Research Article

Multitarget Linear-Quadratic Control Problem: Semi-Infinite Interval

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We consider multitarget linear-quadratic control problem on semi-infinite interval. We show that the problem can be reduced to a simple convex optimization problem on the simplex.

1. Introduction

Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space, Z be its closed vector subspace, $h_1, \dots, h_m$, and c be vectors in H . Consider the following optimization problem:

$$\max_{1 \leq i \leq m} \|h - h_i\| \longrightarrow \min, \quad h \in c + Z. \quad (1.1)$$

Here $\|\cdot\|$ is the norm in H induced by the scalar product $\langle \cdot, \cdot \rangle$. In [1], we analyzed (1.1) using duality theory for infinite-dimensional second-order cone programming. We obtained a reduction of this problem to a finite-dimensional second-order cone programming and applied this result to a multitarget linear-quadratic control problem on a finite time interval. In this paper, we consider a reduction (1.1) to even simpler optimization problem of minimization of convex quadratic function on the $(m - 1)$ dimensional simplex. We then apply this result to the analysis of a multitarget linear-quadratic control problem on semi-infinite time interval. We show that the coefficients of the quadratic function admit a simple expressions in term of the original data.

2. Reduction to a Simple Quadratic Programming Problem

Let $f_i(h) = \|h - h_i\|^2$, $i = 1, 2, \dots, m$. It is obvious that (1.1) is equivalent to the following optimization problem:

$$\begin{aligned} z &\longrightarrow \min, \\ f_i(h) &\leq z, \quad i = 1, 2, \dots, m, \quad h \in c + Z. \end{aligned} \quad (2.1)$$

Consider the Lagrange function

$$\begin{aligned} \mathcal{L}(\lambda_1, \dots, \lambda_m, h, z) &= z + \sum_{i=1}^m \lambda_i (f_i(h) - z) \\ &= z \left(1 - \sum_{i=1}^m \lambda_i \right) + \sum_{i=1}^m \lambda_i f_i(h). \end{aligned} \quad (2.2)$$

Notice that despite the fact that our original problem is infinite dimensional, the usual KKT theorem holds true (see e.g., [2], page 72). It is also clear that Slater conditions are satisfied. Hence, optimality condition for (2.1) takes the form

$$\begin{aligned} \lambda_i &\geq 0, \quad \lambda_i (f_i(h) - z) = 0, \quad i = 0, 1, 2, \dots, m, \\ \frac{\partial \mathcal{L}}{\partial z} &= 0, \quad \sum_{i=1}^m \lambda_i \nabla f_i(h) \in Z^\perp, \end{aligned} \quad (2.3)$$

where $\nabla f_i(h) = 2(h - h_i)$, $i = 1, 2, \dots, m$, $Z^\perp$ is the orthogonal complement of Z in H . Conditions (2.3) lead to

$$\begin{aligned} \sum_{i=0}^m \lambda_i &= 1, \quad \lambda_i \geq 0, \quad i = 1, 2, \dots, m, \\ \pi_Z(h) &= \sum_{i=1}^m \lambda_i (\pi_Z h_i). \end{aligned} \quad (2.4)$$

Here $\pi_Z : H \rightarrow Z$ is the orthogonal projection. Let us form the Lagrange dual of (2.1). Consider

$$\varphi(\lambda_1, \lambda_2, \dots, \lambda_m) = \min \{ \mathcal{L}(\lambda_1, \dots, \lambda_m, h, z) : h \in c + Z, z \in Z \}. \quad (2.5)$$

Using (2.4), we obtain that

$$\varphi(\lambda_1, \lambda_2, \dots, \lambda_m) = \sum_{i=1}^m \lambda_i f_i(h(\lambda_1, \dots, \lambda_m)), \quad (2.6)$$

where

$$h(\lambda_1, \dots, \lambda_m) = \pi_{Z^\perp}(c) + \sum_{i=1}^m \lambda_i \pi_Z(h_i). \quad (2.7)$$

Notice that for any $h \in c + Z$, $\pi_{Z^\perp}(h) = \pi_{Z^\perp}(c)$. Here $\pi_{Z^\perp} : H \rightarrow Z^\perp$ is the orthogonal projection of H onto orthogonal complement $Z^\perp$ of Z . To further simplify (2.6), introduce the notation

$$h(\lambda) = \sum_{i=1}^m \lambda_i h_i. \quad (2.8)$$

Then

$$\begin{aligned} f_j(h(\lambda_1, \dots, \lambda_m)) &= \|\pi_Z(h(\lambda) - h_j) + \pi_{Z^\perp}(c - h_j)\|^2 \\ &= \|\pi_Z(h(\lambda) - \pi_Z(h_j))\|^2 + \|\pi_{Z^\perp}(c - h_j)\|^2 \\ &= \|\pi_Z(h(\lambda))\|^2 + \|\pi_Z(h_j)\|^2 - 2\langle \pi_Z(h(\lambda)), \pi_Z(h_j) \rangle \\ &\quad + \|\pi_{Z^\perp}(c - h_j)\|^2. \end{aligned} \quad (2.9)$$

Hence, according to (2.6), we have the following:

$$\begin{aligned} \varphi(\lambda_1, \dots, \lambda_m) &= \|\pi_Z(h(\lambda))\|^2 + \sum_{j=1}^m \lambda_j \|\pi_Z(h_j)\|^2 \\ &\quad - 2\langle \pi_Z(h(\lambda)), \pi_Z(h(\lambda)) \rangle + \sum_{j=1}^m \lambda_j \|\pi_{Z^\perp}(c - h_j)\|^2. \end{aligned} \quad (2.10)$$

We, hence, arrive at the following expression of φ :

$$\varphi(\lambda_1, \dots, \lambda_m) = -\left\| \pi_Z \left(\sum_{i=1}^m \lambda_i h_i \right) \right\|^2 + \sum_{j=1}^m \lambda_j \left(\|\pi_Z(h_j)\|^2 + \|\pi_{Z^\perp}(c - h_j)\|^2 \right). \quad (2.11)$$

We can simplify (2.11) somewhat. Notice that

$$\|\pi_{Z^\perp}(c - h_j)\|^2 = \|\pi_{Z^\perp}(c)\|^2 + \|\pi_{Z^\perp}(h_j)\|^2 - 2\langle \pi_{Z^\perp}(c), \pi_{Z^\perp}(h_j) \rangle. \quad (2.12)$$

Consequently,

$$\begin{aligned}
 \varphi(\lambda_1, \dots, \lambda_m) &= -\|\pi_Z(h(\lambda))\|^2 + \sum_{j=1}^m \lambda_j \|h_j\|^2 \\
 &\quad - 2\langle \pi_{Z^\perp}(c), \pi_{Z^\perp}(h(\lambda)) \rangle + \|\pi_{Z^\perp}(c)\|^2 \\
 &= -\|h(\lambda)\|^2 + \|\pi_{Z^\perp}(h(\lambda) - c)\|^2 + \sum_{j=1}^m \lambda_j \|h_j\|^2.
 \end{aligned} \tag{2.13}$$

Here,

$$h(\lambda) = \sum_{i=1}^m \lambda_i h_i. \tag{2.14}$$

Hence, the Lagrange dual to (2.1) takes the following form:

$$\begin{aligned}
 \varphi(\lambda_1, \dots, \lambda_m) &\longrightarrow \max, \\
 \sum_{i=1}^m \lambda_i &= 1, \quad \lambda_i \geq 0, \quad i = 1, 2, \dots, m.
 \end{aligned} \tag{2.15}$$

If $(\lambda_1^*, \dots, \lambda_m^*)$ is an optimal solution to (2.15), we can recover the optimal solution of the original problem using the relation (2.7), and $\varphi(\lambda_1^*, \dots, \lambda_m^*)$ gives the optimal value for the original problem (1.1).

3. Linear-Quadratic Case

Denoted by $L_2^n[0, \infty)$, the vector space of square integrable functions $f : [0, \infty) \rightarrow R^n$. Let $H = L_2^n[0, \infty) \times L_2^m[0, \infty)$, and

$$Z = \{(\alpha, \beta) \in H : \alpha \text{ is absolutely continuous on } [0, \infty), \dot{\alpha} = A\alpha + B\beta, \alpha(0) = 0\}. \tag{3.1}$$

Here A (respectively B) is an n by n (respectively n by m) matrix. Observe that

$$\begin{aligned}
 \langle (\alpha_1, \beta_1), (\alpha_2, \beta_2) \rangle &= \int_0^\infty [\alpha_1(t)^T \alpha_2(t) + \beta_1(t)^T \beta_2(t)] dt, \\
 (\alpha_i, \beta_i) &\in H, \quad i = 1, 2.
 \end{aligned} \tag{3.2}$$

In this setting, the problem (1.1) admits a natural interpretation as a linear-quadratic multitarget control problem. An interesting solution for this problem for $m = 2$ is described in [3]. In our approach, we need an explicit computation of the coefficients of the objective function (2.13) which in turn requires an explicit description of orthogonal projection π_Z . Such a description has been found in [4]. We briefly describe it here.

Theorem 3.1. *Let C be an antistable n by n matrix (i.e., real parts of all eigenvalues of C are positive). Consider the following system of linear differential equations:*

$$\dot{x} = Cx + f, \quad (3.3)$$

where $f \in L_2^n[0, \infty)$. Then there exists a unique solution $L(f)$ of (3.3) belonging to $L_2^n[0, \infty)$. Moreover, the map $L : L_2^n[0, \infty) \rightarrow L_2^n[0, \infty)$ is linear and bounded. Explicitly,

$$L(f)(t) = - \int_0^\infty e^{-C\tau} f(t + \tau) d\tau. \quad (3.4)$$

For the proof, see [4].

Consider the algebraic Riccati equation

$$KBB^TK + A^TK + KA - I = 0. \quad (3.5)$$

We assume that (3.5) has a real symmetric solution K_{st} such that the matrix

$$F = A + BB^TK_{st} \quad (3.6)$$

is stable (i.e., real parts of all eigenvalues of F are negative). Notice that such a solution exists if and only if the pair (A, B) is stabilizable. See, for example, [5].

Theorem 3.2. *We have the following:*

$$Z^\perp = \left\{ \left(\dot{p} + A^Tp, B^Tp \right); p \in L_2^n[0, \infty), p \text{ is absolutely continuous, } \dot{p} \in L_2^n[0, \infty) \right\}. \quad (3.7)$$

Given that $(\psi, \varphi) \in H$, we have

$$\psi = x - \left(\dot{p} + A^Tp \right), \quad (3.8)$$

$$\varphi = u - B^Tp, \quad (3.9)$$

where x is the solution of the differential equation

$$\dot{x} = \left(A + BB^TK_{st} \right) x + BB^T\rho + B\varphi, \quad x(0) = 0, \quad (3.10)$$

$$u = B^TK_{st}x + B^T\rho + \varphi, \quad (3.11)$$

$$p = K_{st}x + \rho, \quad (3.12)$$

and ρ is a unique solution to the differential equation

$$\dot{\rho} = - \left(A + BB^TK_{st} \right)^T \rho - K_{st}B\varphi - \psi \quad (3.13)$$

belonging to $L_2^n[0, \infty)$.

In particular, $(x, u) \in Z$, $-(\dot{p} + A^T p, B^T p) \in Z^\perp$, and consequently Z is a closed subspace in H with

$$\pi_Z(\psi, \varphi) = (x, u), \quad \pi_{Z^\perp}(\psi, \varphi) = -(\dot{p} + A^T p, B^T p). \quad (3.14)$$

Remark 3.3. The required solution ρ exists and unique by Theorem 3.1, since the matrix $-(A + BB^T K_{st})$ is antistable.

Sketch of the Proof

Let $p \in L_2^n[0, \infty)$ be absolutely continuous and such that $\dot{p} \in L_2^n[0, \infty)$. Suppose that $(x, u) \in Z$. Then

$$\begin{aligned} \langle (x, u), (\dot{p} + A^T p, B^T p) \rangle &= \int_0^\infty (x^T \dot{p} + x^T A^T p + u B^T p) dt \\ &= \int_0^\infty [x^T \dot{p} + (Ax + Bu)^T p] dt \\ &= \int_0^\infty (x^T \dot{p} + \dot{x}^T p) dt \\ &= \int_0^\infty \frac{d}{dt} (x^T p) dt \\ &= \lim_{\tau \rightarrow \infty} x^T(\tau) p(\tau) - x(0)^T p(0). \end{aligned} \quad (3.15)$$

But $x(\tau), p(\tau) \rightarrow 0$, as $\tau \rightarrow \infty$ (see e.g., [4] for details) and $x(0) = 0$. Hence,

$$\langle (x, u), (\dot{p} + A^T p, B^T p) \rangle = 0. \quad (3.16)$$

Let us now show that the decomposition (3.5) and (3.9) takes place for an arbitrary $(\psi, \varphi) \in H$. Indeed, using (3.12),

$$\dot{p} = K_{st} \dot{x} + \dot{\rho}. \quad (3.17)$$

Hence by (3.10) and (3.13),

$$\dot{p} = K_{st} (A + BB^T K_{st}) x + K_{st} BB^T \rho + K_{st} B \varphi - (A + BB^T K_{st})^T \rho - K_{st} B \varphi - \psi. \quad (3.18)$$

Combining all terms with x and all terms with ρ in two separate groups, we obtain that

$$\begin{aligned}\dot{p} + A^T p &= \dot{p} + A^T K_{st} x + A^T \rho \\ &= \left(K_{st} A + K_{st} B B^T K_{st} + A^T K_{st} \right) x \\ &\quad + \left(K_{st} B B^T - A^T - K_{st} B B^T + A^T \right) \rho - \varphi.\end{aligned}\tag{3.19}$$

Using now the fact that K_{st} satisfies (3.5), we obtain that

$$\dot{p} + A^T p = x - \varphi\tag{3.20}$$

which is (3.8). Using (3.11) and (3.12), we obtain that

$$\begin{aligned}u - B^T p &= B^T K_{st} x + B^T \rho + \varphi - B^T K_{st} x - B^T \rho \\ &= \varphi,\end{aligned}\tag{3.21}$$

which is (3.9). Finally, it is clear that for x and u defined by (3.11) and (3.12), we have

$$\dot{x} = Ax + Bu\tag{3.22}$$

and consequently $(x, u) \in Z$. This completes the proof of Theorem 3.2.

Looking at (2.13), we see that the evaluation of coefficients of the quadratic function requires the knowledge of expressions of the type $\|\pi_{Z^\perp}(h)\|^2$, where $h \in H$.

Theorem 3.4. *Let $h = (\varphi, \rho) \in H$, and $\rho \in L_2^n[0, \infty)$ is the function entering the decomposition (3.8) and (3.9) and described in (3.13). Then*

$$\|\pi_Z(h)\|^2 = \|B^T \rho + \varphi\|^2,\tag{3.23}$$

$$\|\pi_{Z^\perp}(h)\|^2 = \|h\|^2 - \|B^T \rho + \varphi\|^2.\tag{3.24}$$

Proof. Let $(y, v) \in Z$. Let, further,

$$\Delta(y, v) = \left(v - B^T K_{st} y - B^T \rho - \varphi \right)^T \left(v - B^T K_{st} y - B^T \rho - \varphi \right).\tag{3.25}$$

Here for simplicity of notations, we suppressed the dependence on t . Then

$$\Delta(y, v) = \Delta_1 + \Delta_2 + \Delta_3,\tag{3.26}$$

where

$$\Delta_1 = (\nu - \varphi)^T (\nu - \varphi), \quad \Delta_2 = (K_{\text{st}} y + \rho)^T B B^T (K_{\text{st}} y + \rho), \quad \text{and} \quad \Delta_3 = -2(\nu - \varphi)^T (B^T K_{\text{st}} y + \rho). \quad (3.27)$$

Since $(y, \nu) \in Z$, we have

$$\dot{y} = Ay + B\nu, \quad y(0) = 0. \quad (3.28)$$

Hence,

$$\begin{aligned} \Delta_2 &= y^T (K_{\text{st}} B B^T K_{\text{st}}) y + \rho^T B B^T \rho + 2\rho^T B B^T K_{\text{st}} y, \\ \Delta_3 &= -2(B\nu - B\varphi)^T (K_{\text{st}} y + \rho) \\ &= -2(\dot{y} - Ay - B\varphi)^T (K_{\text{st}} y + \rho) \\ &= -2\dot{y}^T K_{\text{st}} y + y^T (A^T K_{\text{st}} + K_{\text{st}} A) y + 2(B\varphi)^T K_{\text{st}} y \\ &\quad - 2\dot{y}^T \rho + 2(Ay)^T \rho + 2(B\varphi)^T \rho. \end{aligned} \quad (3.29)$$

Notice that $\dot{y}^T \rho + y^T \dot{\rho} = (d/dt)(y^T \rho)$. Hence,

$$\begin{aligned} \Delta(y, \nu) &= (\nu - \varphi)^T (\nu - \varphi) + y^T (K_{\text{st}} B B^T K_{\text{st}} + A^T K_{\text{st}} + K_{\text{st}} A) y \\ &\quad + 2y^T (\dot{\rho} + K_{\text{st}} B\varphi + K_{\text{st}} B B^T \rho + A^T \rho) + (B^T \rho)^T (B^T \rho) \\ &\quad + 2\varphi^T (B^T \rho) - \frac{d}{dt}(y^T \rho) - \frac{d}{dt}(y^T K_{\text{st}} y). \end{aligned} \quad (3.30)$$

Using the fact that K_{st} is a solution to (3.5) and (3.13), we obtain that

$$\begin{aligned} \Delta(y, \nu) &= (\nu - \varphi)^T (\nu - \varphi) + y^T y - 2y^T \varphi + (B^T \rho + \varphi)^T (B^T \rho + \varphi) \\ &\quad - \varphi^T \varphi - \frac{d}{dt}(y^T \rho) - \frac{d}{dt}(y^T K_{\text{st}} y) \\ &= (\nu - \varphi)^T (\nu - \varphi) + (y - \varphi)^T (y - \varphi) + (B^T \rho + \varphi)^T (B^T \rho + \varphi) \\ &\quad - \varphi^T \varphi - \varphi^T \varphi - \frac{d}{dt}(y^T \rho) - \frac{d}{dt}(y^T K_{\text{st}} y). \end{aligned} \quad (3.31)$$

Integrating (3.31) from 0 to $+\infty$ and using the fact that $y(0) = 0$, $y(t)$, $\rho(t) \rightarrow 0$ as $t \rightarrow \infty$, we obtain that

$$\int_0^\infty \Delta(y, \nu) dt = \|(y - \varphi, \nu - \varphi)\|^2 - \|(\varphi, \varphi)\|^2 + \|B^T \rho + \varphi\|^2. \quad (3.32)$$

Notice that $\Delta(y, v) \geq 0$ and $\Delta(y, v) = 0$ provided $(y, v) = \pi_Z(\psi, \varphi)$. See (3.11). Consequently, (3.32) implies that

$$\|(\psi, \varphi)\|^2 = \|B^T \rho + \varphi\|^2 + \|\pi_{Z^\perp}(\psi, \varphi)\|^2. \quad (3.33)$$

Hence,

$$\|\pi_Z(\psi, \varphi)\|^2 = \|B^T \rho + \varphi\|^2. \quad (3.34)$$

This completes the proof of Theorem 3.4. $\square$

We can now easily compute the coefficients of the objective function (2.11). Assuming that $h_i = (\psi_i, \varphi_i) \in L_2^n[0, \infty) \times L_2^m[0, \infty)$, $i = 1, 2, \dots, m$, $c = (\alpha, \beta) \in L_2^n[0, \infty) \times L_2^m[0, \infty)$ and noticing that by Theorem 3.4

$$\|\pi_Z(h(\lambda) - c)\|^2 = \int_0^\infty \left[B^T \rho(\lambda) + \varphi(\lambda) \right]^T \left[B^T \rho(\lambda) + \varphi(\lambda) \right] dt, \quad (3.35)$$

where $\rho(\lambda)$ is the solution of the differential equation

$$\frac{d}{dt} \rho(\lambda) = - \left(A + BB^T K_{st} \right)^T \rho(\lambda) - K_{st} B (\varphi(\lambda) - \psi(\lambda)), \quad (3.36)$$

belonging to $L_2^n[0, \infty)$ and

$$\varphi(\lambda) = \sum_{i=1}^m \lambda_i (\varphi_i - \beta), \quad \psi(\lambda) = \sum_{i=1}^m \lambda_i (\psi_i - \alpha). \quad (3.37)$$

Consequently,

$$\rho(\lambda) = \sum_{i=1}^m \lambda_i (\rho_i - \rho_c), \quad (3.38)$$

where ρ_i and ρ_c are $L_2^n[0, \infty)$ solutions of differential equations

$$\begin{aligned} \dot{\rho}_i &= - \left(A + BB^T K_{st} \right) \rho_i - K_{st} B \varphi_i - \psi_i, \quad i = 1, 2, \dots, m, \\ \dot{\rho}_c &= - \left(A + BB^T K_{st} \right) \rho_c - K_{st} B \beta - \alpha, \end{aligned} \quad (3.39)$$

respectively.

Hence,

$$\|\pi_Z(h(\lambda) - c)\|^2 = \int_0^\infty \Gamma(\lambda)^T \Gamma(\lambda) dt, \quad (3.40)$$

where

$$\Gamma(\lambda) = \sum_{i=1}^m \lambda_i \left[B^T (\rho_i - \rho_c) + (\varphi_i - \beta) \right], \quad (3.41)$$

which allows us to easily express the objective function (2.13) in terms of integrals of ρ_i and ρ_c .

4. Concluding Remarks

In this paper, we have shown that multitarget linear-quadratic control problem on semi-infinite interval can be reduced to solving a simple convex optimization on the simplex. The reduction involves solving one standard algebraic Riccati equation and $m + 1$ linear differential equations, where m is the number of targets. Notice that our results can be easily extended to discrete-time systems.

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