Research Article
Chaos Generated by Switching Fractional Systems

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We, for the first time, investigate the basic behaviours of a chaotic switching fractional system via both theoretical and numerical ways. To deeply understand the mechanism of the chaos generation, we also analyse the parameterization of the switching fractional system and the dynamics of the system’s trajectory. Then we try to write down some detailed rules for designing chaotic or chaos-like systems by switching fractional systems, which can be used in the future application. At last, for the first time, we proposed a new switching fractional system, which can generate three attractors with the positive largest Lyapunov exponent.

1. Introduction

Over the last two decades, since chaos has been demonstrated that it can be useful and well controlled [1–3], increasing interests have been found in many areas such as secure communication [4], data encryption [5], nonlinear optimization [6], synchronization [7], and biology [8] (for more areas, see [3]). Therefore, generating chaos, especially generating chaos via simple physical devices, has been paid extensive and massive attention [9–14]. Just like the n-scroll Chua’s circuit [15], switching piecewise-linear function can easily generate various chaos dynamic behaviours. Especially the literatures [16–18] and give great details to generate chaos via switching systems.

On the other hand, fractional calculus is a mathematical branch which has more than 300 years of history but just been interested recently in physics [19], chemistry, biology, and engineering [20]. Since fractional-order calculus can be treated as an expanding concept of integer-order calculus, there are many chaos systems that are expanded from integer-order ones: the fractional-order Chua system [21, 22], the fractional-order Duffing system [23], the fractional-jerk model [24], the fractional-order Lorenz system [25], the fractional-order Chen system [26], the fractional-order Lu system [27], the fractional-order Rössler system [28], the fractional-order Arneodo system [29], the fractional-order Newton-Leipnik system [30],
and the fractional-order Genesio-Tesi system [31]. And also, the fractional-order Ikeda delay system [32], non-integer-order cellular neural networks [33], and the fractional-order systems proposed in [34, 35] all can generate chaos.

Generating a new chaotic or chaos-like system is always on the core of the chaos research with great theoretical and applied meanings. In addition a new chaotic system usually gets more complex dynamic behaviours being less recognized by people who does not catch up with the nonlinear dynamics. Thus, a new chaos system may be popular used in chaotic secure communication and encryption. Not like lots of the existing switching systems literatures [15–18], there are few literatures about switching fractional systems, even generating a fractional chaos easily is an increasing topic of many applied and theoretical fields. Also, as far as we know, there have not been basic dynamical behaviours given to any chaotic or chaos-like switching fractional systems ever. A significant work of switching fractional systems is proposed by S. Mohammad and H. Mohammad in [36]. The paper discussed the switching rules and how to choose the parameters to generate chaos and the rule of designing switching function have been discussed. However, the detailed relationship between the parameters and the behaviours of the chaos has not been discussed, not even being part of some analysis of the basic behaviours of the systems. In the present paper, we try to research the relationship between the parameters of the system and the behaviours of the chaos in both analytic and numerical ways. And we even epitomized some more detailed rules of generating chaotic or chaos-like dynamic behaviours via switching fractional systems then Mohammad did. Under these rules, we can generate chaotic or chaos-like dynamic behaviours easily via simple switching fractional systems. At last, we propose a new chaotic or chaos-like switching fractional system and count out its Lyapunov exponent.

The paper is organized as follows. Section 2 introduces basic definitions and some theories and lemmas, which are useful in following sections. In Section 3, we analyse an existing chaos generator and do numerical simulations to research quantificationally the relationship between the parameter and the dynamic behaviours. And we propose a new switching fractional systems, which can generate chaotic or chaos-like dynamic behaviours. Finally Section 4 concludes the paper.

2. Background of Fractional Calculus

2.1. Basic Definitions

There exits three main definitions of fractional-order derivatives. They are Grünwald-Letnikov fractional derivatives (G-L):

\[
a D_t^\alpha f(t) = \lim_{h \to 0} h^{-\alpha} \sum_{r=0}^{n} (-1)^r \binom{\alpha}{r} f(t - rh) \quad (\alpha > 0),
\]

(2.1)

Riemann-Liouville fractional derivatives (R-L):

\[
a D_t^\alpha f(t) = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{a}^{t} (t-\tau)^{m-\alpha-1} f(\tau) d\tau \quad (m-1 \leq \alpha < m, \alpha > 0),
\]

(2.2)
and Caputo’s fractional derivative:

$$\frac{d}{dt}^\alpha f(t) = \frac{1}{\Gamma(a-n)} \int_a^t f^{(n)}(\tau)d\tau \quad (n-1 \leq \alpha < n, \alpha > 0).$$

And since they can transform to each other, our use of fractional derivatives can be free for all of these three definitions.

### 2.2. Some Theories and Lemmas

For the requirement in the next part, we list several theories and lemmas. All of them come from [37].

We first define $\gamma(\epsilon, \varphi)$ [36]. $\gamma(\epsilon, \varphi)$ ($\epsilon > 0, 0 < \varphi < \pi$) denotes the contour consisting of the following three parts:

(i) $\arg \tau = \varphi, |\tau| \geq \epsilon$,

(ii) $-\varphi \leq \arg \tau \leq \varphi, |\tau| = \epsilon$,

(iii) $\arg \tau = -\varphi, |\tau| \geq \epsilon$.

More details of $\gamma(\epsilon, \varphi)$ can be found in literature [36].

**Lemma 2.1.** If $\alpha < 2, \pi \alpha / 2 < \mu < \min\{\pi, \pi \alpha\}$ and $\epsilon > 0$ is arbitrary, for arbitrary complex $z$ the following expansion holds:

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{\gamma(\epsilon, \mu)} \exp\left(\frac{\varsigma^{1/a}}{\alpha}\right)\varsigma^{(1-z-\alpha)/\alpha}d\varsigma,$$

where $\Gamma(z)$ is Euler’s gamma function.

**Lemma 2.2.** If $\alpha > 0, \beta > 0$, one obtains

$$\int_0^z E_{\alpha, \beta}(\lambda t^\alpha)\lambda^{\beta-1}dt = z^\beta E_{\alpha, \beta+1}(\lambda z^\alpha).$$

**Theorem 2.3.** If $0 < \alpha < 2$, $\beta$ is an arbitrary complex number, and $\mu$ is an arbitrary real number such that

$$\frac{\pi \alpha}{2} < \mu < \min\{\pi, \pi \alpha\},$$

then for an arbitrary integer $p \geq 1$ the following expansion holds:

$$E_{\alpha, \beta}(z) = \sum_{k=0}^{p} \frac{z^{-k}}{\Gamma(\beta - ak)} + I_p(z),$$

$$|z| \to \infty, \quad \mu \leq |\arg(z)| \leq \pi,$$
where

\[ I_p(z) = \frac{1}{2\pi ai} \int_{\gamma(1,\phi)} \exp\left(\zeta^{1/\alpha}\right)\zeta^{(1-\beta)/\alpha+p} d\zeta. \tag{2.8} \]

Using Lemma 2.1 and Theorem 2.3, if \(0 < \alpha < 2, |z| \to \infty, \pi \alpha/2 < |\arg(z)| \leq \pi\), we can quickly obtain the following:

\[ E_{\alpha,\alpha}(z) = \frac{z^{-1}}{\Gamma(-\alpha)}, \tag{2.9} \]
\[ E_{\alpha,\alpha+1}(z) = \left(\frac{1}{\Gamma(1-\alpha)} - 1\right)z^{-1}, \tag{2.10} \]

which will be used in the next section of our paper.

Consider the following initial value problem for a nonhomogeneous fractional differential equation under nonzero initial conditions:

\[ 0D_t^\alpha y(t) - \lambda y(t) = h(t) \quad (t > 0), \]
\[ 0D_t^{\alpha-k} y(t) \bigg|_{t=0} = b_k \quad (k = 1, 2, \ldots, n), \tag{2.11} \]

then we obtain the following solution:

\[ y(t) = \sum_{k=1}^{n} b_k t^{\alpha-k} E_{\alpha,\alpha-k+1}(\lambda^a) + \int_{0}^{t} (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(\lambda(t-\tau)^a) h(\tau) d\tau. \tag{2.12} \]

3. Chaos Generation

3.1. Analysis of an Existing Chaos Generator

First, we discuss the chaos generated by the switching fractional system \(S_1\) and \(S_2\) with the switching function (3.2):

\[
S_1: \begin{cases}
D^{a_{11}} x_1 = a_1 x_1 + b_1 y_1, \\
D^{a_{12}} y_1 = -b_1 x_1 + a_1 y_1, \\
D^{a_{13}} z_1 = -c_1 z_1,
\end{cases}
\]

\[
S_2: \begin{cases}
D^{a_{21}} x_2 = a_2 x_2 + b_2 y_2, \\
D^{a_{22}} y_2 = -b_2 x_2 + a_2 y_2, \\
D^{a_{23}} z_2 = -c_2 z_2 + p,
\end{cases}
\tag{3.1} \]
where $0 < a_{11}, a_{12}, a_{13}, a_{21}, a_{22}, a_{23} < 1$, $a_1, a_2, b_1, b_2, c_1, c_2, p \neq 0$,

$$g(x, y, z) = x^2 + y^2 + z^2 - 1. \quad (3.2)$$

This kind of chaos was proposed by the literature [36]. That paper has discussed how
to design the switching rule to generate chaos, how to choose the parameters to generate
chaos, and how to design the switching function to generate chaos. Here we want to discuss
the dynamic behaviours of the chaos more carefully and research the relationship between
the parameters of the system and the dynamic behaviours of the chaos more carefully and
quantificationally.

Here we design the switching rule as follows: when $S_1$ is active, the system will
switch to $S_2$ at the time $g(x_1(t), y_1(t), z_1(t)) \geq 0$ with the initial condition of $S_2$ being
$(x_1(t), y_1(t), z_1(t))$. Similarly, when $S_2$ is active, the system will switch to $S_1$ at the time
$g(x_2(t), y_2(t), z_2(t)) < 0$ with the initial condition of $S_1$ being $(x_2(t), y_2(t), z_2(t))$.

And we take $S_2$ to be asymptotically stable and take $S_1$ to be unstable. Then we get
the restricted conditions of the parameters: $c_2 > 0$, $|b_2| > a_2 \max \{\tan(a_{21} \pi/2), \tan(a_{22} \pi/2)\}$,
$|p/c_2| < 1$, and $c_1 < 0$ or $|b_1| < a_1 \min \{\tan(a_{11} \pi/2), \tan(a_{12} \pi/2)\}$ and make $c_1 > 0$, so that, under the discussion in paper [36],
the switching fractional system will perform chaos.

Then we want to discuss the dynamic behaviours of the chaos more carefully. The
switching function (3.2) and the switching rules divide the whole space into two regions,
which we denote $\Sigma = \{(x, y, z) \mid x^2 + y^2 + z^2 - 1 \leq 0\}$ and $\Sigma = \{(x, y, z) \mid x^2 + y^2 + z^2 - 1 > 0\}$,
respectively. When the system is in $\Sigma$, $S_2$ is active. Since $S_2$ is asymptotically stable, $S_2$
will converge to its fixed point, which will be discussed in the following. Either the system orbits
reach or does not reach the fixed point, because $g(0, 0, 0, p/c_2) < 0$, as we take above, the system
orbits will go through the plane $x^2 + y^2 + z^2 = 1$ and then switch to $\Sigma$. When the system is in
$\Sigma$, $S_1$ is active. Since $S_1$ is unstable, the system orbits will diverge and go through the plane
$x^2 + y^2 + z^2 = 1$ and then switch to $\Sigma$. We will discuss $S_1$ and $S_2$ in the following analytically,
respectively.

We first rewrite $S_2$ as follows:

$$S_2 : D^q \nu = A \nu + U, \quad (3.3)$$

where

$$\nu = (x_2, y_2, z_2), \quad A = \begin{bmatrix} a_2 & b_2 & 0 \\ -b_2 & a_2 & 0 \\ 0 & 0 & -c_2 \end{bmatrix}, \quad U = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & p \end{bmatrix}. \quad (3.4)$$

And we can obtain the eigenvalues: $\lambda_{21,22} = a_2 \pm ib_2$, $\lambda_{23} = -c_2$. Under the constricted
conditions of the parameters, we can obtain

$$\frac{\alpha_{ij} \pi}{2} < |\arg(\lambda_{ij})| \quad (i = 2, j = 1, 2, 3). \quad (3.5)$$
Since the eigenvalues are different, we obtain a transformation matrix:

\[
T = \begin{bmatrix}
-i & i & 0 \\
1 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

\[
T^{-1} = \begin{bmatrix}
\frac{1}{2i} & \frac{1}{2} & 0 \\
-\frac{1}{2i} & \frac{1}{2} & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

so that we obtain

\[
D^\alpha \mathbf{v}' = \Lambda \mathbf{v}' + \mathbf{U},
\]

where

\[
\mathbf{v}' = T \mathbf{v},
\]

\[
\Lambda = T \mathbf{A} T^{-1} = \text{diag}(\lambda_{21}, \lambda_{22}, \lambda_{23}).
\]

Because of (3.5), we can solve the transformed fractional differential equations (3.7) in the solution (2.12) with (2.9), so that we obtain

\[
x_2' = b_{21} \frac{\lambda_{21} t^{-1}}{\Gamma(-\alpha_{21})},
\]

\[
b_{21} = \left[_0D^\alpha_{\alpha_{21}} x_2(t)\right]_{t = 0'}
\]

\[
y_2' = b_{22} \frac{\lambda_{22} t^{-1}}{\Gamma(-\alpha_{22})},
\]

\[
b_{22} = \left[_0D^\alpha_{\alpha_{22}} y_2(t)\right]_{t = 0'}
\]

\[
z_2' = b_{23} \frac{\lambda_{23} t^{-1}}{\Gamma(-\alpha_{23})} + \left[ \int_0^t (t - \tau)^{\alpha_{23} - 1} E_{\alpha_{23},\alpha_{23}} \left(\lambda_{23}(t - \tau)^{\alpha_{23}}\right) d\tau \right]_{t = 0'} + p\int_0^t (t - \tau)^{\alpha_{21} - 1} E_{\alpha_{21},\alpha_{21}} \left(\lambda_{21}(t - \tau)^{\alpha_{21}}\right) d\tau,
\]

\[
b_{23} = \left[_0D^\alpha_{\alpha_{23}} z_2(t)\right]_{t = 0'}
\]
By using Lemma 2.2, we obtain

\[ z'_2 = b_{23} \frac{\lambda_{23}^{-1} t^{-1}}{\Gamma(-\alpha_{23})} + pt^{a_{23}} E_{a_{23},a_{23}+1}(\lambda_{23} t^{a_{23}}). \]  

(3.16)

Taking (2.10) into account, we obtain

\[ z'_2 = b_{23} \frac{\lambda_{23}^{-1} t^{-1}}{\Gamma(-\alpha_{23})} + \frac{p}{\lambda_{23}} \left( \frac{1}{\Gamma(1-\alpha_{23})} - 1 \right). \]  

(3.17)

Using (3.8), we can obtain

\[ x^2 + y^2 = \frac{1}{2} \left( -x^2 + y^2 \right), \quad z = z'_2, \]  

(3.18)

taking (3.10) and (3.12) into account, we can obtain

\[ x^2 + y^2 = \frac{1}{2} t^{-2} \left( -\frac{1}{\Gamma^2(-\alpha_{21})} \frac{b_{21}^2}{(a_2 + ib_2)^2} + \frac{1}{\Gamma^2(-\alpha_{22})} \frac{b_{22}^2}{(a_2 - ib_2)^2} \right). \]  

(3.19)

when

\[ t \to +\infty, \quad \sqrt{x^2 + y^2} \to 0, \quad z \to \frac{p}{\lambda_{23}} \left( \frac{1}{\Gamma(1-\alpha_{23})} - 1 \right). \]  

(3.20)

So we see that the trajectory of \( S_2 \) is a spiral line. And \((0, 0, (p/\lambda_{23})(1/\Gamma(1-\alpha_{23})-1))\) is the fixed point of \( S_2 \). And we can slightly change the restricted condition of the parameters \(|p/c_2| < 1\) to

\[ \left| \frac{p}{-c_2} \left( \frac{1}{\Gamma(1-\alpha_{23})} - 1 \right) \right| < 1. \]  

(3.21)

For \( S_1 \), taking \( c_1 > 0 \), we can obtain

\[ \frac{\alpha_{13} \pi}{2} < |\arg(\lambda_{13})|. \]  

(3.22)

Then we can quickly write

\[ z_1 = b_{13} \frac{\lambda_{13}^{-1} t^{-1}}{\Gamma(-\alpha_{13})}, \]  

(3.23)

when

\[ t \to +\infty, \quad z_1 \to 0. \]  

(3.24)

So we see that the trajectory of \( S_1 \) will fall into the plane \( z = 0 \). For \( x_1, y_1 \), because of their complexity, we do not give their analytic solutions.
3.2. Numerical Simulations

Now we can do some numerical simulations and research in quantities of relationship between the parameters and the dynamic behaviours of the chaos. We take all the parameters under the restricted conditions we proposed above, so that the switching fractional system (3.1) will show chaos behaviours. Below in this part, we take $a_{11} = a_{12} = a_{13} = a_{21} = a_{22} = a_{23} = 0.9$. Taking the parameters: $a_1 = 1, b_1 = 2, c_1 = 5, a_2 = 0.7, b_2 = 6, c_2 = 1, p = 1$, the switching fractional system (3.1) has a chaotic attractor, which is shown in Figure 1. The maximum Lyapunov exponent of this attractor is $\text{LE} = 0.0155$. And Figure 2 shows the directions of the trajectory of the switching fractional system (3.1) under these parameters: $a_1 = 1, b_1 = 2, c_1 = 5, a_2 = 0.6, b_2 = 5, c_2 = 1, p = 1$. The trajectory is denoted by the arrows.

We first focus on the parameters $a_2$ and $b_2$. In this segment, we fix the parameters: $a_1 = 1, b_1 = 2, c_1 = 5, c_2 = 1, p = 1$. We first take the parameter $a_2 = 0.7$ and increase $b_2$ from 6 to 7 with step 1. Figure 3 shows the phase portraits of the switching fractional system (3.1) under these parameters. From Figure 3, we can conclude that the increase of $b_2$ makes $S_2$ converge faster. This conclusion is established in all conditions, which satisfy $a_2 \in (-\infty, +\infty)$ and $|b_2| > a_2 \max\{\tan(a_{21}\pi/2), \tan(a_{22}\pi/2)\}$. Then we take the parameter $b_2 = 6$ and increase $a_2$ from 0.5 to 0.6 with step 0.1. Figure 4 shows the phase portraits of the switching fractional system (3.1) under these parameters. From Figure 4, we can conclude that the increase of $a_2$ makes $S_2$ converge slower. This conclusion is established in all conditions, which satisfy $a_2 \in (-\infty, +\infty)$ and $|b_2| > a_2 \max\{\tan(a_{21}\pi/2), \tan(a_{22}\pi/2)\}$.

Then we focus on the parameters $c_2$ and $p$. In this segment, we fix the parameters: $a_2 = 0.7, b_2 = 6, a_1 = 1, b_1 = 2, c_1 = 5$. We increase $c_2$ and $p$ simultaneously from 1 to 2 with step 1. Figure 5 shows the phase portraits of the switching fractional system (3.1) under these parameters. From Figure 5, we can conclude that the increase of $c_2$ makes $S_2$ reach the plane faster. This conclusion is established in all conditions, which satisfy $c_2 > 0$.

\[ XYZ' = \left(0, 0, \frac{p}{-c_2} \left(\frac{1}{\Gamma(1 - \alpha_{23})} - 1\right)\right) \quad (3.25) \]
Figure 2: The trajectory of the switching fractional system (3.1).

Figure 3: Phase portraits of the switching fractional system (3.1).

(a) $b_2 = 6$

(b) $b_2 = 7$

Figure 4: Phase portraits of the switching fractional system (3.1).

(a) $a_2 = 0.5$

(b) $a_2 = 0.6$
For $S_1$, we first focus on the parameters $a_1$ and $b_1$. In this segment, we fix the parameters: $c_1 = 4$, $a_2 = 0.7$, $b_2 = 6$, $c_2 = 1$, $p = 1$. We first take the parameter $b_1 = 6$ and increase $a_1$ from 1 to 3 with step 2. Figure 6 shows the phase portraits of the switching fractional system (3.1) under these parameters. From Figure 6, we can conclude that the increase of $a_1$ makes $S_1$ diverge faster. This conclusion is established in all conditions, which satisfy $a_1 \in (-\infty, +\infty)$ and $|b_1| < a_1 \min\{\tan(\alpha_{11}\pi/2), \tan(\alpha_{12}\pi/2)\}$. Then we take the parameter $a_1 = 1$ and increase $b_1$ from 3 to 6 with step 3. Figure 7 shows the phase portraits of the switching fractional system (3.1) under these parameters. From Figure 7, we can conclude that the increase of $b_1$ makes $S_1$ diverge slower. This conclusion is established in all conditions, which satisfy $a_1 \in (-\infty, +\infty)$ and $|b_1| < a_1 \min\{\tan(\alpha_{11}\pi/2), \tan(\alpha_{12}\pi/2)\}$.

Then we focus on the parameters $c_1$. In this segment, we fix the parameters: $a_1 = 1$, $b_1 = 2$, $a_2 = 0.7$, $b_2 = 6$, $c_2 = 1$, $p = 1$. We increase $c_1$ from 1 to 2 with step 1. Figure 8 shows the phase portraits of the switching fractional system (3.1) under these parameters. From Figure 8, we can conclude that the increase of $c_1$ makes $S_1$ reach the plane

$$XYZ = (0, 0, 0)$$

faster. This conclusion is established in all conditions, which satisfy $c_1 > 0$. 

Figure 5: Phase portraits of the switching fractional system (3.1).

Figure 6: Phase portraits of the switching fractional system (3.1).
3.3. More Detailed Rules to Design Chaotic or Chaos-Like Fractional Switching Systems

Here we epitomize some more detailed rules of relationship of the dynamic behaviours of the switching fractional systems and the parameters of the switching fractional systems. Take the switching fractional system (3.1) with the parameters under the restricted conditions of the parameters, which we proposed previously, for example. There are two fixed points of the switching fractional system (3.1). The parameters $c_1$, $c_2$, and $p$ control the speed of the switching fractional system to reach the fixed points, respectively. The parameters $a_1$ and $b_1$ control the speed of $S_1$ to diverge. The parameters $a_2$ and $b_2$ control the speed of $S_2$ to converge. Taking the experiments, we did above, into account, we conclude that if we want the switching fractional system to arrive at the fixed points faster, we can increase the parameters $c_1$, $c_2$, and $p$. However, $c_1$, $c_2$, and $p$ should also not be so large that the orbits of the switching fractional system will cross through the plane $x^2 + y^2 + z^2 = 1$, because the sum $x_1^2 + y_1^2 + z_1^2 - 1$ will be bigger or smaller than 0, respectively; if we want $S_1$ to diverge faster, we can increase the parameter $a_1$ or decrease the parameter $b_1$; if we want $S_2$ to converge faster, we can increase the parameter $b_2$ or decrease the parameter $a_2$. 

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**Figure 7:** Phase portraits of the switching fractional system (3.1).

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**Figure 8:** Phase portraits of the switching fractional system (3.1).
So the rules are that we should balance all the parameters when we apply the rules obtained in the literature [36]. If we find that $S_2$ converges too fast, we may decrease the value of either $b_2$ or $c_2$ or increase $a_2$; it is also very similar to $S_1$. But remember if either $S_1$ or $S_2$ converges or diverges too fast that it cannot reach its own fixed point plane, the pattern we get may not show two attractors. Even if we obey these rules, we can design a chaotic or chaos-like switching fractional system with a wide range of the parameter values.

### 3.4. A New Chaotic or Chaos-Like Switching Fractional System

Under the rules above, we here propose a new chaotic or chaos-like switching fractional system $S_1$, $S_2$, and $S_3$ with the switching function (3.2). Actually, under the rules above, lots of chaotic or chaos-like switching fractional systems can also be proposed:

$$
S_1: \begin{aligned}
D^{\alpha_1}x_1 &= a_1x_1 + b_1y_1, \\
D^{\alpha_2}y_1 &= -b_1x_1 + a_1y_1, \\
D^{\alpha_3}z_1 &= -c_1z_1 + p_1,
\end{aligned}
$$

$$
S_2: \begin{aligned}
D^{\alpha_1}x_2 &= a_2x_2 + b_2y_2, \\
D^{\alpha_2}y_2 &= -b_2x_2 + a_2y_2, \\
D^{\alpha_3}z_2 &= -c_2z_2,
\end{aligned}
$$

$$
S_3: \begin{aligned}
D^{\alpha_1}x_3 &= a_3x_3 + b_3y_3, \\
D^{\alpha_2}y_3 &= -b_3x_3 + a_3y_3, \\
D^{\alpha_3}z_3 &= -c_3z_3 + p_3,
\end{aligned}
$$

where $0 < \alpha_1, \alpha_2, \alpha_3, \alpha_21, \alpha_22, \alpha_23, \alpha_31, \alpha_32, \alpha_33 < 1$, $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3, p_1, p_2 \neq 0$,

$$
g_1(x, y, z) = x^2 + y^2 + z^2 - 1, \quad g_2(x, y) = xy. \quad (3.28)
$$

We take the switching rule as follows: when $S_2$ is active, the system will switch to $S_1$ at the time $g_1(x_2(t), y_2(t), z_2(t)) \geq 0$ and $g_2(x_2(t), y_2(t)) \geq 0$ with the initial condition of $S_1$ being $(x_2(t), y_2(t), z_2(t))$ and the system will switch to $S_3$ at the time $g_1(x_2(t), y_2(t), z_2(t)) \geq 0$ and $g_2(x_2(t), y_2(t)) < 0$ with the initial condition of $S_1$ being $(x_2(t), y_2(t), z_2(t))$. When $S_1$ is active, the system will switch to $S_2$ at the time $g_1(x_1(t), y_1(t), z_1(t)) < 0$ with the initial condition of $S_2$ being $(x_1(t), y_1(t), z_1(t))$. When $S_3$ is active, the system will switch to $S_2$ at the time $g_1(x_3(t), y_3(t), z_3(t)) < 0$ with the initial condition of $S_2$ being $(x_3(t), y_3(t), z_3(t))$. 


Lyapunov exponent of this attractor is \( LE \). Fractional systems based on the basic rules given in the literature bracketleftmath 

Down some more detailed rules for designing chaotic or chaos-like systems by switching fractional system and the dynamics of the system’s trajectory. Then we try to write system both theoretically and numerically. We also analyse the parameterization of the

In this paper, we for the first time study the basic behaviours of a chaotic switching fractional system. Then we can obtain that the switching fractional system parenleftmath 

determine the parameters:

\[
\begin{align*}
|b_1| & > a_1 \max \left\{ \tan \left( \frac{\alpha_{11} \pi}{2} \right), \tan \left( \frac{\alpha_{12} \pi}{2} \right) \right\}, \quad c_1 > 0, \quad \left| \frac{p_1}{-c_1} \left( \frac{1}{\Gamma(1 - \alpha_{13})} - 1 \right) \right| < 1, \\
|b_2| & < a_2 \min \left\{ \tan \left( \frac{\alpha_{21} \pi}{2} \right), \tan \left( \frac{\alpha_{22} \pi}{2} \right) \right\}, \quad c_2 > 0, \\
|b_3| & > a_3 \max \left\{ \tan \left( \frac{\alpha_{31} \pi}{2} \right), \tan \left( \frac{\alpha_{32} \pi}{2} \right) \right\}, \quad c_3 > 0, \quad \left| \frac{p_3}{-c_3} \left( \frac{1}{\Gamma(1 - \alpha_{33})} - 1 \right) \right| < 1.
\end{align*}
\]

Figure 9: The chaotic attractor generated by the switching fractional system (3.27).

We take \( S_1, S_3 \) to be asymptotically stable, while we take \( S_2 \) to be unstable. Then we determine the parameters:

We take the parameters: \( a_1 = 0.6, b_1 = 9, c_1 = 5, p_1 = 5, a_2 = 1, b_2 = 2, c_2 = 5, a_3 = 0.6, b_3 = 9, c_3 = 5, p_2 = -5, a_{11} = a_{12} = a_{13} = a_{21} = a_{22} = a_{23} = a_{31} = a_{32} = a_{33} = 0.9 \). The switching fractional system (3.27), has a chaotic attractor, which is shown in Figure 9. The maximum Lyapunov exponent of this attractor is \( LE = 0.0434 \).

We take the parameters: \( a_1 = 0.6, b_1 = 9, c_1 = 5, p_1 = 5, a_2 = 1, b_2 = 2, c_2 = 5, a_3 = 0.6, b_3 = 9, c_3 = 5, p_2 = -5, a_{11} = a_{12} = a_{13} = a_{21} = a_{22} = a_{23} = a_{31} = a_{32} = a_{33} = 0.9, a_{13} = a_{23} = a_{33} = 0.5 \). The switching fractional system (3.27), has a chaotic attractor, which is shown in Figure 10. The maximum Lyapunov exponent of this attractor is \( LE = 0.0255 \). From Figure 10, we can obtain that the switching fractional system (3.27), under variant parameters \( a_{ij} (i = 1, 2, 3, \ j = 1, 2, 3) \) can also perform chaotic or chaos-like dynamic behaviours.

4. Conclusions

In this paper, we for the first time study the basic behaviours of a chaotic switching fractional system both theoretically and numerically. We also analyse the parameterization of the switching fractional system and the dynamics of the system’s trajectory. Then we try to write down some more detailed rules for designing chaotic or chaos-like systems by switching fractional systems based on the basic rules given in the literature [36]. At last, for the first
time, we proposed a new switching fractional system, which can generate three attractors with the positive largest Lyapunov exponent.

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