Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2012, Article ID 617390, 13 pages
doi:10.1155/2012/617390

Research Article

New Preconditioners with Two Variable Relaxation Parameters for the Discretized Time-Harmonic Maxwell Equations in Mixed Form

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Received 4 May 2012; Revised 3 July 2012; Accepted 4 July 2012

Academic Editor: Ion Zaballa

We provide new preconditioners with two variable relaxation parameters for the saddle point linear systems arising from finite element discretization of time-harmonic Maxwell equations in mixed form. The new preconditioners are of block-triangular forms and Schur complement-free. They are extensions of the results in Cheng et al., 2009, Grief and Schötzau, 2007, and Huang et al., 2009. Theoretical analysis shows that all eigenvalues of the preconditioned matrices are tightly clustered, and numerical tests confirm our analysis.

1. Introduction

We consider the preconditioning techniques for solving the saddle point linear systems arising from finite element discretization of the following time-harmonic Maxwell equations in mixed form [1–5]: find the vector field u and the Lagrangian multiplier p such that

$$\begin{gathered}
\nabla \times \nabla \times u - k^2 u + \nabla p = f \quad \text{in } \Omega, \\
\nabla \cdot u = 0 \quad \text{in } \Omega, \\
u \times \vec{n} = 0 \quad \text{on } \partial\Omega, \\
p = 0 \quad \text{on } \partial\Omega.
\end{gathered} \tag{1.1}$$

Here, $\Omega \subset \mathbb{R}^2$ is a simply connected polyhedron domain with a connected boundary $\partial\Omega$, and $\vec{n}$ denotes the outward unit normal on $\partial\Omega$. The datum f is a given source (not necessarily divergence-free), and the wave number $k^2 = \omega^2 \epsilon \mu$, where $\omega \geq 0$ is the frequency, and ϵ and μ are positive permittivity and permeability parameters, respectively.

In recent years, there have been many techniques for solving Maxwell equations, such as the geometry multigrid methods [6–8], algebraic multigrid methods [9], domain decomposition methods [4, 10–13], Nodal auxiliary space preconditioning methods [14], and the solution methods to the corresponding saddle-point linear systems [2, 3, 15]. We can also use Uzawa-type iterative methods [16, 17] and preconditioned Krylov subspace methods [18–24] to solve the saddle-point linear systems. Based on the previous works in [2, 3, 15], we will further study solution methods for the saddle-point linear systems in this paper.

Using Nédélec elements of the first kind [25–27] for the approximation of the vector field and standard nodal elements for the Lagrangian multiplier yields the following saddle-point linear system:

$$\mathcal{A}x \equiv \begin{pmatrix} A - k^2 M & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ p \end{pmatrix} = \begin{pmatrix} g \\ 0 \end{pmatrix} \equiv b, \tag{1.2}$$

where $u \in \mathbf{R}^n$ and $p \in \mathbf{R}^m$ are finite arrays, and $g \in \mathbf{R}^n$ is a load vector associated with f. The matrix $A \in \mathbf{R}^{n \times n}$ is symmetric positive semidefinite with nullity m and corresponds to the curl-curl operator; $B \in \mathbf{R}^{m \times n}$ is a discrete divergence operator with full-row rank, and $M \in \mathbf{R}^{n \times n}$ is a vector mass matrix.

For convenience, we denote the standard Euclidean inner product of vectors by $\langle \cdot, \cdot \rangle$ and the null space of a matrix by null $(\cdot)$. For a given positive (semi)definite matrix W and a vector x, we define the (semi)norm:

$$|x|_W = \sqrt{\langle Wx, x \rangle}. \tag{1.3}$$

The matrices A and B have the following stability properties [3]. Let $\langle Au, u \rangle = |u|_A^2$. Then there exists an α, $0 < \alpha < 1$, such that

$$|u|_A^2 \geq \overline{\alpha} |u|_M^2, \quad u \in \text{null}(B), \tag{1.4}$$

where $\overline{\alpha} = \alpha / (1 - \alpha)$. Matrix B satisfies the discrete inf-sup condition:

$$\inf_{0 \neq q \in \mathbf{R}^m} \sup_{0 \neq v \in \text{ null}(A)} \frac{\langle Bv, q \rangle}{|v|_M |q|_L} \geq \beta > 0, \tag{1.5}$$

where the inf-sup constant $\beta > 0$ is only dependent on the domain Ω.

If the wave number $k^2 > 0$, then the $(1,1)$ block of (1.2) is indefinite. For difficulty and corresponding solution methods of this problem, we refer to [18, 28]. Recently, by using the spectral equivalent properties similar to [4], Grief and Schötzau [3] construct the block-diagonal preconditioner:

$$\mathcal{M}_k = \begin{pmatrix} A - k^2 M + B^T L^{-1} B & 0 \\ 0 & L \end{pmatrix}, \tag{1.6}$$

where $L \in \mathbf{R}^{m \times m}$ is the discrete Laplace operator introduced in [3], $k^2 < 1$, and $\mathcal{M}_k$ is a symmetric positive definite block-diagonal matrix. As L is augmentation-free and Schur complement-free, this approach overcomes the difficulty in forming the Schur complement in general. However, the computational work of $B^T L^{-1} B$ may be too large. Using the fact that the matrices $A + B^T L^{-1} B$ and $A + M$ are spectrally equivalent, [3] considers the following preconditioner:

$$\widehat{\mathcal{M}}_k = \begin{pmatrix} A + (1 - k^2) M & 0 \\ 0 & L \end{pmatrix}, \tag{1.7}$$

and shows that the eigenvalues of the preconditioned matrix are tightly clustered.

Based on the work of Grief and Schötzau [3], [2] gives block-triangular Schur complement-free preconditioners for the linear system (1.2). And it is shown that all eigenvalues of the proposed block-triangular preconditioning matrices are more tightly clustered. Compared with the restriction $k^2 < 1$ in [3], [2] considers the general case $k^2 \in \mathbf{R}^+$. Furthermore, [15] provides block-triangular preconditioners when $k^2 = 0$ with two variable relaxation parameters.

Based on the previous work [2, 3, 15], mentioned above, in this paper we are devoted to give new preconditioners with two scaling parameters. The new block triangular preconditioners in the general case $k^2 \in \mathbf{R}^+$ contain the preconditioners discussed in [2]. Theoretical analysis shows that all eigenvalues of the preconditioned matrices are tightly clustered. Numerical experiments demonstrate efficiency of the new method and show that preconditioner $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}$ is more efficient than $\widehat{\mathcal{M}}_{k,t}$.

The remainder of the paper is as follows. In Section 2, we establish new block-triangular preconditioners for the linear systems (1.2) in the general case $k^2 \in \mathbf{R}^+$, and then the corresponding spectral analysis is presented. In Section 3, we provide numerical examples to examine our analysis. Finally, some conclusions are drawn in Section 4.

2. New Block-Triangular Preconditioners for Any k^2

We consider the saddle-point linear system (1.2) arising from the discretized time-harmonic Maxwell equations in mixed form (1.1) and assume that k^2 is not an eigenvalue and $k^2 \in \mathbf{R}^+$.

Grief and Schötzau [3] provide the block-diagonal Schur complement-free preconditioner $\mathcal{M}_k$ as in (1.6). Using the fact that the matrices $A + B^T L^{-1} B$ and $A + M$ are spectrally equivalent, the argumentation-free and Schur complement-free preconditioner $\widehat{\mathcal{M}}_k$ is defined in (1.7). Spectral analysis shows that the eigenvalues of the preconditioned saddle-point matrices $\mathcal{M}_k^{-1}\mathcal{A}$ and $\widehat{\mathcal{M}}_k^{-1}\mathcal{A}$ are strongly clustered when k^2 is small.

Reference [2] provides the block-triangular Schur complement-free preconditioners for the linear system (1.2). In particular, they considered preconditioning matrices for the general case $k^2 \in \mathbf{R}^+$ with

$$\begin{gathered} \mathcal{M}_{k,t} = \begin{pmatrix} A - k^2 M + t B^T L^{-1} B & t B^T \\ 0 & \dfrac{1-t}{t} L \end{pmatrix}, \\ \widehat{\mathcal{M}}_{k,t} = \begin{pmatrix} A + (t - k^2) M & t B^T \\ 0 & \dfrac{1-t}{t} L \end{pmatrix}, \end{gathered} \tag{2.1}$$

where $1 \neq t > k^2$.

For $k^2 = 0$, [15] provides the block-triangular preconditioner for linear system (1.2):

$$\mathcal{P}_{\eta,\varepsilon} = \begin{pmatrix} A + \eta B^T L^{-1} B & (1-\eta\varepsilon)B^T \\ 0 & \varepsilon L \end{pmatrix}, \tag{2.2}$$

where $\eta > 0$ and $\varepsilon > 0$.

Based on the works in [2, 3, 15], we provide the following new block-triangular Schur complement-free preconditioners $\mathcal{M}_{k,\eta,\varepsilon}$ and $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}$:

$$\mathcal{M}_{k,\eta,\varepsilon} = \begin{pmatrix} A - k^2 M + \eta B^T L^{-1} B & (1-\eta\varepsilon)B^T \\ 0 & \varepsilon L \end{pmatrix}, \tag{2.3}$$

$$\widehat{\mathcal{M}}_{k,\eta,\varepsilon} = \begin{pmatrix} A + (\eta - k^2) M & (1-\eta\varepsilon)B^T \\ 0 & \varepsilon L \end{pmatrix}, \tag{2.4}$$

where $\eta > k^2$ and $\varepsilon \neq 0$ are scaling parameters. It is interesting to note that when parameters $\eta = t$ and $\varepsilon = (1-t)/t$, $\mathcal{M}_{k,\eta,\varepsilon}$ and $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}$ apparently reduce to $\mathcal{M}_{k,t}$ and $\widehat{\mathcal{M}}_{k,t}$, respectively. We also see that when $k^2 = 0$, the preconditioner $\mathcal{M}_{k,\eta,\varepsilon}$ in (2.3) ($\varepsilon \neq 0$) is different from $\mathcal{P}_{\eta,\varepsilon}$ in (2.2) ($\varepsilon > 0$).

We stress that $\mathcal{M}_{k,\eta,\varepsilon}$ is not the preconditioner we eventually use in actual computation. It is only introduced to lay theoretical basis and motivation for the preconditioner $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}$ in (2.4), which we will use in practice. We note that the (1,1) block $A - k^2 M + \eta B^T L^{-1} B$ in $\mathcal{M}_{k,\eta,\varepsilon}$ is symmetric positive definite for k is sufficiently small [3]. But this is not true when k is large enough. However, this situation may not appear in the actual preconditioner $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}$. The (1,1) block $A + (\eta - k^2)M$ in $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}$ is always symmetric positive definite when $\eta > k^2$. In this paper, we will apply the BiCGSTAB with the preconditioner $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}$ as an outer solver for the saddle-point system (1.2). Then, the overall computational cost of solution procedure relies on how to efficiently solve the linear systems $A + \tau M (\tau = \eta - k^2)$ and L, which are called by inner solvers. For the linear system L arising from a standard scalar elliptic problem, many efficient solution methods exist. On the other hand, for solving the linear system $A + \tau M$, we refer to [6, 8, 9, 14], and some detailed numerical examples are provided in [29].

For the spectral analysis, we recall some results which are contained in the following lemma.

Lemma 2.1 (see [3]). *The following relations hold:*

(i) $\mathcal{R}^n = \text{null}(A) \oplus \text{null}(B)$;

(ii) $\langle M u_A, u_B \rangle = 0$ for any $u_A \in \text{null}(A)$ and any $u_B \in \text{null}(B)$;

(iii) $\langle B^T L^{-1} B u_A, u_A \rangle = \langle M u_A, u_A \rangle$ for any $u_A \in \text{null}(A)$.

Theorem 2.2. *Let $\mathcal{A}$ be the saddle-point matrix in* (1.2). *Then the matrix $\mathcal{M}_{k,\eta,\varepsilon}^{-1}\mathcal{A}$ has two distinct eigenvalues, which are given by*

$$\lambda_1 = 1, \qquad \lambda_2 = -\frac{1}{\varepsilon(\eta - k^2)}, \tag{2.5}$$

with the algebraic multiplicities n and m, respectively.

Proof. Suppose that λ is an eigenvalue of $\mathcal{M}_{k,\eta,\varepsilon}^{-1}\mathcal{A}$, whose eigenvector is $\binom{v}{q}$. Then the corresponding eigenvalue problem is

$$\begin{pmatrix} A - k^2 M & B^T \\ B & 0 \end{pmatrix}\begin{pmatrix} v \\ q \end{pmatrix} = \lambda \begin{pmatrix} A - k^2 M + \eta B^T L^{-1} B & (1-\eta\varepsilon)B^T \\ 0 & \varepsilon L \end{pmatrix}\begin{pmatrix} v \\ q \end{pmatrix}. \tag{2.6}$$

From the second row we can obtain $q = (1/\lambda\varepsilon)L^{-1}Bv$. By substituting it into the first row we have

$$(1-\lambda)\left[\lambda\left(A - k^2 M\right)v + \left(\frac{1}{\varepsilon} + \lambda\eta\right)B^T L^{-1} B v\right] = 0. \tag{2.7}$$

It is straightforward to see that any vector $v \in \mathcal{R}^n$ satisfies (2.7) with $\lambda = 1$, so $\lambda = 1$ is an eigenvalue of $\mathcal{M}_{k,\eta,\varepsilon}^{-1}\mathcal{A}$. By the similar technique of linear independence considerations from [3], we can demonstrate that the eigenvalue $\lambda = 1$ has algebraic multiplicity n.

Since there are m linearly independent null vectors of A, by Lemma 2.1,

$$v = v_A + v_B (v_A \neq 0), \quad v_A \in \operatorname{null}(A),\ v_B \in \operatorname{null}(B). \tag{2.8}$$

By Lemma 2.1 (ii) and (iii), and using the inner product in (2.7) with v_A, we have

$$(1-\lambda)\left[\frac{1}{\varepsilon} + \lambda\left(\eta - k^2\right)\right]|v_A|_M^2 = 0. \tag{2.9}$$

Since $v_A \neq 0$, from (2.9) we can obtain that $\lambda = -1/\varepsilon(\eta - k^2)$ is another eigenvalue of $\mathcal{M}_{k,\eta,\varepsilon}^{-1}\mathcal{A}$, and we claim that the eigenvalue $\lambda = -1/\varepsilon(\eta - k^2)$ has algebraic multiplicity m. □

Corollary 2.3. *Let $-1/\varepsilon = \eta - k^2$. Then the corresponding preconditioned matrix $\mathcal{M}_{k,\eta,\varepsilon}^{-1}\mathcal{A}$ has only one eigenvalue $\lambda = 1$ of algebraic multiplicity $n + m$.*

Proof. From Theorem 2.2, we can easily obtain the corresponding conclusion. □

Remark 2.4. From Theorem 2.2, we demonstrate that the preconditioned matrix $\mathcal{M}_{k,\eta,\varepsilon}^{-1}\mathcal{A}$ has precisely two distinct eigenvalues. Then if Krylov subspace methods are used to solve (1.2) with $\mathcal{M}_{k,\eta,\varepsilon}$ as a preconditioner, the iteration will require merely two steps if round-off errors are ignored [30]. And from Corollary 2.3, for any η, we can find a number ε which makes the preconditioned matrix $\mathcal{M}_{k,\eta,\varepsilon}^{-1}\mathcal{A}$ have only one eigenvalue. Therefore, we can demonstrate that our preconditioners are more efficient than the block-triangular preconditioner proposed in [2].

Remark 2.5. From (2.3) we know that if $\eta\varepsilon = 1$, the new preconditioner reduces to the diagonal preconditioner $\mathcal{M}_{k,\eta}$:

$$\mathcal{M}_{k,\eta} = \begin{pmatrix} A - k^2 M + \eta B^T L^{-1} B & 0 \\ 0 & \dfrac{1}{\eta} L \end{pmatrix}. \tag{2.10}$$

Then we can use MINRES to solve the linear system (1.2).

Theorem 2.6. *Let $\mathcal{A}$ be the saddle-point matrix in* (1.2). *Then*

$$\lambda_1 = 1, \qquad \lambda_2 = -\frac{1}{\varepsilon(\eta - k^2)} \tag{2.11}$$

are the eigenvalues of $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}^{-1}\mathcal{A}$, having algebraic multiplicity m. The rest of the eigenvalues satisfy

$$\frac{\overline{\alpha} - k^2}{\overline{\alpha} + \eta - k^2} \leq \lambda < 1, \tag{2.12}$$

where $\overline{\alpha}$ is defined as in (1.4).

Proof. Suppose that λ is an eigenvalue of $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}^{-1}\mathcal{A}$, whose eigenvector is $\binom{v}{q}$. Then the corresponding eigenvalue problem is

$$\begin{pmatrix} A - k^2 M & B^T \\ B & 0 \end{pmatrix}\begin{pmatrix} v \\ q \end{pmatrix} = \lambda \begin{pmatrix} A + (\eta - k^2) M & (1 - \eta\varepsilon) B^T \\ 0 & \varepsilon L \end{pmatrix}\begin{pmatrix} v \\ q \end{pmatrix}. \tag{2.13}$$

From the second row we can obtain $q = (1/\varepsilon\lambda)(L^{-1}Bv)$. By substituting it into the first row we have

$$\lambda\left(A - k^2 M\right)v + \frac{1}{\varepsilon}B^T L^{-1}Bv = \lambda^2\left(A + \left(\eta - k^2\right)M\right)v + \frac{\lambda}{\varepsilon}(1 - \eta\varepsilon)B^T L^{-1}Bv. \tag{2.14}$$

Consider the m linearly independent null vectors of A, by Lemma 2.1 (i),

$$v = v_A + v_B \quad (v_A \neq 0), \tag{2.15}$$

where $v_A \in \text{null}(A)$ and $v_B \in \text{null}(B)$. By Lemma 2.1 (ii) and (iii), and taking the inner product in (2.14) with v_A, we obtain

$$(1 - \lambda)\left[\frac{1}{\varepsilon} + \lambda\left(\eta - k^2\right)\right]|v_A|_M^2 = 0. \tag{2.16}$$

Since $|v_A|_M \neq 0$, $\lambda_1 = 1$ and $\lambda_2 = -1/(\varepsilon(\eta - k^2))$ are two eigenvalues of $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}^{-1}\mathcal{A}$ and by the similar technique of linear independence considerations from [3], we claim that each eigenvalue has algebraic multiplicity m.

For the rest of eigenvectors we have $v_B \neq 0$. Noting that

$$\left\langle B^T L^{-1} B v_A, v_B \right\rangle = \left\langle L^{-1} B v_A, B v_B \right\rangle = 0, \tag{2.17}$$

by Lemma 2.1 (ii) and by taking the inner product in (2.14) with v_B and using (2.17), we obtain

$$(1 - \lambda)|v_B|_A^2 = \left(\lambda\left(\eta - k^2\right) + k^2\right)|v_B|_M^2. \tag{2.18}$$

It is impossible to have $\lambda = 1$, since (2.18) leads to $|v_B|_M = 0$, which contradicts with $v_B \neq 0$. We cannot have $\lambda > 1$, since the left-hand side is negative but the right-hand side is positive (because we assume $\eta > k^2$). Thus, we claim that $\lambda < 1$.

From (1.4), we recall that for any $u \in \text{null}(B)$, $|u|_A^2 \geq \overline{\alpha}|u|_M^2$ with $\overline{\alpha} = \alpha/(1-\alpha)$. Applying this to (2.18), we have $(\lambda(\eta - k^2) + k^2)/(1-\lambda) \geq \overline{\alpha}$. Since $\eta > k^2 > 0$, we have $\overline{\alpha} + \eta - k^2 > 0$ and

$$\frac{\overline{\alpha} - k^2}{\overline{\alpha} + \eta - k^2} \leq \lambda < 1. \tag{2.19}$$

□

Corollary 2.7. *Let $-1/\varepsilon = \eta - k^2$. Then the corresponding preconditioned matrix $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}\mathcal{A}$ has only one eigenvalue $\lambda = 1$ with algebraic multiplicity 2m. The remaining eigenvalues satisfy* (2.12).

Remark 2.8. From (2.4) we know that when $\eta\varepsilon = 1$, the new preconditioner reduces to the diagonal preconditioner $\widehat{\mathcal{M}}_{k,\eta}$:

$$\widehat{\mathcal{M}}_{k,\eta} = \begin{pmatrix} A + (\eta - k^2)M & 0 \\ 0 & \dfrac{1}{\eta}L \end{pmatrix}. \tag{2.20}$$

Then we can use MINRES to solve the linear system (1.2).

Remark 2.9. From the proof of Theorem 2.6, we easily see that the new preconditioner $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}$ is also efficient for $k^2 = 0$. Then from (2.12), we conclude that if $-1/\varepsilon = \eta - k^2$ and $k^2 = 0$, then the closer η is to 0 and the closer $(\overline{\alpha} - k^2)/(\overline{\alpha} + \eta - k^2) = \overline{\alpha}/(\overline{\alpha} + \eta)$ is to 1; that is, the preconditioned matrix has more tightly clustered eigenvalues. For a general case of $k^2 \in \mathbb{R}^+$, we can only obtain similar results when $\overline{\alpha} > k^2$. The following numerical experiments show that the closer η is to k^2, the less iteration counts we have used for a fixed $k^2 \in \mathbb{R}^+$. However, choosing $\eta - k^2$ too small may result in too large ε, then result in ill-conditioning of $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}$. So we choose $\eta - k^2$ to be moderate size in practice.

3. Numerical Experiments

The test problem is a two-dimensional time-harmonic Maxwell equations in mixed form (1.1) in a square domain $\Omega = (0 < x < 1; 0 < y < 1)$. We set the right-hand side function so that the exact solution is given by

$$u(x,y) = \begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix} = \begin{pmatrix} y(1-y) \\ x(1-x) \end{pmatrix} \tag{3.1}$$

and $p \equiv 0$.

We consider five uniformly refined meshes, which are constructed by subsequently splitting each triangle into four triangles by joining the midpoints of the edges of the triangle. Two of five mesh grids are depicted in Figures 1 and 2. The lowest order elements are used to discretize equations. The matrix sizes on different meshes are given in Table 1.

Table 1: Values of matrix size of the linear system for five meshes.

Mesh	n	m	$n + m$
8×8	176	49	225
16×16	736	225	961
32×32	3008	961	3969
64×64	12160	3969	16129
128×128	48896	16129	65025

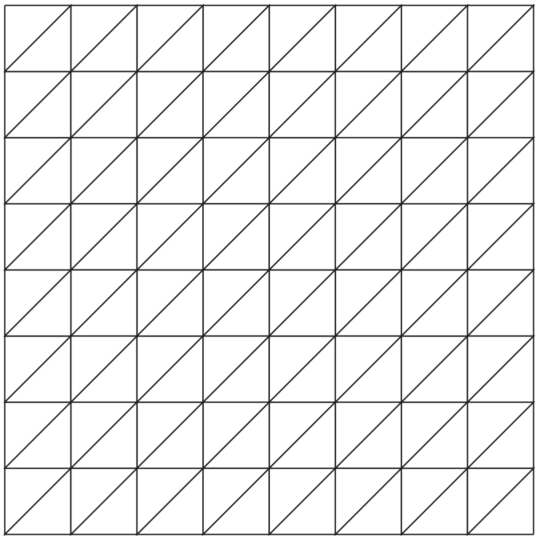

Figure 1: 8×8 mesh.

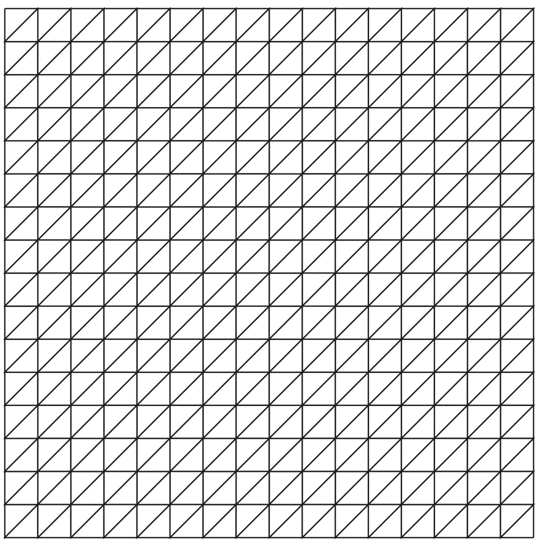

Figure 2: 16×16 mesh.

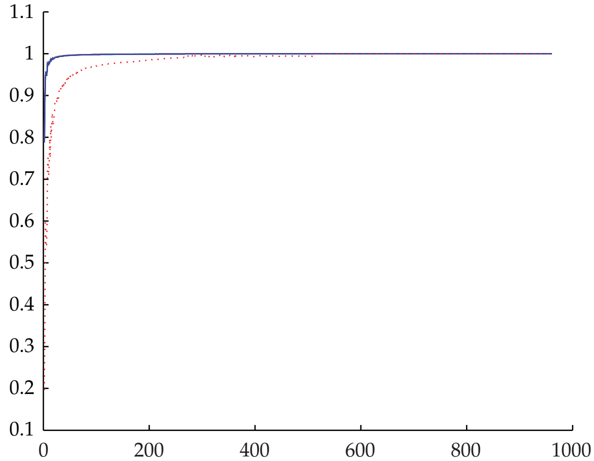

Figure 3: The eigenvalue distribution of the preconditioned matrix $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}^{-1}\mathcal{A}$ with $k^2 = 2$, $\eta = 2.1$ and $\eta = 32$, respectively, and $m + n = 961$. The case $\eta = 2.1$ is indicated by the solid line while the case $\eta = 32$ is indicated by the dotted line.

Our numerical experiments were performed using MATLAB. The machine is a PC-Intel (R), Pentium(R)Dual CPU E2200 2.20 GHz, 1.00 G of RAM. The purpose of our experiments is to investigate the convergence behavior of preconditioned BiCGSTAB by choosing different parameters η and and ε in the preconditioner $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}$. Thus, we apply exact inner solver, and the outer iteration is used as a zero initial guess and stopped when $\|r^{(k)}(= b - \mathcal{A}x^{(k)})\|_2/\|r^{(0)}(= b)\|_2 \leq 5 \times 10^{-10}$.

From Theorem 2.6 and Corollary 2.7 we know that the preconditioned matrix $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}^{-1}\mathcal{A}$ has one eigenvalue $\lambda = 1$, and the remaining eigenvalues are satisfying (2.12). Figure 3 depicts the eigenvalues of the preconditioned matrix $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}^{-1}\mathcal{A}$ with $k^2 = 2$, where η and ε satisfy $-1/\varepsilon = \eta - k^2$, and $m + n = 961$. From it we observe that the eigenvalue distribution of preconditioned matrix $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}^{-1}\mathcal{A}$ with $\eta = k^2 + 0.1 = 2.1$ denoted by solid line is more tightly clustered than with $\eta = k^2 + 30 = 32$ denoted by dotted line. From Remark 2.9 we know that the new preconditioner $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}$ is also efficient for $k^2 = 0$. Figures 4, 5, 6, and 7 show the eigenvalue distribution of the preconditioned matrix $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}^{-1}\mathcal{A}$ for different η with $k^2 = 0$, where η and ε satisfy $-1/\varepsilon = \eta - k^2$, and $m + n = 961$. From Figures 4–7 we know that the closer the parameter η is to 0, the more tightly clustered the eigenvalues of the preconditioned matrix will be.

Table 2 shows the outer iteration counts for different k^2 and η, applying BiCGSTAB with the block-triangular preconditioner, where η and ε satisfy $-1/\varepsilon = \eta - k^2$, and $m + n = 16129$. The iteration counts are denoted by Iter. We observe that for a fixed k^2, the closer η is to k^2, the less iteration counts are produced. For comparison, we also give the outer iteration counts for $\eta = t^* = 1 + (k^2 + \sqrt{1 + k^4})/2$. We refer the definition of t^* to [2]. It shows that the preconditioner $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}$ is more efficient than $\widehat{\mathcal{M}}_{k,t}$.

Tables 3 and 4 show the outer iteration numbers for different meshes, applying BiCGSTAB with the preconditioner $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}$, where η are set to be $\eta - k^2 = 0.1$ and $\eta - k^2 = 6$, and $-1/\varepsilon = \eta - k^2$. We observe that the outer iteration numbers of the preconditioned BiCGSTAB are hardly sensitive to the changes in the mesh size.

Table 2: Iteration counts for different k^2 and η, using BiCGSTAB with the preconditioner $\widehat{M}_{k,\eta,\varepsilon}$, and $m+n = 16129$. The iteration was stopped once $\|r^{(k)}\|/\|r^{(0)}\| \le 5 \times 10^{-10}$.

η	k^2	0.1	0.2	0.5	1	2	3	4	5
$k^2 + 0.001$	Iter	2	2	2.5	2.5	3	3.5	4	4.5
$k^2 + 0.01$	Iter	2	2	2.5	2.5	3	3.5	4	4.5
$k^2 + 0.1$	Iter	2	2	2.5	2.5	3	3.5	4	4.5
$k^2 + 0.25$	Iter	2.5	2.5	2.5	2.5	3.5	3.5	4	4.5
$k^2 + 0.5$	Iter	2.5	2.5	2.5	2.5	3.5	4	4	4.5
$k^2 + 1$	Iter	2.5	2.5	2.5	3	3.5	4	4	4.5
$k^2 + 1.5$	Iter	2.5	2.5	3	3	3.5	4	4.5	4.5
$k^2 + 2$	Iter	3	3	3	3.5	3.5	4	4.5	4.5
$k^2 + 5$	Iter	3.5	3.5	3.5	4	4.5	4.5	4.5	5
$k^2 + 15$	Iter	4.5	5	5	5	5	5.5	5.5	5.5
$k^2 + 30$	Iter	5.5	5.5	5.5	6	6	6	6	6
$t^* = 1 + ((k^2 + \sqrt{1+k^4})/2)$	Iter	3	3	3	3	3.5	4	4	4.5

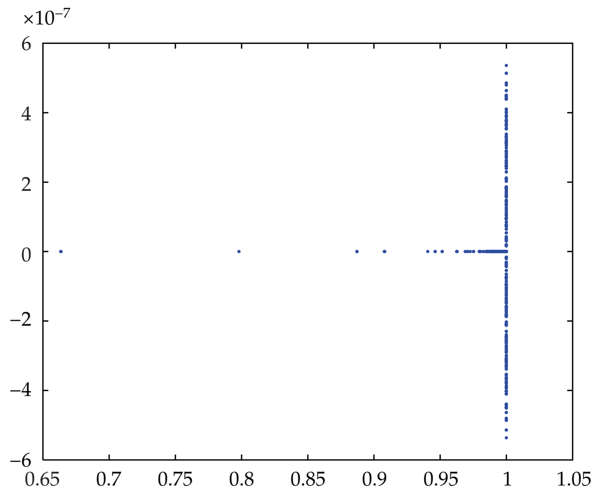

Figure 4: The eigenvalue distribution of the preconditioned matrix $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}^{-1}\mathcal{A}$ with $k^2 = 0$, $\eta = 5$ and $m + n = 961$.

4. Conclusions

We have investigated the use of new block-triangular preconditioners with two variable relaxation parameters for solving the mixed formulation of the time-harmonic Maxwell equations. Our results are extensions of the work in [2, 3, 15]. The preconditioned matrices are demonstrated to have clustering eigenvalues. We have shown experimentally that the outer iteration numbers of BiCGSTAB with the new preconditioner are hardly any sensitive to the changes in the mesh size.

Acknowledgments

The authors thank the anonymous referees for their valuable comments and suggestions which lead to an improved presentation of this paper. This work is supported by the Chinese

Table 3: Iteration counts for different meshes, using BiCGSTAB with the preconditioner $\widehat{M}_{k,\eta,\varepsilon}$ satisfying $\eta - k^2 = 0.1$ and $-1/\varepsilon = \eta - k^2$. The iteration was stopped once $\|r^{(k)}\|/\|r^{(0)}\| \leq 5 \times 10^{-10}$.

Mesh	$k^2 = 0$	$k^2 = 0.25$	$k^2 = 0.5$	$k^2 = 1$	$k^2 = 3$	$k^2 = 4$	$k^2 = 6$	$k^2 = 10$
8×8	2	2.5	2.5	2.5	3.5	4	4.5	5.5
16×16	2	2.5	2.5	2.5	3.5	4	4.5	5
32×32	2	2.5	2.5	2.5	3.5	4	4.5	5.5
64×64	2	2.5	2.5	2.5	3.5	4	5	5.5
128×128	2	2.5	3	3	4	4.5	5	6

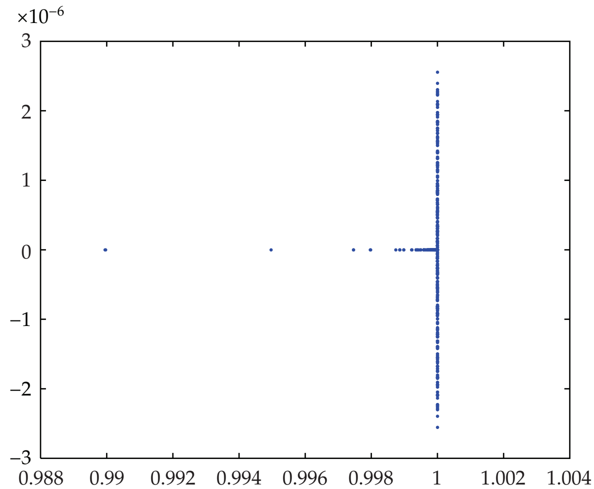

Figure 5: The eigenvalue distribution of the preconditioned matrix $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}^{-1}\mathcal{A}$ with $k^2 = 0$, $\eta = 0.1$ and $m + n = 961$.

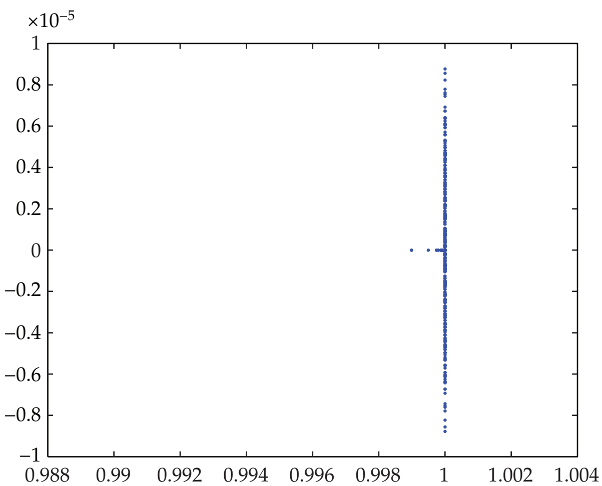

Figure 6: The eigenvalue distribution of the preconditioned matrix $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}^{-1}\mathcal{A}$ with $k^2 = 0$, $\eta = 0.01$ and $m + n = 961$.

Table 4: Iteration counts for different meshes, using BiCGSTAB with the preconditioner $\widehat{M}_{k,\eta,\varepsilon}$ satisfying $\eta - k^2 = 6$ and $-1/\varepsilon = \eta - k^2$. The iteration was stopped once $\|r^{(k)}\|/\|r^{(0)}\| \leq 5 \times 10^{-10}$.

Mesh	$k^2 = 0$	$k^2 = 0.25$	$k^2 = 0.5$	$k^2 = 1$	$k^2 = 3$	$k^2 = 4$	$k^2 = 6$	$k^2 = 10$
8×8	3.5	3.5	3.5	4	4.5	4.5	5	6.5
16×16	3.5	4	4	4	4.5	4.5	5.5	6.5
32×32	3.5	4	4	4	4.5	4.5	5.5	6
64×64	3.5	4	4	4	4.5	5	5.5	6
128×128	3.5	4	4.5	4.5	5	5	6	6.5

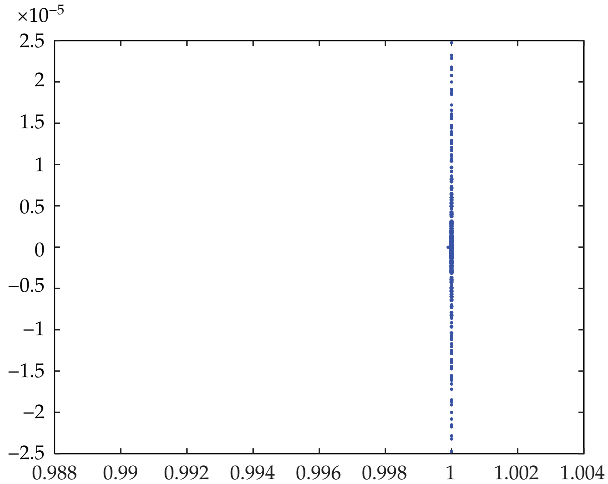

Figure 7: The eigenvalue distribution of the preconditioned matrix $\widehat{\mathcal{M}}_{k,\eta,\varepsilon}^{-1}\mathcal{A}$ with $k^2 = 0$, $\eta = 0.001$ and $m + n = 961$.

National Science Foundation Project (11161014), the Science and Technology Development Foundation of Guangxi (Grant no. 0731018), and Innovation Project of Guangxi Graduate Education (Grant no. ZYC0430).

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