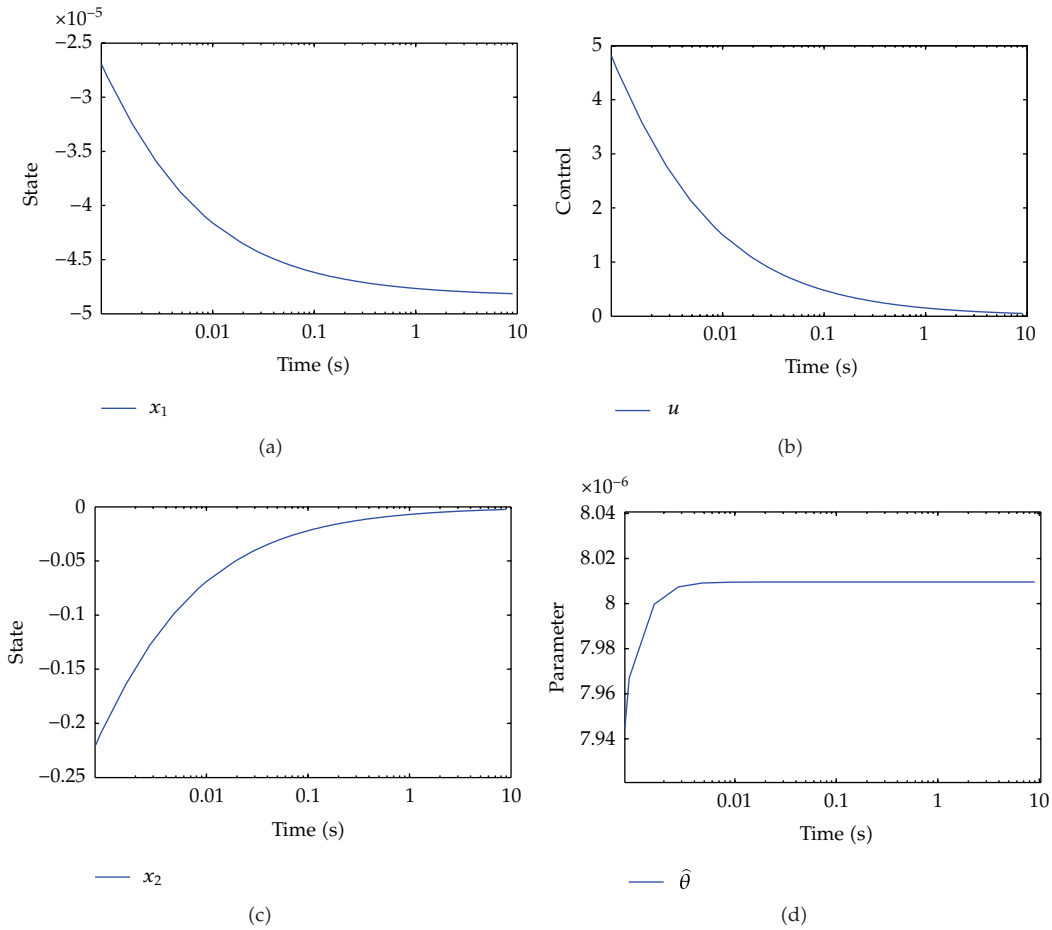

Figure 1: The responses of closed-loop system (4.1)–(4.3), (4.8).

Choosing the smooth adaptive controller

$$\begin{gathered} u = -z_2\beta_2\left(\overline{z}_2, \widehat{\theta}\right), \qquad \dot{\widehat{\theta}} = \tau_2(\overline{z}_2), \\ \tau_2(\overline{z}_2) = \Gamma z_1^6 \mu_{15}(z_1) + \Gamma z_2^6\left(\mu_{24}\left(\overline{z}_2, \widehat{\theta}\right) + \mu_{25}\left(\overline{z}_2, \widehat{\theta}\right)\right), \\ \beta_2\left(\overline{z}_2, \widehat{\theta}\right) = \left(c_{22} + \mu_{21} + \mu_{22} + \mu_{23} + \mu_{26} + (\mu_{24} + \mu_{25})\left(\sqrt{1 + \widehat{\theta}^2} + \Gamma z_2^3 \frac{\partial \alpha_1}{\partial \widehat{\theta}}\right)\right)^{1/3}, \end{gathered} \tag{4.8}$$

and substituting (4.6)–(4.8) into (4.5), one has

$$\mathcal{L}V_2 \le -c_{12} z_1^6 - c_{22} z_2^6, \tag{4.9}$$

where $c_{12} = c_{11} - \xi_{21} - \xi_{221} - \xi_{231} - \xi_{241} - \xi_{251} - \xi_{261} > 0$.

In simulation, we choose $\Sigma(t) \equiv 1$, the parameters $\Gamma = 1$, $c_{11} = 11$, $c_{22} = 1$, $d_{11} = 1$, $d_{12} = 1$, $d_{13} = 1$, $d_{21} = 1$, $d_{22} = 1$, $d_{23} = 1$, $d_{31} = 0.01$, $d_{32} = 1$, $d_{41} = 1$, $d_{51} = 1$, $d_{61} = 1$, the initial values $\theta(0) = 0$, $x_1(0) = 0$, $x_2(0) = -0.5$, the sampling period = 0.01. Figure 1 verifies the effectiveness of the control scheme.

5. Conclusion

In this paper, we further consider more general high-order stochastic nonlinear system driven by noise of unknown covariance and its adaptive state-feedback stabilization problem.

There is a still remaining problem to be investigated: under current investigation, how to design an output feedback controller for system (1.1) with Assumption 3.1?

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