Research Article

$L_\infty$ Control with Finite-Time Stability for Switched Systems under Asynchronous Switching

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1. Introduction

Switched systems are a class of hybrid systems consisting of subsystems and a switching law, which defines a specific subsystem being activated during a certain interval of time. Many real-world processes and systems can be modeled as switched systems such as chemical processes and computer controlled systems. Besides, switched systems are widely applied in many domains, including mechanical systems, automotive industry, aircraft and air traffic control, and many other fields [1–3].

At early time, the issue of stability of switched systems which has attracted most of the attention is one basic research topic. Lyapunov stability theory and its variations or generalizations had played an important role in this research field. Common Lyapunov function method and multiple Lyapunov functions method for switched system are
presented by researchers [4-8]. For most switched systems, it is hard to find a common Lyapunov function; however, we can guarantee the switched system is still stable under some properly chosen switching signals which are found by using the multiple Lyapunov functions technique. In addition, more researchers pay attention to average dwell-time control of switched systems [9, 10]. In particular, the average dwell-time approach is employed to deal with the control, observe, and filtering problem of switched delay systems or network control systems [11–14].

As we know, a large number of literatures related to stability of switched systems focus on Lyapunov asymptotic stability, which is defined over an infinite time interval. In many practical applications, however, the main concern is the behavior of the system over a fixed finite-time interval, for instance to avoid saturations or the excitation of nonlinear dynamics. It should be clear that a finite-time stable system may not be Lyapunov asymptotical stable, and a Lyapunov asymptotical stable system may not be finite-time stable since the transient of a system response may exceed the bound. Recently, there have been some literatures discussing the finite-time stability analysis of switched systems [15–17]. In [18], finite-time bounded and finite-time weighted $L_2$-gain for a class of switched delay systems with time-varying external disturbances is addressed. Reference [19] investigated finite-time control for switched discrete-time system. Considering the potential faults in a system, [20] studied fault-tolerant control with finite-time stability for switched linear systems. Delay-dependent observer-based $H_{\infty}$ finite-time control for switched systems with time-varying delay was studied in [21]. In [22], the problems of finite-time stability analysis and stabilization for switched nonlinear discrete-time systems are investigated, and then the results are extended to $H_{\infty}$ finite-time bounded. However, in many applications, external disturbance is always persistent bounded with infinite energy. $H_{\infty}$ control cannot be employed to deal with a system with persistent bounded disturbance. In this situation, it is more appealing to develop $L_{\infty}$ control for switched systems with disturbances of this type. So far, however, compared with research results on $H_{\infty}$ finite-time stability, few results on $L_{\infty}$ finite-time stability of switched systems have been given in the literature.

Additionally, in actual operation, there inevitably exists asynchronous switching between the controllers and the practical subsystems, that is, the real switching time of controllers exceeds or lags behind that of the practical subsystems, which will deteriorate performance of systems, even makes system out of control. Up to now, there have been a number of literatures on asynchronous switching control research of switched system [23–28]. But it is worth to point that all of these studies focus on designing the controller to guarantee the Lyapunov asymptotical stable or exponential stable of the system. To the best of our knowledge, the finite-time stabilization issue of switched system under asynchronous switching has not been fully investigated, which is quite an important issue for the switched system. This motivates us to carry out present work. In this paper, we deal with the problem of $L_{\infty}$ finite-time stabilization for switched systems under asynchronous switching.

The main contributions of this paper are that several sufficient conditions ensuring the finite-time bounded and $L_{\infty}$ finite-time stability are proposed with asynchronous switching between the controllers and the practical subsystems. The result shows that it is unnecessary to guarantee each subsystem can be finite-time stabilizable with $L_{\infty}$ performance by the designed asynchronous switching controller. During the finite-time interval, the switching frequency only needs to be limited in some value, then the switched system is finite-time stable with $L_{\infty}$ performance by the designed controller despite of the asynchronous switching between the controllers and the practical subsystems.
This paper is organized as follows. In Section 2, some preliminary definitions are
provided, and the problem we deal with is precisely stated. Section 3 provides, the main
results of this paper: a sufficient condition for the existence of a state feedback controller
guaranteeing the finite-time stability under asynchronous switching between the controllers
and the practical subsystems. Moreover, \( L_\infty \) control with finite-time stability for switched
systems under asynchronous switching is provided in Section 4. Finally, a numerical example
is presented by using LMI toolbox to illustrate the efficiency of the proposed method in
Section 5. Our conclusions are drawn in Section 6.

Notation. Throughout this paper, \( A^T \) denotes transpose of matrix \( A \), \( L_\infty \) denotes space of
functions with bounded amplitude, \( \|x(t)\| \) denotes the usually 2-norm. \( \lambda_{\text{max}}(P) \), and \( \lambda_{\text{min}}(P) \)
denote the maximum and minimum eigenvalues of matrix \( P \), respectively, \( I \) is an identity
matrix with appropriate dimension. \( S > 0 \) denotes \( S \) is a positive definite symmetric matrix.
\( Z \) denotes the integer set and \( Z^* \) denotes the positive integer set.

2. Problem Formulation and Preliminary

A switched system is considered as follows:

\[
\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + G_{\sigma(t)}w(t),
\]

(2.1)

where \( x(t) \in \mathbb{R}^n \) is the system state. \( u(t) \in \mathbb{R}^p \) is the control input, \( x(t_0) = x_0 \) is the initial
state of the system. \( w(t) \in \mathbb{R}^q \) is the measurement noise over the interval \([t_0, T_f]\), which
satisfies \( \sup_{t \in [t_0, T_f]} \|w(t)\| < \infty \), \( \sigma(t) : Z^* \to \mathbb{N} = \{1, 2, \ldots, N\} \) is a switching signal which is
a piecewise constant function depending on time \( t \) or state \( x(t) \), and \( \mathbb{N} \) denotes the number
of subsystems. Moreover, \( \sigma(t) = i \) means that the \( i \)th subsystem is activated. \( A_i \in \mathbb{R}^{n \times n} \), \( B_i \in \mathbb{R}^{n \times p} \), \( G_i \in \mathbb{R}^{n \times q} \) for \( i \in \mathbb{N} \) are real-valued matrices with appropriate dimensions.

Assume that the state of the switched system (2.1) does not jump at the switching
instants, that is, the trajectory \( x(t) \) is everywhere continuous. The switching law \( \sigma(t) : Z^* \to \mathbb{N} = \{1, 2, \ldots, N\} \) discussed in this paper is time dependent, that is, \( \sigma(t) : \{(t_0, \sigma(t_0)), (t_1, \sigma(t_1)), \ldots, (t_k, \sigma(t_k))\}, k \in \mathbb{Z}, \) where \( t_0 \) is the initial switching instant, and \( t_k \)
denotes the \( k \)th switching instant.

Owing to asynchronous switching, the practical switching instant of controller is
different from that of systems. For convenience, \( \sigma'(t) \) is used to denote the practical switching
signal of controller, \( \sigma'(t) \) can be written as \( \sigma'(t) : \{(t_0 + \Delta_0, \sigma(t_0)), (t_1 + \Delta_1, \sigma(t_1)), \ldots, (t_k + \Delta_k, \sigma(t_k))\}, k \in \mathbb{Z}, \) where \( |\Delta_k| < \inf_{k \geq 0} (t_{k+1} - t_k) \), \( \Delta_k > 0 \) (or \( |\Delta_k| < \inf_{k \geq 0} (t_k - t_{k-1}) \), \( \Delta_k < 0 \); \( \Delta_k \) represents the delayed period of the controller switching or the exceeded period of the
controller switching). In both cases, the period \( \Delta_k \) is said to be the mismatched period
between the controller and the system.

Remark 2.1. Mismatched period \( \Delta_k \) guarantees that there always exists a period that the
controller and the system operate synchronously, which makes it possible to design the
stabilizable controller for the system.

Under the asynchronous switching, the switched controller can be written as

\[
u(t) = K_{\sigma(t)}x(t).
\]

(2.2)
If we substitute the $u(t) = K_{\sigma(t)}x(t)$ into system (2.1), we can obtain that

$$\dot{x}(t) = (A_{\sigma(t)} + B_{\sigma(t)}K_{\sigma(t)})x(t) + G_{\sigma(t)}w(t).$$

(2.3)

The following lemma will be useful for the design of controller.

**Lemma 2.2** (see [29]). If a real scalar function $\varphi(t), v(t)$ satisfies the following differential inequality:

$$\varphi(t) \leq \varphi(t) + \kappa v(t),$$

(2.4)

then we have

$$\varphi(t) \leq e^{\zeta(t-t_0)}\varphi(t_0) + \kappa \int_{t_0}^{t} e^{\zeta(t-t_0)} v(t) dt,$$

(2.5)

where $\zeta \in \mathbb{R}, \kappa \in \mathbb{R}, t \geq t_0$.

Let us review the definition of average dwell-time, which will be useful in designing the stabilization controller to guarantee the system finite-time stable.

**Definition 2.3** (see [30]). For any $T_2 > T_1 \geq 0$, let $N_\sigma(T_1, T_2)$ denote the switching number of $\sigma(t)$ on an interval $(T_1, T_2)$, if

$$N_\sigma(T_1, T_2) \leq N_0 + \frac{T_2 - T_1}{\tau_a}$$

(2.6)

holds for given $N_0 \geq 0, \tau_a > 0$. Then the constant $\tau_a$ is called the average dwell time, and $N_0$ is the chatter bound.

For switched system, the general conception of finite-time stability concerns the boundness of continuous state $x(t)$ over finite-time interval $[t_0, T_f]$ with respect to given initial condition $x_0$. This conception can be formulized through following definition.

**Definition 2.4.** The switched linear system (2.1) with $G_{\sigma(t)} \equiv 0$ is said to be finite-time stabilizable under the asynchronous switching control mode with respect to $(c_1, c_2, T_f, \sigma(t), \sigma'(t))$ with $c_1 < c_2$ and a given switching signal $\sigma(t)$, if $\|x(t)\| \leq c_2$, for all $t \in [t_0, T_f]$, whenever $\|x_0\| \leq c_1$.

**Definition 2.5.** Switched system (2.1) is said to be $L_\infty$ finite-time stabilizable with respect to $(c_1, c_2, T_f, \sigma(t), \sigma'(t))$ where $c_1 < c_2$, $\sigma(t)$ is a switching signal of the system, and $\sigma'(t)$ is a switching signal of the controller, the following conditions should be satisfied.

(i) Switched linear system (2.1) with $G_{\sigma(t)} \equiv 0$ is finite-time stabilizable.

(ii) Under zero-initial condition $x(t_0) = 0$, the following inequality holds:

$$\sup_{t \in [t_0, T_f]} \|x(t)\| \leq \gamma \sup_{t \in [t_0, T_f]} \|w(t)\|, \quad \forall w(t) : \sup_{t \in [t_0, T_f]} \|w(t)\| < \infty.$$ 

(2.7)

The main issue in this paper is given as follows.
It is assumed that the switching system (2.1) is finite-time stabilizable with respect to \((c_1, c_2, T_f, \sigma(t), \sigma'(t))\) under the asynchronous switching control mode, then the result will be extended to the \(L_\infty\) controller design for system (2.1).

### 3. Finite-Time Stabilization under the Asynchronous Switching

It is assumed that the \(i\)th subsystem switched to the \(j\)th subsystem at the switching instant \(t_k\). Owing to asynchronous switching, the switching instant of the controller is \(t_k + \Delta_k\), then there exists mismatched period at time interval \([t_k, t_k + \Delta_k]\), \(\Delta_k > 0\) (or \((t_k + \Delta_k, t_k), \Delta_k < 0\)). In this period, the controller \(K_i\) affected the \(j\)th subsystem (or the controller \(K_j\) affected the \(i\)th subsystem).

**Remark 3.1.** We consider the case of \(\Delta_k > 0\), that is to say, the switching time of the controller is lag of the switching time of the system. Figure 1 illustrates the asynchronous switching mode between the controller and the subsystems. From Figure 1, we can see that the controller \(K_i\) of the \(i\)th subsystem affects the \(i\)th subsystem in the matched period \([t_k - \Delta_k, t_k]\) and affects the \(j\)th subsystem in the mismatched period \([t_k, t_k + \Delta_k]\).

The following theorem presents the finite-time stabilization design method of the system (2.1) under asynchronous switching.

**Theorem 3.2.** If there exist matrices \(P_i > 0, P_{ij} > 0, K_i\) and scalars \(\mu_1 > 1, \mu_2 > 1, \lambda^+ > 0, \lambda^- > 0\) such that

\[
P_i < \mu_1 P_{ij}, \quad P_{ij} < \mu_2 P_i,
\]

\[
(A_i + B_i K_i)^T P_i + P_i (A_i + B_i K_i) < \lambda^- P_i,
\]

\[
(A_j + B_j K_i)^T P_{ij} + P_{ij} (A_j + B_j K_i) < \lambda^+ P_{ij},
\]

\[
\tau_a > \frac{(T_f - t_0) \ln(\mu_1 \mu_2)}{\ln\left(\left(\frac{\varepsilon^2}{\delta^2}\right) \cdot B \cdot \left(\mu_2 / (\mu_1 \mu_2)^{N_0}\right)\right) - \lambda^+ T^+(t_0, T_f) - \lambda^- T^-(t_0, T_f)},
\]

where \(B\) denotes \(\inf_{i,j \in \mathbb{N}} \{\lambda_{\min}(P_i), \lambda_{\min}(P_{ij})\} / \sup_{i,j \in \mathbb{N}} \{\lambda_{\max}(P_i), \lambda_{\max}(P_{ij})\}\), then switched system (2.1) is finite-time stabilizable with respect to \((\delta, \varepsilon, T_f, \sigma(t), \sigma'(t))\) under the feedback controller \(u(t) = K_{\sigma(t)} x(t)\), where \(T^+(t_0, T_f)\) and \(T^-(t_0, T_f)\) denote the matched period and the mismatched period in finite-time interval \([t_0, T_f]\), respectively.
Proof. Here, we only discuss the situation of $\Delta_k > 0$. For $\Delta_k < 0$, the proof method is similar, and we can reach the same conclusion.

When $t \in [t_{k-1} + \Delta_{k-1}, t_k)$, for the $i$th subsystem, the state feedback controller $u(t) = K_i x(t)$. So the state equation of closed-loop system can be written as

$$\dot{x}(t) = (A_i + B_i K_i)x(t). \tag{3.5}$$

Choose a switching Lyapunov function as follows:

$$V_i(t) = x^T(t)P_i x(t). \tag{3.6}$$

By (3.2), it implies that

$$\dot{V}_i(t) < \lambda^{-1} V_i(t). \tag{3.7}$$

When $t \in [t_k, t_k + \Delta_k)$, for the $j$th subsystem, the state feedback controller is still $u(t) = K_j x(t)$. So the closed-loop system can be described as

$$\dot{x}(t) = (A_j + B_j K_j)x(t). \tag{3.8}$$

Consider the Lyapunov function candidate as follows:

$$V_{ij}(t) = x^T(t)P_{ij} x(t). \tag{3.9}$$

By (3.3), we can obtain that

$$\dot{V}_{ij}(t) < \lambda^{-1} V_{ij}(t). \tag{3.10}$$

Notice that the Lyapunov function (3.6) and (3.9) can be rewritten as

$$V_i(t) = x^T(t)P_i x(t), \quad t \in [t_{k-1} + \Delta_{k-1}, t_k), \quad k = 1, 2, \ldots,$$

$$V_{ij}(t) = x^T(t)P_{ij} x(t), \quad t \in [t_k, t_k + \Delta_k), \quad k = 0, 1, \ldots. \tag{3.11}$$

Let $t_0 < t_1 < t_2 < \cdots < t_k = T_f$ is the switching time in the period $[t_0, T_f]$, we define the following piecewise Lyapunov function:

$$V(t) = \begin{cases} 
  x^T(t)P_i x(t), & t \in [t_r + \Delta_r, t_{r+1}), \quad r = 0, 1, \ldots, k-1, \\
  x^T(t)P_{ij} x(t), & t \in [t_r, t_r + \Delta_r), \quad r = 0, 1, \ldots, k-1.
\end{cases} \tag{3.12}$$
By (3.7) and (3.10), we can obtain that

\[
V(t) < e^{\lambda(t^{-t_{k-1}}-\Delta_{k-1})}V(t_{k-1} + \Delta_{k-1})
\]
\[
< \mu_1 e^{\lambda(t^{-t_{k-1}}-\Delta_{k-1})}V((t_{k-1} + \Delta_{k-1})^{-})
\]
\[
< \mu_1 e^{\lambda \Delta_{k-1}} e^{\lambda(t^{-t_{k-1}}-\Delta_{k-1})}V(t_{k-1})
\]
\[
< \mu_1 \mu_2 e^{\lambda \Delta_{k-1}} e^{\lambda(t^{-t_{k-1}}-\Delta_{k-1})}V(t_{k-1}^{-})
\]
\[
< \mu_1 \mu_2 e^{\lambda \Delta_{k-1}} e^{\lambda(t^{-t_{k-1}}-\Delta_{k-1})} e^{\lambda(t_{k-1}^{-} - t_{k-2}^{-} - \Delta_{k-2})}V(t_{k-2} + \Delta_{k-2})
\]
\[
< \mu_1 \mu_2 e^{\lambda \Delta_{k-1}} e^{\lambda(t^{-t_{k-1}}-\Delta_{k-1})} e^{\lambda(t_{k-1}^{-} - t_{k-2}^{-} - \Delta_{k-2})} V((t_{k-2} + \Delta_{k-2})^{-})
\]
\[
< \mu_1 \mu_2 e^{\lambda \Delta_{k-1}} e^{\lambda(t^{-t_{k-1}}-\Delta_{k-1})} e^{\lambda(t_{k-1}^{-} - t_{k-2}^{-} - \Delta_{k-2})} e^{\lambda(t_{k-2}^{-} - t_{k-3}^{-} - \Delta_{k-3})} V(t_{k-2})
\]
\[
\vdots
\]
\[
< \mu_1 \mu_2 e^{\lambda \Delta_{k-1}} e^{\lambda(t^{-t_{k-1}}-\Delta_{k-1})} e^{\lambda(t_{k-1}^{-} - t_{k-2}^{-} - \Delta_{k-2})} e^{\lambda(t_{k-2}^{-} - t_{k-3}^{-} - \Delta_{k-3})} \cdots e^{\lambda(t_{k-2}^{-} - t_{k-3}^{-} - \Delta_{k-3})} e^{\lambda(t_{k-2}^{-} - t_{k-3}^{-} - \Delta_{k-3})} e^{\lambda(t_{k-2}^{-} - t_{k-3}^{-} - \Delta_{k-3})} \cdots e^{\lambda(t_{k-2}^{-} - t_{k-3}^{-} - \Delta_{k-3})} V(t_0)
\]
\[
< \mu_2^{-1} (\mu_1 \mu_2)^{k_{[0,T_f]}} e^{\lambda T^-(t_0,T_f) + \lambda T^+(t_0,T_f))} V(t_0),
\]

where \(T^+(t_0,T_f)\) denotes the sum of the matched period between the controllers and subsystem in \((t_0,T_f)\). \(T^-(t_0,T_f)\) denotes the sum of the mismatched period between the controllers and subsystem in \([t_0,T_f]\).

And from (3.12) we have

\[
V(t) \geq \inf_{i,j \in \mathbb{N}} \{\lambda_{\min}(P_i), \lambda_{\min}(P_{ij})\} \|x(t)\|^2.
\]

(3.14)

On the other hand, for \(i \in \mathbb{N}\), we have

\[
V(t_0) \leq \sup_{i,j \in \mathbb{N}} \{\lambda_{\max}(P_i), \lambda_{\max}(P_{ij})\} \|x(t_0)\|^2.
\]

(3.15)

Using the fact

\[
\|x(t_0)\| \leq \delta,
\]

(3.16)

we get

\[
V(t_0) \leq \sup_{i,j \in \mathbb{N}} \{\lambda_{\max}(P_i), \lambda_{\max}(P_{ij})\} \delta^2.
\]

(3.17)
Altogether (3.13)–(3.17), the following inequality can be derived

\[
\|x(t)\|^2 \leq \mu_2^{-1}(\mu_1 \mu_2) k_{[t_0,T_f]} e^{\lambda T^*(t_0,T_f)+\lambda^- T^-(t_0,T_f)} \frac{\sup_{i,j \in N} \{\lambda_{\max}(P_i), \lambda_{\max}(P_{ij})\}}{\inf_{i,j \in N} \{\lambda_{\min}(P_i), \lambda_{\min}(P_{ij})\}} \delta^2. \tag{3.18}
\]

From the Definition 2.3, we know that \(k_{[t_0,T_f]} = N_\sigma\), then we have the relation

\[
k_{[t_0,T_f]} \leq N_0 + \frac{T_f - t_0}{\tau_a}. \tag{3.19}
\]

From (3.4) and (3.19), we get

\[
\mu_2^{-1}(\mu_1 \mu_2) k_{[t_0,T_f]} e^{\lambda T^*(t_0,T_f)+\lambda^- T^-(t_0,T_f)} \frac{\sup_{i,j \in N} \{\lambda_{\max}(P_i), \lambda_{\max}(P_{ij})\}}{\inf_{i,j \in N} \{\lambda_{\min}(P_i), \lambda_{\min}(P_{ij})\}} \delta^2 < \varepsilon. \tag{3.20}
\]

According to (3.18) and (3.20), we have

\[
\|x(t)\| < \varepsilon. \tag{3.21}
\]

The proof is completed. \(\Box\)

Remark 3.3. From (3.2) and (3.3), we know that for finite-time stabilization issue, the subsystem needs not to be stabilized in finite-time interval, that is to say, the designed asynchronous switching controller needs not to stabilize the subsystem in the matched period and the mismatched period in finite-time interval \([t_0, T_f]\), but the whole system is finite-time stabilizable. Reference [31] gives the exponential stabilization condition under asynchronous switching, which requests that the subsystem can be exponentially stabilized in the matched period. But as to the problem of finite-time stabilization, it is unnecessary to request that the subsystem can be stabilized in the matched period or mismatched period. In particular, when \(\lambda^+ = \lambda^- = \lambda\) in (3.2) and (3.3), (3.4) becomes

\[
\tau_a > \frac{(T_f - t_0) \ln(\mu_1 \mu_2)}{\ln\left(\frac{\varepsilon^2}{\delta^2} \cdot \mathcal{B} \cdot \left(\mu_2/(\mu_1 \mu_2)^{N_0}\right)\right) - \lambda (T_f - t_0)} \tag{3.22}
\]

which is independent of \(T^*(t_0,T_f)\) and \(T^-(t_0,T_f)\).

Remark 3.4. In fact, (3.4) in Theorem 3.2 implies that if switching sequence \(\sigma(t) : \{(t_0, \sigma(t_0)), (t_1, \sigma(t_1)), \ldots, (t_k, \sigma(t_k))\}\) of the system can be prespecified, that is, \(\tau_a\) is a known constant,


the matched period $T^-(t_0,T_f)$ and the mismatched period $T^+(t_0,T_f)$ should satisfy the following relation:

$$\lambda^*T^+(t_0,T_f) + \lambda^-T^-(t_0,T_f) < \ln\left(\frac{\varepsilon^2}{\sigma^2} \cdot \inf_{i,j \in \mathbb{N}} \left\{ \frac{\lambda_{\min}(P_i)}{\lambda_{\max}(P_i)} \right\} \cdot \frac{\mu_2}{(\mu_1\mu_2)^{N_0}} \right) - \frac{(T_f - t_0) \ln(\mu_1\mu_2)}{\tau_a}.$$  

(3.23)

**Remark 3.5.** Reference [31] gives the design method of exponential stabilization controller under asynchronous switching. The condition implies that the ratio of the mismatched period and the matched period should be less than some value which means that the matched period should be large enough to stabilize the subsystem. However, from the condition of Theorem 3.2, we know that when the switching sequence is unknown, the ratio of the mismatched period and the matched period can be designed freely to guarantee the finite-time stability of the system by the asynchronous switched controller. But if switching sequence of the system is prespecified, the ratio of the mismatched period and the matched period may need to be limited. On the other hand, the average dwell-time scheme with Lyapunov stability limits the dwell-time $\tau_a$ and the ratio of $T^+(t_0,T_f)$ and $T^-(t_0,T_f)$ to satisfy the proposed condition in [31] at the same time. But for the average dwell-time scheme with finite-time stability, we can predetermine one value among two parameters of the dwell-time $\tau_a$ and the ratio of $T^+(t_0,T_f)$ and $T^-(t_0,T_f)$, then the other value can be determined by the condition (3.4).

**Remark 3.6.** In order to get the solution of the asynchronous switched controller $K_i$, we denote $X_i = P_i^{-1}$, $X_{ij} = P_{ij}^{-1}$, $W_i = K_iP_i^{-1}$, then (3.1) to (3.3) can be written as

$$\mu_1X_i > X_{ij}, \quad \mu_2X_{ij} > X_i, \quad (A_iX_i + B_iW_i)^T + (A_iX_i + B_iW_i) < \lambda^*X_i,$$  

(3.24)

$$X_{ij} \left( A_j + B_jW_jX_j^{-1} \right)^T + \left( A_j + B_jW_jX_j^{-1} \right)X_{ij} < \lambda^*X_{ij},$$  

(3.25)

$$X_{ij} \left( A_j + B_jW_jX_j^{-1} \right)^T + \left( A_j + B_jW_jX_j^{-1} \right)X_{ij} < \lambda^*X_{ij}.$$  

(3.26)

It is noticed that the matrix inequalities (3.24), (3.25), and (3.26) are coupled. Therefore, we can firstly solve the linear matrix inequality (3.25) to obtain the solution to matrices $X_i$ and $W_i$. Then we solve the matrix inequality (3.24), (3.26) by substituting $X_i$ and $W_i$ into (3.24), (3.26). By adjusting the parameter $\mu_1, \mu_2$, and $\lambda^*$ appropriately, we seek the feasible solutions $X_i, W_i$ and $X_{ij}$ such that the matrix inequalities (3.24) and (3.26) hold. If the chosen parameters $\mu_1, \mu_2$, and $\lambda^*$ have no feasible solution, we can adjust $\mu_1, \mu_2$, or $\lambda^*$ to be larger. Following this guideline, the solution to the matrix inequalities (3.24) to (3.26) will be found.

4. $L_\infty$ Finite-Time Stabilization under the Asynchronous Switching

Now, we are in a position to investigate $L_\infty$ finite-time stabilization design method of the system (2.1) under asynchronous switching.
Theorem 4.1. If there exist matrices $P_l > 0$, $P_{ij} > 0$, $K_i$ and scalars $\mu_1 > 1$, $\mu_2 > 1$, $\lambda^+ > 0$, $\lambda^- > 0$ such that

$$P_l < \mu_1 P_{ij}, \quad P_{ij} < \mu_2 P_l,$$

$$\left[ (A_i + B_iK_i)^T P_l + P_l(A_i + B_iK_i) - \lambda^- P_l \begin{bmatrix} P_lG_i \\ G_i^T P_l \end{bmatrix} - \varepsilon_i I \right] < 0, \quad \tau_0 > \frac{(T_f - t_0) \ln (\mu_1/\mu_2)}{\ln \left( (e^2/\delta^2) \cdot B \cdot \left( \mu_2 / (\mu_1 \mu_2)^N \right) \right) - \lambda^+ T^+(t_0, T_f) - \lambda^- T^-(t_0, T_f)}.$$

$L_\infty$ disturbance attenuation performance $\gamma^- \leq \sqrt{\varepsilon_i (e^{\lambda^+ T^-(t_0, T_f)} - 1) / \lambda^- \lambda_{\min}(P_l)}$ during the matched period and $\gamma^+ \leq \sqrt{\varepsilon_{ij} (e^{\lambda^+ T^+(t_0, T_f)} - 1) / \lambda^+ \lambda_{\min}(P_{ij})}$ during the mismatched period, then switched system (2.1) is finite-time stabilizable of $L_\infty$ disturbance attenuation performance with respect to $(\delta, \varepsilon, T_f, \sigma(t), \sigma'(t))$ under the feedback controller $u(t) = K_{\sigma(t)} x(t)$, where $T^-(t_0, T_f)$ and $T^+(t_0, T_f)$ denote the matched period and the mismatched period in finite-time interval $[t_0, T_f]$, respectively.

Proof. It can be concluded from Theorem 4.1 that system (2.1) is finite-time stable under the feedback controller $u(t) = K_{\sigma(t)} x(t)$.

When $t \in [t_{k-1} + \Delta_{k-1}, t_k)$, for the $i$th subsystem, the state feedback controller $u(t) = K_i x(t)$. So the state equation of closed-loop system can be written as

$$\dot{x}(t) = (A_i + B_iK_i)x(t) + G_i \omega(t).$$

Choose a switching Lyapunov function as follows:

$$V_i(t) = x^T(t) P_l x(t), \quad t \in [t_{k-1} + \Delta_{k-1}, t_k), \quad k = 1, 2, \ldots.$$ 

By (4.2), it implies that

$$V_i(t) \leq \lambda^- V_i(t) + \varepsilon_i \omega^T(t) \omega(t).$$

With zero initial conditions, by Lemma 2.2, we have

$$V_i(t) \leq \varepsilon_i \int_0^{t - t_{k-1} - \Delta_{k-1}} e^{\lambda^- \tau} \omega^T(t - \tau) \omega(t - \tau) d\tau.$$ 

Note that

$$V_i(t) \geq \lambda_{\min}(P_i) \| x(t) \|^2.$$
From (4.8) and (4.9), we can obtain

\[
\lambda_{\min}(P_i) \sup_{t \in [t_{k-1} + \Delta_k, t_k]} \|x(t)\|^2 \leq \frac{\epsilon_i \left( e^{\lambda^+ T (t_0, T_f)} - 1 \right)}{\lambda^-} \sup_{t \in [t_{k-1} + \Delta_k, t_k]} \|w(t)\|^2. \tag{4.10}
\]

From (4.10), we have

\[
\frac{\sup_{t \in [t_{k-1} + \Delta_k, t_k]} \|x(t)\|}{\sup_{t \in [t_{k-1} + \Delta_k, t_k]} \|w(t)\|} \leq \sqrt{\frac{\epsilon_i \left( e^{\lambda^+ T (t_0, T_f)} - 1 \right)}{\lambda^- \lambda_{\min}(P_i)}}. \tag{4.11}
\]

When \( t \in [t_k, t_k + \Delta_k] \), for the \( j \)th subsystem, the state feedback controller is still \( u(t) = K_i x(t) \). So the closed-loop system can be described as

\[
\dot{x}(t) = (A_j + B_j K_i) x(t) + G_j w(t). \tag{4.12}
\]

Consider the Lyapunov function candidate as follows:

\[
V_{ij}(t) = x^T(t) P_{ij} x(t), \quad t \in [t_k, t_k + \Delta_k], \quad k = 0, 1, \ldots. \tag{4.13}
\]

By (4.3), it implies that

\[
\dot{V}_{ij}(t) \leq \lambda^+ V_{ij}(t) + \epsilon_{ij} w^T(t) w(t). \tag{4.14}
\]

With zero initial conditions, by Lemma 2.2, we have

\[
V_{ij}(t) \leq \epsilon_{ij} \int_0^{t-t_k} e^{\lambda^- \tau} w^T(t-\tau) w(t-\tau) d\tau. \tag{4.15}
\]

Notice that

\[
V_{ij}(t) \geq \lambda_{\min}(P_{ij}) \|x(t)\|^2. \tag{4.16}
\]

From (4.15) and (4.16), we can obtain

\[
\lambda_{\min}(P_{ij}) \sup_{t \in [t_k, t_k + \Delta_k]} \|x(t)\|^2 \leq \frac{\epsilon_{ij} \left( e^{\lambda^+ T (t_0, T_f)} - 1 \right)}{\lambda^-} \sup_{t \in [t_k, t_k + \Delta_k]} \|w(t)\|^2. \tag{4.17}
\]

From (4.17), we have

\[
\frac{\sup_{t \in [t_k, t_k + \Delta_k]} \|x(t)\|}{\sup_{t \in [t_k, t_k + \Delta_k]} \|w(t)\|} \leq \sqrt{\frac{\epsilon_{ij} \left( e^{\lambda^+ T (t_0, T_f)} - 1 \right)}{\lambda^- \lambda_{\min}(P_{ij})}}. \tag{4.18}
\]
By (4.11) and (4.18), during the finite-time \([t_0, T_f] = \bigcup_{r=0}^{k-1} [t_r, t_r + \Delta r] \cup [t_r + \Delta r, t_{r+1})\), we can obtain

\[
\sup_{t \in [t_0, T_f]} \|x(t)\| \leq \max \left\{ \sqrt{\frac{e^{\lambda^- t} T^- (t_0, T_f)}{\lambda^-} - 1} \max_{i \in \mathbb{N}} \left( \frac{\varepsilon_i}{\lambda_{\min}(P_i)} \right), \sqrt{\frac{e^{\lambda^+ t} T^+ (t_0, T_f)}{\lambda^+} - 1} \max_{ij \in \mathbb{N}} \left( \frac{\varepsilon_{ij}}{\lambda_{\min}(P_{ij})} \right) \right\}
\]

(4.19)

By the definition of \(L_\infty\) finite-time stabilization, we can obtain that the designed controller \(u(t) = K_{\sigma(t)} x(t)\) can guarantee the finite-time stability of \(L_\infty\) disturbance attenuation performance. This completes the proof.

**Remark 4.2.** Theorem 4.1 represents that if each subsystem satisfies \(L_\infty\) disturbance attenuation performance during the mismatched period and the matched period, the designed asynchronous switched controller \(u(t) = K_{\sigma(t)} x(t)\) can guarantee the whole system has \(L_\infty\) disturbance attenuation performance. However, the condition of each subsystem satisfying \(L_\infty\) disturbance attenuation performance during the mismatched period and the matched period seems to be more conservative, and in fact through the following theorem, this condition is not essential.

**Remark 4.3.** Although Theorem 4.1 gives the method of finite-time stabilization with \(L_\infty\) disturbance attenuation performance, the matched period \(T^-(t_0, T_f)\) and the mismatched period \(T^+(t_0, T_f)\) need to be prespecified in order to obtain \(L_\infty\) disturbance attenuation performance of the system. However, in practical engineering it is difficult to obtain the matched period \(T^-(t_0, T_f)\) and the mismatched period \(T^+(t_0, T_f)\) before designing the controller. Based on these, the following result can be derived.

**Theorem 4.4.** If there exist matrices \(P_i > 0, P_{ij} > 0, K_i\) and scalars \(\mu_1 > 1, \mu_2 > 1, \lambda^+ > 0, \lambda^- > 0\) such that

\[
P_i < \mu_1 P_{ij}, \quad P_{ij} < \mu_2 P_i,
\]

(4.20)

\[
\begin{bmatrix}
(A_i + B_i K_i)^T P_i + P_i (A_i + B_i K_i) - \lambda^- P_i & P_i G_i \\
G_i^T P_i & -\varepsilon_i I
\end{bmatrix} < 0,
\]

(4.21)

\[
\begin{bmatrix}
(A_j + B_j K_j)^T P_{ij} + P_{ij} (A_j + B_j K_j) - \lambda^+ P_{ij} & P_{ij} G_j \\
G_j^T P_{ij} & -\varepsilon_{ij} I
\end{bmatrix} < 0,
\]

(4.22)

\[
T_\sigma > \frac{(T_f - t_0) \ln(\mu_1 \mu_2)}{\ln\left(\frac{(\varepsilon^2 / G^2) \cdot B \cdot (\mu_2 / (\mu_1 \mu_2)^{N_0})}{\lambda^+ T^+(t_0, T_f) - \lambda^- T^-(t_0, T_f)}\right)}
\]

(4.23)

and in finite-time interval \([t_0, T_f]\) the measurement noise \(w(t)\) satisfies \(\sup_{t \in [t_0, T_f]} \|w(t)\| < \infty\), then switched system (2.1) is finite-time stabilizable of \(L_\infty\) disturbance attenuation performance \(\gamma = \sqrt{\max_{i,j \in \mathbb{N}}(\varepsilon_i, \varepsilon_{ij})} (e^{\max(\lambda^-,\lambda^-)(T_f-t_0)} - 1)/ \max(\lambda^+,\lambda^-)\min_{i,j \in \mathbb{N}}(\lambda_{\min}(P_i), \lambda_{\min}(P_{ij}))\) with respect to \((\delta, \varepsilon, T_f, \sigma(t), \sigma'(t))\) under the feedback controller \(u(t) = K_{\sigma(t)} x(t)\), where \(T^-(t_0, T_f)\) and \(T^+(t_0, T_f)\) denote the matched period and the mismatched period in finite-time interval \([t_0, T_f]\), respectively.
Proof. At first, from Theorem 4.4, system (2.1) is finite-time stable under the feedback controller \( u(t) = K_{x(t)}x(t) \).

Then following the proof line of Theorem 4.1 and considering (4.6) and (4.13), we can define piecewise Lyapunov function

\[
V(t) = \begin{cases} 
    x^T(t)P_rx(t), & t \in [t_r + \Delta_r, t_{r+1}), \quad r = 0, 1, \ldots, k - 1, \\
    x^T(t)P_tx(t), & t \in [t_0, t_r + \Delta_r), \quad r = 0, 1, \ldots, k - 1.
\end{cases}
\]  

(4.24)

By (4.21) and (4.22), it implies that

\[
\dot{V}(t) \leq \max(\lambda^+, \lambda^-) V(t) + \max_{i,j \in N}(\varepsilon_i, \varepsilon_{ij}) w^T(t)w(t).
\]  

(4.25)

With zero initial conditions, by Lemma 2.2, we have

\[
V(t) \leq \max_{i,j \in N}(\varepsilon_i, \varepsilon_{ij}) \int_0^{T_f-t_0} e^{\max(\lambda^+, \lambda^-) \tau} w^T(t-\tau)w(t-\tau) d\tau.
\]  

(4.26)

Notice that

\[
V(t) \geq \min_{i,j \in N}(\lambda_{\min}(P_i), \lambda_{\min}(P_{ij})) \|x(t)\|^2.
\]  

(4.27)

From (4.26) and (4.27), we can obtain

\[
\min_{i,j \in N}(\lambda_{\min}(P_i), \lambda_{\min}(P_{ij})) \sup_{t \in [t_0, T_f]} \|x(t)\|^2 \leq \frac{\max_{i,j \in N}(\varepsilon_i, \varepsilon_{ij})(e^{\max(\lambda^+, \lambda^-)(T_f-t_0)} - 1)}{\max(\lambda^+, \lambda^-)} \sup_{t \in [t_0, T_f]} \|w(t)\|^2.
\]  

(4.28)

From (4.28), we have

\[
\frac{\sup_{t \in [t_0, T_f]} \|x(t)\|}{\sup_{t \in [t_0, T_f]} \|w(t)\|} \leq \frac{\max_{i,j \in N}(\varepsilon_i, \varepsilon_{ij})(e^{\max(\lambda^+, \lambda^-)(T_f-t_0)} - 1)}{\max(\lambda^+, \lambda^-)\min_{i,j \in N}(\lambda_{\min}(P_i), \lambda_{\min}(P_{ij}))}.
\]  

(4.29)

By the definition of \( L_\infty \) finite-time stabilization, we can obtain that the designed controller \( u(t) = K_{x(t)}x(t) \) can guarantee the finite-time stability of \( L_\infty \) disturbance attenuation performance. This completes the proof. \( \square \)

Remark 4.5. It should be pointed out that the conditions in Theorems 4.4 are not standard LMIs conditions. However, through the variable substitution, (4.20) to (4.22) can be solved following the method proposed in Remark 3.6.

Remark 4.6. Theorem 4.4 presents that if the measurement noise \( w(t) \) is magnitude bounded during finite-time interval \([t_0, T_f]\), then we can design the asynchronous switching controller.
then from \( \text{parenleftmath} \begin{bmatrix} 0 & -1 \\ -0.1 & 0 \end{bmatrix} \text{parenrightmath} \) as \( u \).

When there exists asynchronous switching between the controller and the system, the system has 

\[ L_{\infty} \]

such that the system has \( L_{\infty} \) disturbance attenuation performance. However, it is unnecessary to guarantee \( L_{\infty} \) disturbance attenuation performance during the mismatched period and the matched period by the designed controller which is less conservative than Theorem 4.1.

5. Numerical Example

We consider an example to illustrate the main result. Consider the switched linear system given by the system (2.1) with 

\[ u(t) = K_{\sigma(t)} x(t), \]

\[ \dot{x}(t) = (A_{\sigma(t)} + B_{\sigma(t)} K_{\sigma(t)}) x(t) + G_{\sigma(t)} w(t), \]  

(5.1)

where 

\[ A_1 = \begin{bmatrix} -1 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.1 & 0.1 \\ 0 & 0.2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & 0.14 \\ 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0.3 \end{bmatrix}, \quad G_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}. \]

Applying Theorem 4.4 and solving corresponding matrix inequalities lead to feasible solutions, when \( \sigma = 0.1, \quad \varepsilon = 10, \quad \varepsilon_1 = \varepsilon_2 = 100, \quad \varepsilon_{12} = \varepsilon_{21} = 10, \quad \mu_1 = \mu_2 = 20, \quad \lambda^+ = 100, \quad \lambda^- = 10, \quad T_f = 0.005, \quad t_0 = 0, \quad N_0 = 0, \quad \tau_0 = 0.00375.

\[
K_1 = \begin{bmatrix} 9.6364 & 1.4424 \\ -10.3539 & 0.4207 \end{bmatrix}, \quad K_2 = \begin{bmatrix} 2.1337 & 0.4083 \\ -0.6623 & 3.5807 \end{bmatrix},
X_1 = \begin{bmatrix} 8.2146 & -14.6028 \\ -14.6028 & 86.9322 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 92.6569 & 14.6028 \\ 14.6028 & 13.9393 \end{bmatrix},
X_{12} = \begin{bmatrix} 7.9844 & -0.3851 \\ -0.3851 & 9.9854 \end{bmatrix}, \quad X_{21} = \begin{bmatrix} 10.1766 & 0.1461 \\ 0.1461 & 8.7611 \end{bmatrix}.
\]

Then from (3.23), we know that the matched period \( T^-(t_0, T_f) \) and the mismatched period \( T^+(t_0, T_f) \) satisfy the following relation:

\[ 100T^+(t_0, T_f) + 10T^-(t_0, T_f) < 0.36. \]

(5.3)

Notice that \( T^+(t_0, T_f) + T^-(t_0, T_f) = 0.005 \), then we have

\[ T^+(t_0, T_f) < 0.003, \]

\[ 0.003 < T^-(t_0, T_f) < 0.005. \]

(5.4)

Then, \( L_{\infty} \) state feedback controller \( K_1, K_2 \) can guarantee that system (5.1) is finite-time stabilizable with respect to \( (0.1, 10, 0.005, \sigma(t), \sigma'(t)) \) under the asynchronous switching where \( L_{\infty} \) disturbance attenuation performance \( \gamma = 7.8 \).

6. Conclusions

The \( L_{\infty} \) finite-time stabilization problems for switched linear system are addressed in this paper. When there exists asynchronous switching between the controller and the system, a sufficient condition for the existence of stabilizing switching law for the addressed
switched system is derived. It is proved that the switched system is finite-time stabilizable under asynchronous switching satisfying the average dwell-time condition. Furthermore, the problem of $L_\infty$ control for switched systems under asynchronous switching is also investigated. At last, a numerical example is given to illustrate the effectiveness of the proposed method.

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**References**


