

## Research Article

# New Analytic Solution to the Lane-Emden Equation of Index 2

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We present two new analytic methods that are used for solving initial value problems that model polytropic and stellar structures in astrophysics and mathematical physics. The applicability, effectiveness, and reliability of the methods are assessed on the Lane-Emden equation which is described by a second-order nonlinear differential equation. The results obtained in this work are also compared with numerical results of Horedt (1986) which are widely used as a benchmark for testing new methods of solution. Good agreement is observed between the present results and the numerical results. Comparison is also made between the proposed new methods and existing analytical methods and it is found that the new methods are more efficient and have several advantages over some of the existing analytical methods.

## 1. Introduction

In this work we investigate the solution of nonlinear initial value problem that has applications in mathematical physics and astrophysics. We consider the Lane-Emden equation of the form

$$y''(x) + \frac{2}{x}y'(x) + y^s = 0, \quad (1.1)$$

with initial conditions

$$y(0) = 1, \quad y'(0) = 0, \quad (1.2)$$

where  $s$  is a constant. This equation is very useful in astrophysics in the study of polytropic models and stellar structures [1, 2]. For the special case when  $s = 0, 1, 5$  exact analytical

solutions were obtained by Chandrasekhar [1]. For all other values of  $s$  approximate analytical methods and numerical methods are used to approximate the solution of the Lane-Emden or related equations. Analytical approaches that have recently been applied in solving the Lane-Emden equations include the Adomian decomposition method [3, 4], differential transformation method [5], homotopy perturbation method [6], He's Energy Balance Method (HEBM) [7], homotopy analysis method [8, 9], power series expansions [10–14], and variational iteration method [15, 16]. Generally, when all the above cited analytical approaches are used to solve Lane-Emden equation, a truncated power series solution of the true solution is obtained. This solution converges rapidly in a very small region ( $0 < x < 1$ ). For  $x > 1$  convergence is very slow and the solutions are inaccurate even when using a large number of terms. Convergence acceleration methods such as Pad'e approximations may be used to improve the convergence of the resulting series or to enlarge their domains of convergence. The homotopy analysis method [9, 17] has a unique advantage over the other analytic approximation methods because it has a convergence controlling parameter that can be adjusted to improve the region of convergence of the resulting series. An important physical parameter associated with the Lane-Emden function is the location of its first positive real zero. The first zero of  $y(x)$  is defined as the smallest positive value  $x_0$  for which  $y(x_0) = 0$ . This value is important because it gives the radius of a polytropic star. The analytic approaches on their own are not very useful in solving for  $x_0$  because their region of convergence is usually less than  $x_0$ . Recently, there has been a surge in the number of numerical methods that have been proposed to find solutions of the Lane-Emden equations. Recent numerical methods that have been proposed include the Legendre Tau method [18] and the sinc-collocation method [19], the Lagrangian approach [20], and the successive linearization method [21]. Accurate results for the Lane-Emden function have previously been reported in [22] where the Runge-Kutta routine with self-adapting step was used to generate seven digit tables of Lane-Emden functions. These tables are now widely used as a benchmark for testing the accuracy of new methods of solving the Lane-Emden equations.

In this study, we propose two new analytic methods for solving the Lane-Emden equation of the form (1.1). The first method is a modification of the successive linearisation method (SLM) that has been recently reported and successfully utilized in solving boundary value problems [23–30]. An attempt adapts the SLM in solving initial value problems such as Lane-Emden equation has recently been made in [21]. The SLM approach is based on transforming an ordinary nonlinear differential equation into an iterative scheme made up of linear equations which are then solved using numerical methods such as the Chebyshev spectral method. This method works very well in problems defined on finite domains. For initial value problems, the method may not be very useful. For instance, in applying the method in [21] the domain of the Lane-Emden equations solved was defined to be  $[0, L]$ , where  $L$  was conveniently chosen to be close to  $x_0$ , the first zero of the Lane-Emden equation. That is, a rough estimate of  $x_0$  had to be known before the SLM was applied in [21]. In this paper we propose a modification of the SLM approach and use it to solve the Lane-Emden equation (1.1). Unlike the SLM [21], the modified SLM, hereinafter referred to as the MSLM, results in reduced differential equations which are solved analytically to give series solutions which are highly convergent and can be used to find the first zero  $x_0$  of the Lane-Emden equation. Excellent agreement is observed between the MSLM results and the numerical results of [22]. The second method proposed in this work is an innovative technique that blends the SLM method with the Adomian decomposition method (ADM) to result in a hybrid method, hereinafter referred to as the ADM-SLM, that is superior to both the ADM and SLM methods. The results of the ADM-SLM are also compared with the numerical results [22] and excellent agreement is

observed. The performance between the three methods in terms of their total number iterations, run times of each algorithm, and rate of convergence is assessed. The main aim of the analysis presented in this paper is to introduce the two new analytic approaches which are presented as an alternative way of improving the convergence of the ADM without resorting to convergence accelerating techniques such as the Padé approach.

## 2. Outline of Methods of Solution

In this section, we describe the methods of solution that are used to solve the governing equation (1.1). Three methods, namely, the Adomian decomposition method (ADM), the modified successive linearization method (SLM), and a hybrid method that blends the ADM and the SLM, are used in this study. The modified successive linearization method, hereinafter reference to as the MSLM, is an alternative implementation of the SLM method that has been recently introduced in [23–30] for finding solutions of various boundary value problems. In this work, the SLM is modified and adapted to be usable in initial value problems of the type (1.1). The blend between the ADM and SLM uses ideas of both the ADM and the SLM to yield a more powerful hybrid method which is referred to as the ADM-SLM in this paper.

### 2.1. Adomian Decomposition Method (ADM)

In this section we give a brief description of the implementation of the ADM in solving the governing equation (1.1). We begin by writing (1.1) in operator form as

$$\mathcal{L}y = -y^s, \quad (2.1)$$

where  $\mathcal{L}$  is a linear operator described as

$$\mathcal{L} = x^{-2} \frac{d}{dx} \left( x^2 \frac{d}{dx} \right). \quad (2.2)$$

This type of operator was suggested in [3, 4] and has been found to give results which converge much faster than the original ADM method of [31, 32]. The inverse operator is the two-fold integral operator given as

$$\mathcal{L}^{-1}(\cdot) = \int_0^x x^{-2} \int_0^x x^2 (\cdot) dx dx. \quad (2.3)$$

If we operate  $\mathcal{L}^{-1}$  on (2.1) and apply the boundary conditions (1.2) we obtain

$$y(x) = y(0) + y'(0)x - \mathcal{L}^{-1}[y^s]. \quad (2.4)$$

The basic idea behind the ADM is the representation of the solution  $y(x)$  as an infinite series of the form

$$y(x) = \sum_{n=0}^{\infty} y_n(x), \quad (2.5)$$

and the nonlinear function  $y^s$  by the following infinite series of polynomials:

$$y^s = \sum_{n=0}^{\infty} A_n. \quad (2.6)$$

The components  $y_n(x)$  of the solution of  $y(x)$  are determined recurrently using the Adomian polynomials  $A_n$  that can be calculated for various classes of nonlinearity according to algorithms set out by Adomian and Rach [33–35] and more recently in [3, 4]. For a nonlinear function  $F(u)$ , the first few polynomials are given by

$$\begin{aligned} A_0 &= F(u_0), \\ A_1 &= u_1 F'(u_0), \\ A_2 &= u_2 F'(u_0) + \frac{u_1^2}{2!} F''(u_0), \\ A_3 &= u_3 F'(u_0) + u_1 u_2 F''(u_0) + \frac{u_1^3}{3!} F'''(u_0), \\ &\vdots \end{aligned} \quad (2.7)$$

By substituting (2.5) and (2.6) in (2.4) we obtain

$$\sum_{n=0}^{\infty} y_n = 1 - \mathcal{L}^{-1} \sum_{n=0}^{\infty} A_n. \quad (2.8)$$

To find the components  $y_n$ , the ADM suggests the use of the following recursive relationships:

$$\begin{aligned} y_0(x) &= 1, \\ y_{k+1}(x) &= -\mathcal{L}^{-1} A_k, \quad k \geq 0. \end{aligned} \quad (2.9)$$

For numerical purposes, the solution of the governing equation (1.1) can be approximated by the  $n$ -term approximate series given as

$$\phi_n(x) = \sum_{k=0}^{n-1} y_k(x). \quad (2.10)$$

If the series (2.10) converges, then

$$y(x) = \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} y_k(x). \quad (2.11)$$

## 2.2. Modified Successive Linearization Method (MSLM)

The original SLM approach (see [23–30] for details) assumes that the solution  $y(x)$  of the non-linear equation (1.1) can be expanded as

$$y(x) \approx y_i(x) + \sum_{m=0}^{i-1} y_m(x), \quad i = 1, 2, 3, \dots, \quad (2.12)$$

where  $y_i$  are unknown functions whose solutions are obtained by recursively solving the linearised equation that results from the substitution of (2.12) into the governing equation (1.1). The SLM algorithm starts with an initial approximation  $y_0(x) = 1$  which is chosen to satisfy the initial conditions (1.2) of the problem. Thus substituting equation (2.12) into the governing equation (1.1) and neglecting all nonlinear terms in  $y_i$  gives

$$y_i'' + \frac{2}{x}y_i' + h_{i-1}y_i = r_{i-1}, \quad (2.13)$$

subject to the initial conditions

$$y_i(0) = 0, \quad y_i'(0) = 0, \quad (2.14)$$

where  $h_{i-1}$  and  $r_{i-1}$  are known functions of  $x$  at the  $i$ th iteration and are defined as

$$h_{i-1}(x) = s \left( \sum_{m=0}^{i-1} y_m \right)^{s-1}, \quad (2.15)$$

$$r_{i-1}(x) = - \left[ \sum_{m=0}^{i-1} y_m'' + \frac{2}{x} \sum_{m=0}^{i-1} y_m' + \left( \sum_{m=0}^{i-1} y_m \right)^s \right].$$

We note that in its current form, SLM equation (2.13) cannot be solved to give a closed form solution. In the original implementation of the SLM (see e.g., [23–30]), the linearised equations, such as (2.13), are solved using numerical methods. Some Lane-Emden type equations are singular in nature and standard numerical methods may not be suitable for finding the solution of such equations. For this reason, we seek to find analytical approaches to solve (2.13). In this paper we introduce the modified successive linearisation method (MSLM) which is based on the implementation of the assumption that  $y_i$  becomes increasingly smaller with an increase in the number of iterations. Thus, assuming that enough iterations are used in the algorithm, in addition to neglecting nonlinear terms in  $y_i$  we also neglect all terms that are multiplied by  $y_i$  (i.e., we set  $y_i = 0$ ) in (2.13). This reduces the equation into one whose closed form analytic solution can easily be found by integrating the resulting equation. The reduced equation is given as

$$y_i'' + \frac{2}{x}y_i' = r_{i-1}. \quad (2.16)$$

Since the right-hand side of (2.16) is known at each iteration, the equation can be solved exactly to obtain

$$y_i(x) = \int_0^x x^{-2} \int_0^x x^2 r_{i-1}(x) dx dx. \quad (2.17)$$

Starting from the initial approximation  $y_0(x)$ , the  $n$ th-order approximate solution for  $y(x)$  is thus given by

$$\psi_n(x) = \sum_{m=0}^n y_m(x). \quad (2.18)$$

The initial approximation  $y_0(x)$  is obtained by solving the following equation:

$$y_0'' = 0 \quad (2.19)$$

subject to the underlying initial conditions. If the series (2.18) converges, then

$$y(x) = \lim_{n \rightarrow \infty} \sum_{m=0}^n y_m(x). \quad (2.20)$$

### **2.3. Hybrid Adomian Decomposition Method-Successive Linearisation Method (ADM-SLM)**

The ADM-SLM is derived from by solving the SLM governing iteration scheme (2.13) using the ADM. We remark that the idea behind the application of the ADM is to ensure that the equation is solvable analytically. Rewriting (2.13) in operator form gives

$$\mathcal{L}y_i + h_{i-1}y_i = r_{i-1}. \quad (2.21)$$

The approximate solution of (2.21) is found using the ADM. Thus, operating with  $\mathcal{L}^{-1}$  on (2.21) we get

$$y_i = \mathcal{L}^{-1}[r_{i-1}(x)] - \mathcal{L}^{-1}[h_{i-1}(x)y_i]. \quad (2.22)$$

Implementing the ideas of the ADM on (2.22) results in the following recurrence relations:

$$y_{i,0} = \mathcal{L}^{-1}[r_{i-1}(x)], \quad (2.23)$$

$$y_{i,k+1} = -\mathcal{L}^{-1}[h_{i-1}(x)y_{i,k}], \quad k \geq 0. \quad (2.24)$$

The solution for  $y_i(x)$ , at the  $n$ th term, can be approximated by

$$y_i(x) = \sum_{k=0}^{n-1} y_{i,k}(x). \quad (2.25)$$

Thus, starting from the initial approximation  $y_{0,0} = y_0$ , the  $[i, n]$  ADM-SLM approximate solution for  $y(x)$  is given by

$$\chi_{i,n} = \sum_{m=0}^i \sum_{k=0}^{n-1} y_{m,k}. \quad (2.26)$$

It can be observed that (2.23) is the same as (2.17). Thus, we remark that if  $n = 0$ , corresponding to the ADM step not being taken, (2.26) reduces to the MSLM solution. The ADM-SLM uses an initial approximation that is obtained using the MSLM as described in the previous section and implements the ADM on a new modified governing equation that has been linearised using the SLM approach. In essence, the ADM-SLM solution improves on the traditional ADM approach by implementing the ADM on the SLM-linearized governing equation. The ADM-SLM is essentially a two-dimensional recursive method that implements the SLM iteration in one direction and the ADM iteration in the other direction. In order to assess the difference in performance between the three methods, their accuracy, total number iterations, run time of each algorithm and rate of convergence must be considered. To compare the total number of iterations between the MSLM and ADM-SLM we note that, in approximating  $y(x)$  by  $\chi_{i,n}$  in the ADM-SLM, there are  $n$  iterations for each  $i$ th iteration. Thus the total number of iterations of the ADM-SLM approach can be considered to be  $i \times (n + 1)$ . If  $n = 0$ , meaning that the ADM component of the method is not implemented, the ADM-SLM method becomes equivalent to the MSLM.

### 3. Convergence Analysis

Solving the Lane-Emden type of equations discussed in this paper results in power series of the form

$$y(x) \approx \sum_{k=0}^{\infty} a_k x^{2k}. \quad (3.1)$$

The series of the type (3.1) has also been obtained by several other researchers using various analytical approximation methods (see e.g., [4, 10, 11, 13, 14]) in solving Lane-Emden type equations. According to the standard ratio test, a series of the form (3.1) converges for  $x < R$ , where

$$R = \lim_{k \rightarrow \infty} \left| \frac{a_k}{a_{k+1}} \right|^{1/2}. \quad (3.2)$$

The quantity  $R$  is known as the radius of convergence. The convergence analysis of the ADM has been discussed in [36] in terms of the fixed-point iteration. Recently, a simple method of establishing the convergence of the ADM was presented in [31] for the following functional equation:

$$y = N(y) + f. \quad (3.3)$$

Just like the ADM, the implementation of the MSLM and the ADM-SLM is equivalent to determining the sequence

$$S_n = y_1 + y_2 + \cdots + y_n, \quad (3.4)$$

by using the iterative scheme

$$S_{n+1} = N(y_0 + S_n), \quad S_0 = 0. \quad (3.5)$$

The above iterative scheme results from the associated functional equation

$$S = N(y_0 + S). \quad (3.6)$$

For the purposes of comparing the convergence rates of the three methods presented in this paper the approach used in [31] has been used. In this paper, we only present the relevant theory and corollary associated with proving the convergence and obtaining the rate of convergence of sequences of the form (3.5). For details of the proof, interested readers may refer to [31].

**Theorem 3.1.** *Let  $N$  be an operator from a Hilbert space  $H$  into  $H$  and  $y$  be the exact solution of (3.3). The sum,  $\sum_{i=0}^{\infty} y_i$ , which is obtained by (2.9) in the ADM, (2.17) in the MSLM, and (2.25) in the ADM-SLM, converges to  $y$  when  $\exists 0 \leq \alpha < 1, \|y_{k+1}\| \leq \alpha \|y_k\|$ , for all  $k \in N \cup 0$ .*

*Definition 3.2.* For every  $i \in N \cup 0$  we define

$$\alpha_i = \begin{cases} \frac{\|y_{i+1}\|}{\|y_i\|}, & \|y_i\| \neq 0, \\ 0, & \|y_i\| = 0. \end{cases} \quad (3.7)$$

**Corollary 3.3.** *In Theorem 3.1,  $\sum_{i=0}^{\infty} y_i$  converges to exact solution  $y$ , when  $0 \leq \alpha_i < 1, i = 1, 2, 3, \dots$*

**Corollary 3.4.** *If  $y_i$  and  $\tilde{y}_i$  are obtained by two different methods and their associated ratios  $\alpha_i$ 's and  $\tilde{\alpha}_i$ 's are both less than one, then if  $\alpha_i < \tilde{\alpha}_i$  for all  $i$ , it follows that the rate of convergence of  $\sum_{i=0}^{\infty} y_i$  to the exact solution is higher than  $\sum_{i=0}^{\infty} \tilde{y}_i$ .*

We remark that the series solutions of type (3.1) obtained using the three methods presented in this study do not provide us with explicit expressions that would enable us to calculate the radius of convergence and convergence rates analytically. The convergence rates and radius of convergence discussed in this paper we computed numerically for different values of  $k$  and their trend for large values of  $k$  was noted.

#### 4. Application on an Illustrative Example

In this section numerical experiments are performed on an illustrative example to show the difference between the methods described in the last section. The performance of the methods is measured in terms of rate of convergence, radius of convergence, accuracy, and efficiency.

Consider the nonlinear Lane-Emden equation of index 2 given as

$$y'' + \frac{2}{x}y' + y^2 = 0, \quad y(0) = 1, \quad y'(0) = 0. \quad (4.1)$$

This equation is very useful in astrophysics in the study of polytropic models.

### 4.1. ADM Solution

Using (2.9), the ADM iterates are obtained from

$$\begin{aligned} y_0(x) &= 1, \\ y_{k+1}(x) &= -\mathcal{L}^{-1}(A_k), \quad k \geq 0, \end{aligned} \quad (4.2)$$

where the first few Adomian polynomials are given as

$$\begin{aligned} A_0 &= y_0^2, \\ A_1 &= 2y_1y_0, \\ A_2 &= 2y_2y_0 + y_1^2, \\ A_3 &= 2y_3y_0 + 2y_1y_2 \\ &\vdots \end{aligned} \quad (4.3)$$

Thus, the first three solutions are given by

$$\begin{aligned} y_1 &= -\frac{1}{6}x^2, \\ y_2 &= \frac{1}{60}x^4, \\ y_3 &= -\frac{11}{7560}x^6, \\ y_4 &= \frac{1}{8505}x^8. \end{aligned} \quad (4.4)$$

Thus, the approximate solution for  $y(x)$  obtained using the  $k = 1, 2, \dots, 7$  (i.e.,  $\phi_8$ ) is given by

$$\phi_8(x) = 1 - \frac{x^2}{6} + \frac{x^4}{60} - \frac{11x^6}{7560} + \frac{x^8}{8505} - \frac{97x^{10}}{10692000} + \frac{457x^{12}}{673596000} - \frac{98239x^{14}}{1980372240000}. \quad (4.5)$$

We note that this ADM solution is exactly the same as the power series solution whose relation, according to [14], is

$$y(x) = \sum_{k=0}^{\infty} a_k x^{2k}, \quad \text{with } a_k = -\frac{1}{(2k)(2k+1)} \sum_{j=0}^{k-1} a_j a_{k-j-1}, \quad a_0 = 1, \quad k \geq 1. \quad (4.6)$$

The radius of convergence for the Lane-Emden equation (4.1) with series approximation (4.6) [10, 11, 13] is about  $R = \sqrt{15.7179} = 3.9646$ . The first zero of  $y(x)$  is defined as the smallest positive value  $x_0$  for which  $y(x_0) = 0$ . This value is important because it gives the radius of a polytropic star. The first zero for the Lane-Emden equation (4.1) has been calculated [12, 22, 37] to be 4.3528745959. Since the radius of convergence of the ADM is less than the first zero  $x_0$  it follows that the ADM on its own can not be used to determine  $x_0$ .

**Table 1:** Approximation of  $y(x)$  for the present method at selected values of  $x$  using the ADM and the corresponding run times in seconds.

$x$	10th order	20th order	40th order	60th order	Numerical [22]
0.5	0.9593527	0.9593527	0.9593527	0.9593527	0.9593527
1.0	0.8486541	0.8486541	0.8486541	0.8486541	0.8486541
1.5	0.6953672	0.6953671	0.6953671	0.6953671	0.6953671
2.0	0.5298405	0.5298364	0.5298364	0.5298364	0.5298364
2.5	0.3752294	0.3747394	0.3747393	0.3747393	0.3747393
3.0	0.2657006	0.2419943	0.2418241	0.2418241	0.2418241
3.5	0.7567158	0.2311776	0.1352586	0.1339821	0.1339690
4.0	10.3508034	23.34921747	64.6821008	137.0780406	0.0488402
Run time	0.016	0.016	0.110	0.375	

**Table 2:** Errors of the ADM method at selected values of  $x$ .

$x$	10th order	20th order	40th order	60th order
0.5	0.0000000	0.0000000	0.0000000	0.0000000
1.0	0.0000000	0.0000000	0.0000000	0.0000000
1.5	0.0000001	0.0000000	0.0000000	0.0000000
2.0	0.0000041	0.0000000	0.0000000	0.0000000
2.5	0.0004901	0.0000001	0.0000000	0.0000000
3.0	0.0238765	0.0001702	0.0000000	0.0000000
3.5	0.6227468	0.0972086	0.0012896	0.0000131
4.0	10.3019632	23.3003773	64.6332606	137.0292005

Table 1 gives a comparison between the ADM results, which have been computed up to 60th order, and the numerical results of [22] for selected values of  $x$ . The run times for the algorithm are also given in the table. We see from Table 1 that the run times the ADM algorithm are a fraction of a second even when using many iterations. Table 2 gives the absolute errors between the ADM and the seven-digit accurate numerical results of [22]. It can be seen from the tables that the ADM results are accurate for small values of  $x$ . Accuracy of the ADM results generally increases when the number of iterations is increased. But this is true for  $x$  less than the convergence region of the method. Since the ADM solution of 36 is equivalent to the series solution reported in [14], the convergence radius  $R$  is about 3.9646 [10, 11, 13]. Tables 1 and 2 also indicate that the convergence of the method becomes very slow when  $x$  approaches  $R$  and that above  $R$  the method diverges.

In Figure 1 we plot the ADM solution of Example of (4.1) for orders up to  $n = 64$ . We see that, as we increase the order of the ADM, the interval of convergence increases rapidly when  $x$  is small then increases slowly when  $x$  is closer to  $R$ , the radius of convergence, even when a large number of iterations are used. The ADM solution diverges and does not converge to the numerical results in the region near  $x_0$ .

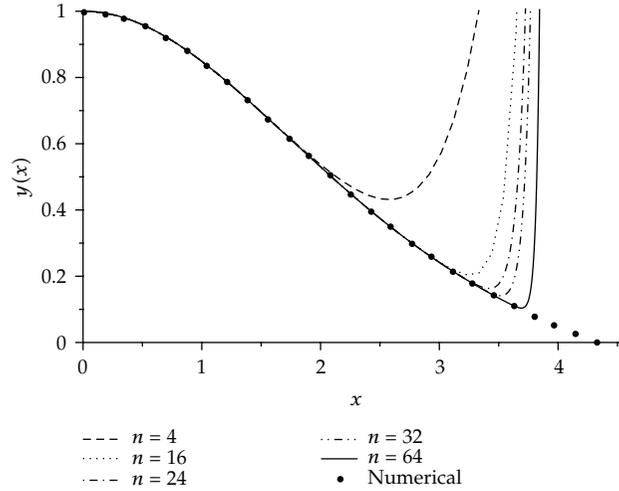


Figure 1: Solutions of the Lane-Emden equation at different orders of the ADM approximate solution.

### 4.2. MSLM Solution

Using (2.17), the MSLM iterates are given by

$$y_i(x) = - \int_0^x x^{-2} \int_0^x x^2 \left[ \sum_{m=0}^{i-1} y_m'' + \frac{2}{x} \sum_{m=0}^{i-1} y_m' + \left( \sum_{m=0}^{i-1} y_m \right)^2 \right] dx dx. \quad (4.7)$$

Equation (4.7) gives the iteration scheme for obtaining the solutions for  $y_i$  starting from the initial approximation  $y_0(x)$  which is in this example is chosen to be  $y_0(x) = 1$ . The first four solutions for  $y_i$  are given as

$$\begin{aligned} y_1 &= -\frac{1}{6}x^2, \\ y_2 &= \frac{1}{60}x^4 - \frac{1}{1512}x^6, \\ y_3 &= -\frac{1}{1260}x^6 + \frac{13}{136080}x^8 - \frac{113}{24948000}x^{10} + \frac{1}{7076160}x^{12} - \frac{1}{480090240}x^{14}. \end{aligned} \quad (4.8)$$

Thus, the approximate solution for  $y(x)$  obtained using the first 3 terms is

$$\psi_3(x) = 1 - \frac{x^2}{6} + \frac{x^4}{60} - \frac{11x^6}{7560} + \frac{13x^8}{136080} - \frac{113x^{10}}{24948000} + \frac{x^{12}}{7076160} - \frac{x^{14}}{480090240}. \quad (4.9)$$

By obtaining the first few terms of  $y_i(x)$  it is found that in general  $y_i(x)$  can be expressed as

$$y_i(x) = \sum_{j=1}^{2^i-i} a_{i,j} x^{2i+2j-2}. \quad (4.10)$$

The coefficients  $a_{i,j}$  can be determined by substituting (4.10) in (4.7) and equating powers of  $x$ . The following recursive formula is obtained:

$$a_{i,j} = - \sum_{m=m_1}^{m_2} X a_{m,j+i-m} - \frac{1}{(2i+2j-1)(2i+2j-2)} \sum_{t=t_1}^{t_2} \sum_{n=n_1}^{n_2} \sum_{m=m_3}^{m_4} a_{m,j+i-t-m} a_{n,t-n}, \quad (4.11)$$

where

$$\begin{aligned} t_1 &= \max(1, j+i-2^{i-1}), & t_2 &= \min(i+j-1, 2^{i-1}), \\ m_1 &= \max\left(0, \left\lceil \frac{\ln(j+i)}{\ln(2)} \right\rceil_c\right), & m_2 &= \min(i-1, j+i-1), \\ n_1 &= \max\left(0, \left\lceil \frac{\ln(t)}{\ln(2)} \right\rceil_c\right), & n_2 &= \min(i-1, t-1), \\ m_3 &= \max\left(0, \left\lceil \frac{\ln(j+i-t)}{\ln(2)} \right\rceil_c\right), & m_4 &= \min(i-1, j+i-t-1), \\ X &= \begin{cases} 1, & j \leq 2^{i-1} - i, \\ 0, & j > 2^{i-1} - i, \end{cases} \end{aligned} \quad (4.12)$$

where the subscript  $c$  represents the function *ceil* where  $\text{ceil}(K)$  rounds the elements of  $K$  to the nearest integers greater than or equal to  $K$ . The MSLM approximate solution for  $y(x)$  is therefore given by

$$y(x) = \sum_{k=0}^{\infty} y_k(x). \quad (4.13)$$

We remark that when (4.13) is expanded we get the series form

$$y(x) = \sum_{k=0}^{\infty} b_k x^{2k}, \quad \text{with } a_0 = 1, \quad k \geq 1. \quad (4.14)$$

which has the same form as (4.6) but with  $a_k \neq b_k$ . Comparing (4.5) and (4.9) we note that only the first three terms of the ADM coincide with the MSLM solution. The MSLM iteration results in more terms than the ADM and the number of terms increases exponentially with an increase in the number of iterations. In Table 3 we present values of the first zero  $y(x)$  at different orders of the MSLM which are compared with the numerical results of Horedt [22] which are widely used as a benchmark for testing the accuracy of new methods of solution. It can be seen from Table 3 that the MSLM method approximates the first zero  $x_0$  of  $y(x)$  with great accuracy after only a few iterations. Seven-digit accuracy is achieved after ten iterations. Compared with the ADM and power series approach we see that the MSLM is more useful as it can be used to estimate  $x_0$  and the ADM diverges before  $x$  reaches  $x_0$ . We remark that the original SLM solution of the Lane-Emden equation (4.36) reported in [21], which was

**Table 3:** Comparison of the first zero of  $y(x)$  between the present MSLM method, numerical values of [22], and results of [12, 37].

4th order	6th order	8th order	10th order	Numerical [22]	References [12, 37]
4.48977895	4.35410947	4.35287815	4.35287460	4.35287460	4.3528745959

**Table 4:** Approximation of  $y(x)$  at selected values of  $x$  using the MSLM and the corresponding run times in seconds.

$x$	4th order	6th order	8th order	10th order	Numerical [22]
0.5	0.9593527	0.9593527	0.9593527	0.9593527	0.9593527
1.0	0.8486544	0.8486541	0.8486541	0.8486541	0.8486541
1.5	0.6953802	0.6953672	0.6953671	0.6953671	0.6953671
2.0	0.5299902	0.5298366	0.5298364	0.5298364	0.5298364
2.5	0.3756041	0.3747415	0.3747393	0.3747393	0.3747393
3.0	0.2448057	0.2418367	0.2418241	0.2418241	0.2418241
3.5	0.1411111	0.1340130	0.1339691	0.1339690	0.1339690
4.0	0.0614816	0.0489446	0.0488404	0.0488402	0.0488402
Run Time(s)	0.031	0.047	0.328	18.610	

a numerical approach, gave results which converged to 4.35287458 even after increasing the number of iterations. The small error of the SLM is presumably due to the inherent errors of the numerical methods such as those that result from discretization of the domain or stability issues. Being an analytical approach, the MSLM does not suffer from the shortfalls associated with its numerical relative (SLM).

Tables 4 and 5 give the MSLM results for the values of  $y(x)$  at selected values of  $x$  and their corresponding errors when compared with the numerical results of [22]. It can be seen from these tables that the MSLM results rapidly converge to the numerical results. Unlike the ADM results, the MSLM results will converge to the numerical results even when  $x$  is close to the first zero  $x_0$ .

In Figure 2 we show the MSLM solution at different orders plotted against the numerical solution of [22]. It be seen from the figure that the MSLM rapidly converges to the numerical solution. After only eight iterations the MSLM matches with the numerical results.

### 4.3. ADM-SLM Solution

Using (2.23) and (2.24) the ADM-SLM iterates for solving Example of (4.1) are given by

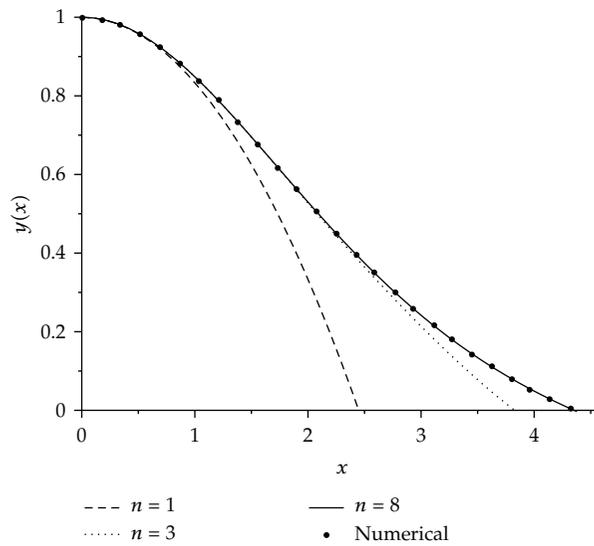
$$y_{i,0}(x) = - \int_0^x x^{-2} \int_0^x x^2 \left[ \sum_{m=0}^{i-1} y''_m + \frac{2}{x} \sum_{m=0}^{i-1} y'_m + \left( \sum_{m=0}^{i-1} y_m \right)^2 \right] dx dx, \tag{4.15}$$

$$y_{i,k+1}(x) = - \int_0^x x^{-2} \int_0^x x^2 \left[ 2 \sum_{m=0}^{i-1} y_m \right] y_{i,k} dx dx. \tag{4.16}$$

We observe that (4.15) is exactly the same as (4.7) for the MSLM iterations. This indicates that the ADM-SLM approach uses the MSLM solution as an initial guess and improves on it by

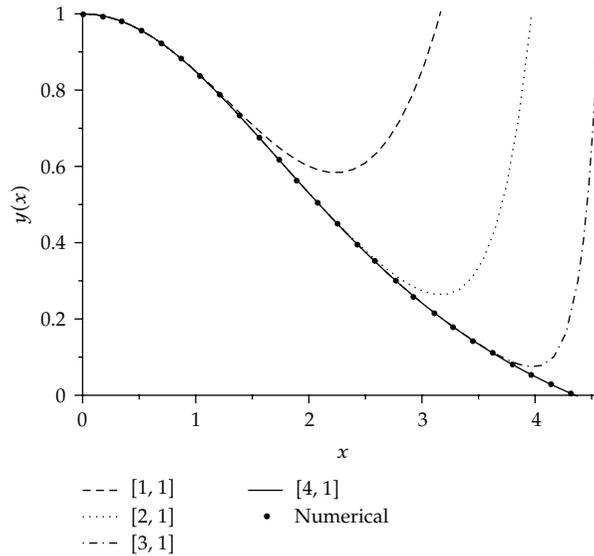
**Table 5:** Errors of the MSLM at selected values of  $x$ .

$x$	4th order	6th order	8th order	10th order
0.5	0.0000000	0.0000000	0.0000000	0.0000000
1.0	0.0000003	0.0000000	0.0000000	0.0000000
1.5	0.0000131	0.0000001	0.0000000	0.0000000
2.0	0.0001538	0.0000002	0.0000000	0.0000000
2.5	0.0008648	0.0000022	0.0000000	0.0000000
3.0	0.0029816	0.0000126	0.0000000	0.0000000
3.5	0.0071421	0.0000440	0.0000001	0.0000000
4.0	0.0126415	0.0001044	0.0000003	0.0000000

**Figure 2:** Solutions of the Lane-Emden equation at different orders of the MSLM approximate solution.

implementing the ADM approach to the resulting linearised equation. Thus, starting from the initial approximation  $y_0(x)$ , (4.15) is used to find the initial approximation to be used in the ADM-SLM iterative scheme (4.16). The approximate solution for  $y(x)$  is then obtained using (2.26). The first few solutions for  $y_{i,n}$  are obtained as

$$\begin{aligned}
 y_{1,0} &= -\frac{1}{6}x^2, \\
 y_{1,1} &= \frac{1}{60}x^4, \\
 y_{2,0} &= -\frac{11}{7560}x^6 + \frac{1}{12960}x^8 - \frac{1}{396000}x^{10}, \\
 y_{2,1} &= \frac{11x^8}{272160} - \frac{29x^{10}}{4989600} + \frac{5933x^{12}}{11675664000} - \frac{73x^{14}}{4490640000} + \frac{x^{16}}{3231360000}.
 \end{aligned} \tag{4.17}$$



**Figure 3:** Solutions of the Lane-Emden equation at different orders of the SLM-ADM approximate solution.

**Table 6:** Comparison of the first zero of  $y(x)$  between the present ADM-SLM method, numerical values of [22] and [12, 37].

$y_{4,1}$	$y_{5,1}$	$y_{6,1}$	Numerical [38]	Reference [12, 37]
4.356001615	4.352874609	4.352874596	4.35287460	4.3528745959

Thus, the approximate solution for  $y(x)$  obtained using 2 iterations in the SLM direction and 1 iteration in the ADM direction, that is,  $\chi_{2,1}$ , is

$$y(x) \approx \chi_{2,1}(x) = y_0 + y_1 + y_2 = y_0 + y_{1,0} + y_{1,1} + y_{2,0} + y_{2,1}, \tag{4.18}$$

$$= 1 - \frac{x^2}{6} + \frac{x^4}{60} - \frac{11x^6}{7560} + \frac{x^8}{8505} - \frac{13x^{10}}{1559250} + \frac{5933x^{12}}{11675664000} - \frac{73x^{14}}{4490640000} + \frac{x^{16}}{3231360000}. \tag{4.19}$$

Comparing the ADM solution (4.5) and the MSLM solution (4.9) with the ADM-SLM solution (4.19) for the first few terms, it can be seen that the three solutions are different.

In Figure 3 we plot the approximate solution  $y(x)$  of Example of (4.1) for the first four iterations of the ADM-SLM. The approximate results are compared against the numerical results of [22]. It can be seen from Figure 3 that with each additional iteration, the radius of convergence of the ADM-SLM rapidly increases. Unlike in the case of the ADM, as can be seen in Figure 1, the ADM-SLM solution converges to the numerical solutions after only [4, 1] iterations, that is a total of eight iterations like in the MSLM case.

In Table 6 we present values of the first zero  $y(x)$  at different orders of the ADM-SLM. The results are compared with the seven-digit accurate numerical results of Horedt [22] and ten-digit numerical results reported in [12, 37]. It can be seen from Table 6 that the MSLM method approximates the first zero  $x_0$  of  $y(x)$  with great accuracy after only a few iterations.

**Table 7:** Approximation of  $y(x)$  at selected values of  $x$  using the ADM-SLM approach and the corresponding run times in seconds.

$x$	[3, 1]	[4, 1]	[5, 1]	[6, 1]	Numerical [22]
0.5	0.9593527	0.9593527	0.9593527	0.9593527	0.9593527
1.0	0.8486541	0.8486541	0.8486541	0.8486541	0.8486541
1.5	0.6953672	0.6953671	0.6953671	0.6953671	0.6953671
2.0	0.5298369	0.5298364	0.5298364	0.5298364	0.5298364
2.5	0.3747452	0.3747393	0.3747393	0.3747393	0.3747393
3.0	0.2418815	0.2418241	0.2418241	0.2418241	0.2418241
3.5	0.1349246	0.1339693	0.1339690	0.1339690	0.1339690
4.0	0.0745177	0.0488451	0.0488402	0.0488402	0.0488402
Run Time(s)	0.032	0.087	0.891	45.203	

**Table 8:** Errors of the ADM-SLM at selected values of  $x$ .

$x$	[3, 1]	[4, 1]	[5, 1]	[6, 1]
0.5	0.0000000	0.0000000	0.0000000	0.0000000
1.0	0.0000000	0.0000000	0.0000000	0.0000000
1.5	0.0000001	0.0000000	0.0000000	0.0000000
2.0	0.0000005	0.0000000	0.0000000	0.0000000
2.5	0.0000059	0.0000000	0.0000000	0.0000000
3.0	0.0000574	0.0000000	0.0000000	0.0000000
3.5	0.0009556	0.0000003	0.0000000	0.0000000
4.0	0.0256775	0.0000049	0.0000000	0.0000000

It can be seen from Table 6 that the ADM-SLM results are accurate to seven digits after [5, 1] iterations (total of 10 iterations) and accurate to nine digits after [6, 1] iterations. This illustrates that this approach converges much more rapidly to the numerical solutions than the MSLM in which seven-digit accuracy was achieved after 10 iterations.

Tables 7 and 8 give the ADM-SLM results for the values of  $y(x)$  at selected values of  $x$  and their corresponding errors when compared with the numerical results of [22]. These tables indicate that the ADM-SLM results converge rapidly to the numerical results.

Comparing the run times between the MSLM and ADM-SLM, for the number of iterations that give seven-digit accuracy, from Tables 4 and 7, respectively, it can be seen that seven-digit accuracy in the range  $0 < x < 4$  is achieved after only 0.891 seconds and [5, 1] iterations (total of 10 iterations) in the ADM-SLM compared to 18.61 seconds and 10 iterations in the case of MSLM. This result shows that the ADM-SLM is much more computationally efficient than the MSLM.

In Table 9 we give a comparison between the rates of convergence of the three methods. It can be seen from the table that the ADM with the largest  $\alpha_i$  converges the slowest among the three methods and the ADM-SLM converges the fastest.

## 5. Conclusion

In this work, we presented two new reliable algorithms for solving Lane-Emden type equations that model polytropic stars. The methods considered are the modified successive

**Table 9:** Comparison between the convergence rates of ADM, MSLM and ADM-SLM for  $[i, 1]$ .

$\alpha_i$	ADM	MSLM	ADM-SLM
$\alpha_0$	0.16666667	0.16666667	0.16666667
$\alpha_1$	0.10000000	0.10000000	0.00873016
$\alpha_2$	0.08730159	0.04761905	0.00050505
$\alpha_3$	0.08080808	0.02777778	0.00012210

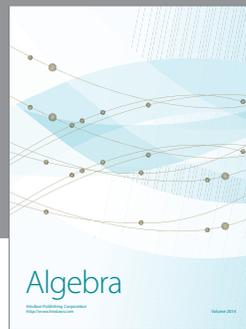
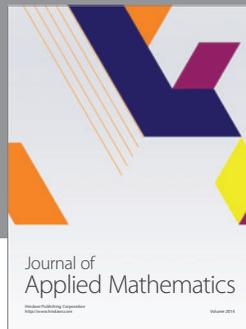
linearisation method (MSLM) and a hybrid method (ADM-SLM) that blends the Adomian decomposition method (ADM) and the successive linearization method (SLM). The applicability of these methods was illustrated by solving the Lane-Emden equation of index two. The Lane-Emden equation of index two was used as an example to illustrate the applicability of the methods of solution and assess their performance in terms of accuracy, convergence, effectiveness, and validity. When considering the convergence rates it was found that the ADM-SLM converged faster than the MSLM which in turn converged faster than then ADM. Both the ADM-SLM and the MSLM were found to be effective in finding the first zero  $x_0$  of equation governing the polytropic model. On the other hand, the region of convergence of the ADM was found to be less than  $x_0$  which means the ADM on its own cannot be used to estimate  $x_0$ . The MSLM and ADM-SLM were compared with previously published numerical results and they were found to quickly converge to the numerical results. Unlike the ADM, the ADM-SLM and MSLM were found to converge to the true solution for all  $x$  not just small  $x < x_0$ . This verifies the validity and reliability of the methods in giving accurate results. In terms of computational efficiency, the ADM-MSLM was found to be better than the MSLM because the implementation of its algorithm took much less time than the MSLM algorithm. We conclude that both the MSLM and ADM-SLM are promising tools for solving both linear and nonlinear initial value problems of the Lane-Emden type.

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