Research Article

Network-Based Robust $H_{\infty}$ Filtering for the Uncertain Systems with Sensor Failures and Noise Disturbance

Mingang Hua,1, 2, 3 Pei Cheng,4 Juntao Fei,1 Jianyong Zhang,5 and Junfeng Chen1

1 College of Computer and Information, Hohai University, Changzhou 213022, China
2 Changzhou Key Laboratory of Sensor Networks and Environmental Sensing, Changzhou 213022, China
3 Jiangsu Key Laboratory of Power Transmission and Distribution Equipment Technology, Changzhou 213022, China
4 School of Mathematical Sciences, Anhui University, Hefei 230601, China
5 Department of Mathematics and Physics, Hohai University, Changzhou 213022, China

Correspondence should be addressed to Mingang Hua, mghua@yahoo.cn

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The network-based robust $H_{\infty}$ filtering for the uncertain system with sensor failures and the noise is considered in this paper. The uncertain system under consideration is also subject to parameter uncertainties and delay varying in an interval. Sufficient conditions are derived for a linear filter such that the filtering error systems are robust globally asymptotically stable while the disturbance rejection attenuation is constrained to a given level by means of the $H_{\infty}$ performance index. These conditions are characterized in terms of the feasibility of a set of linear matrix inequalities (LMIs), and then the explicit expression is then given for the desired filter parameters. Two numerical examples are exploited to show the usefulness and effectiveness of the proposed filter design method.

1. Introduction

Networked control system (NCS) is a new control system structure where sensor-controller and controller-actuator signal link is through a shared communication network. Therefore, networked control systems have become an active research area in recent years in [1–3]. Recently, the filter design for networked systems become an active research area due to the advantages of using networked media in many aspects such as low cost, reduced weight and power requirements, simple installation and maintenance, and high reliability
in [4–10]. On the other hand, with the increasing of the working time in the domain of industry, some parts of the control system (e.g., actuator and sensor) can always be invalid. By time-scale decomposition, the reliable $H_\infty$ control for linear time-invariant multiplicities singularly perturbed systems against sensor failures is studied in [11]. For systems with both state and input time delays, a novel state and sensor fault observer is proposed to estimate system states and sensor faults simultaneously in [12]. A new robust $H_\infty$ filtering problem is investigated for a class of time-varying nonlinear system with norm-bounded parameter uncertainties, bounded state delay, sector-bounded nonlinearity, and probabilistic sensor gain faults in [13]. The robust output feedback controller design for uncertain delayed systems with sensor failure and time delay is considered in [14]. The problems of robust fault estimation and fault-tolerant control for Takagi-Sugeno fuzzy systems with time delays and unknown sensor faults are addressed in [15]. The robust filtering problem for a class of discrete time-varying Markovian jump systems with randomly occurring nonlinearities and sensor saturation is studied in [16]. The robust $H_\infty$ infinite-horizon filtering problem for a class of uncertain nonlinear discrete time-varying stochastic systems with multiple missing measurements and error variance constraints is considered in [17]. The problem of distributed $H_\infty$ filtering in sensor networks using a stochastic sampled-data approach is investigated in [18]. The problems of stability analysis, $H_\infty$ performance analysis, and robust $H_\infty$ filter design for uncertain Markovian jump linear systems with time-varying delays are considered in [19].

In distributed industrial and military NCSs, sensors can be in a hostile environment and subject to failure and malfunction. Recently, the $H_\infty$ filtering problem for NCSs has received considerable attention. The problem of designing $H_\infty$ filters for a class of nonlinear networked control systems with transmission delays and packet losses is investigated in [20]. The control problems of networked control system with fault/failure of sensors and actuators are also received attention. The reliable control of a class of nonlinear NCSs via T-S fuzzy model with probabilistic sensor and actuator faults/failures, measurement distortion, time-varying delay, packet loss, and norm-bounded parameter uncertainties is investigated in [20]. Recently, based on T-S fuzzy model, the robust and reliable $H_\infty$ filter design for a class of nonlinear networked control systems is investigated with probabilistic sensor failure in [21]. The reliable filtering problem for network-based linear continuous-time system with sensor failures has been studied in [22]. However, the proposed filter design approach [21, 22] do not consider the systems with uncertainty. The time delay has restriction when the rate of delay is differential, which is only applicable to unknown rate of time delay. No delay-dependent $H_\infty$ filtering results on the uncertain networked control systems with sensor failures and disturbance noise are available in the literature, which motivates the present study.

In this paper, based on the delay-dependent stability criteria proposed in [23], a delay-dependent $H_\infty$ performance analysis result is established for the filtering error systems. A new sensor failure model with uncertainties is proposed, and a new different Lyapunov functional is then employed to deal with systems with sensor failures and uncertainties. As a result, the $H_\infty$ filter is designed in terms of linear matrix inequalities (LMIs), which involves fewer matrix variables but has less conservatism. The resulting filters can ensure that the filtering error system is asymptotically stable and the estimation error is bounded by a prescribed level for all possible bounded energy disturbances, which has advantages over the results of [22] in that it involves fewer matrix variables but has less conservatism. Meanwhile, the parameter uncertainties for system with sensor failures and the noise are considered in this paper, which are more general cases. Finally, two examples are given to show the effectiveness of the proposed method. This paper is organized as follows. Section 2
describes the system model and presents the definition and some lemmas. The robust $H_{\infty}$ filter design method is derived in Section 3. Section 4 includes two simulation examples.

2. Problem Description

Throughout this paper, $\mathbb{R}^n$ denotes the $n$-dimensional Euclidean space, and $\mathbb{R}^{n \times m}$ is the set of $n \times m$ real matrices. $I$ is the identity matrix, $\| \cdot \|$ stands for the induced matrix 2 norm, and $M^T$ stands for the transpose of the matrix $M$. For symmetric matrices $X$ and $Y$, the notation $X > Y$ (resp., $X \geq Y$) means that $X - Y$ is positive definite (resp., positive semidefinite). $\ast$ denotes a block that is readily inferred by symmetry.

Consider the following uncertain systems:

$$
\begin{align*}
    x(t) &= (A + \Delta A(t))x(t) + (B + \Delta B(t))w(t), \\
    y(t) &= (C + \Delta C(t))x(t), \\
    z(t) &= Lx(t),
\end{align*}
$$

(2.1)

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^d$ is the measurable output vector, the noise disturbance $w(t) \in \mathbb{R}^d$ is external plant belongs to $L_2[0, \infty]$, and $z(t)$ is a signal to be estimated. $A$, $B$, $C$, and $L$ are known real constant matrices, $\Delta A(t)$, $\Delta B(t)$, and $\Delta C(t)$ are unknown matrices representing time-varying parameter uncertainties, and the admissible uncertainties are assumed to be modeled in the form:

$$
[\Delta A(t) \ \Delta B(t)] = M_1 F(t) [N_1 \ N_2], \quad \Delta C(t) = M_2 F(t) N_3,
$$

(2.2)

where $M_1, M_2, N_1, N_2$, and $N_3$ are known constant matrices, $F(t)$ is unknown time-varying matrices with Lebesgue measurable elements bounded by

$$
F^T(t)F(t) \leq I.
$$

(2.3)

Assumption 1

The considered NCS consists of a time-driven sensor.

Assumption 2

There exist some sensor failures in the feedback channel.

Considering the effect of the common network on the data transmission, the filter can be expressed as

$$
\begin{align*}
    \dot{x}_f(t) &= A_f x_f(t) + B_f y_s(i_k h) \quad t \in [i_k h + \tau_{ik}, i_{k+1} h + \tau_{ik+1}), \\
    z_f(t) &= C_f x_f(t) \quad k = 1, 2, \ldots,
\end{align*}
$$

(2.4)

where $x_f(t)$ represents the state estimate, $y_s(i_k h)$ is the output with sensor failures, $z_f(t)$ is the estimated output, $A_f$, $B_f$, and $C_f$ are the filter parameters to be designed. $h$ denotes the
sampling period, and $i_k (k = 1, 2, 3, \ldots)$ are some integers such that $\{i_1, i_2, i_3, \ldots\} \subset \{1, 2, 3, \ldots\}$. $\tau_{ik}$ is the time from the instant $i_k h$ when sensors sample from the plant to the instant when actuators send control actions to the plant. Obviously, $\bigcup_{k=1}^{\infty} [i_k h + \tau_{ik}, i_{k+1} h + \tau_{ik+1}) = [t_0, \infty)$, $t \geq 0$.

**Remark 2.1.** In (2.4), $\{i_1, i_2, i_3, \ldots\}$ is a subset of $\{1, 2, 3, \ldots\}$. Moreover, it is not required that $i_{k+1} > i_k$. When $\{i_1, i_2, i_3, \ldots\} = \{1, 2, 3, \ldots\}$, it means that no packet dropout occurs in the transmission. If $\tau_{k+1} > \tau_k + 1$, there are dropped packets but the received packets are in ordered sequence. If $\tau_{k+1} < \tau_k + 1$, it means out-of-order packet arrival sequences occur. If $\tau_{k+1} = \tau_k + 1$, it implies that $h + \tau_{k+1} > \tau_k$, which includes $\tau_k = \overline{\tau}$ and $\tau_k < h$ as special cases, where $\overline{\tau}$ is a constant.

**Remark 2.2.** Since $i_k h = t - (t - i_k h)$, define $\tau(t) = t - i_k h, t \in [i_k h + \tau_{ik}, i_{k+1} h + \tau_{ik+1})$, which denotes the time-varying delay in the control signal. Obviously,

$$\tau_k \leq \tau(t) \leq (i_{k+1} - i_k) h + \tau_{k+1}, \quad t \in [i_k h + \tau_{ik}, i_{k+1} h + \tau_{ik+1})$$

(2.5)

which implies that

$$0 \leq \tau_1 \leq \tau(t) \leq \tau_2,$$

(2.6)

where $\tau_1$ and $\tau_2$ denote infimum of $\tau_k$ and supremum of $[(i_{k+1} - i_k) h + \tau_{k+1}]$, respectively.

Our aim in this paper is to design a robust $H_{\infty}$ filter in the form of (2.4) such that

(i) system (2.1) is said to be robust globally asymptotically stable, subject to $w(t) = 0$, for all admissible uncertainties satisfying (2.2)-(2.3);

(ii) for the given disturbance attenuation level $\gamma > 0$ and under zero initial condition, the performance index $\gamma$ satisfies the following inequality:

$$\|z(t)\|_2 < \|\gamma w(t)\|_2.$$  

(2.7)

For an easy exposition of our results, we first consider the following systems with no uncertain parameters:

$$x(t) = Ax(t) + Bw(t),$$

$$y(t) = Cx(t),$$

$$z(t) = Lx(t).$$

(2.8)

The switch matrix $G$ for filter (2.4) is introduced against sensor failures as follows:

$$G = \text{diag}(g_1, g_2, \ldots, g_n),$$

(2.9)
where
\[
g_i = \begin{cases} 
1 & \text{the } i\text{th sensor is complete normal} \\
\alpha & (0 < \alpha < 1) \\
0 & \text{the } i\text{th sensor completely fails, } i = 1, 2, \ldots, n
\end{cases} \tag{2.10}
\]

From above analysis and (2.4), then we can get the output of the sensor failures
\[
y_s(t) = y_s(i_k h) = GCx(t - \tau(t)) \quad t \in [i_k h + \tau_{i_k}, i_{k+1} h + \tau_{i_{k+1}}) \tag{2.11}
\]
so the filter (2.4) under consideration is of the following structure:
\[
\begin{align*}
\dot{x}_f(t) &= A_f x_f(t) + B_f G C x(t - \tau(t)), \\
z_f(t) &= C_f x_f(t).
\end{align*} \tag{2.12}
\]

Define \( \varsigma(t) = \begin{bmatrix} x(t) \\ x_f(t) \end{bmatrix} \) and the filter errors \( e(t) = z(t) - z_f(t) \), then the filtering error system can be represented as follows:
\[
\begin{align*}
\dot{\varsigma}(t) &= \tilde{A}_\varsigma(t) + \tilde{A}_d x(t - \tau(t)) + \tilde{B} w(t), \\
e(t) &= \tilde{L}_\varsigma(t),
\end{align*} \tag{2.13}
\]
where
\[
\begin{align*}
\tilde{A} &= \begin{bmatrix} A & 0 \\ 0 & A_f \end{bmatrix}, & \tilde{A}_d &= \begin{bmatrix} 0 \\ B_f G C \end{bmatrix}, & \tilde{B} &= \begin{bmatrix} B \\ 0 \end{bmatrix}, & \tilde{L} &= \begin{bmatrix} L & -C_f G C \end{bmatrix}.
\end{align*} \tag{2.14}
\]

Throughout this paper, we use the following lemmas.

**Lemma 2.3** (see [24]). Given constant matrices \( \Gamma_1, \Gamma_2, \text{ and } \Gamma_3 \) with appropriate dimensions, where \( \Gamma_1^T = \Gamma_1 \) and \( \Gamma_2^T = \Gamma_2 > 0 \), then
\[
\Gamma_1 + \Gamma_3^T \Gamma_2^{-1} \Gamma_3 < 0 \tag{2.15}
\]
if and only if
\[
\begin{bmatrix} \Gamma_1 & \Gamma_3^T \\ \Gamma_3 & -\Gamma_2 \end{bmatrix} < 0, \quad \text{or} \quad \begin{bmatrix} -\Gamma_2 & \Gamma_3 \\ \Gamma_3^T & \Gamma_1 \end{bmatrix} < 0. \tag{2.16}
\]
Lemma 2.4 (see [25]). For any constant matrix $M \in \mathbb{R}^{n \times n}$, $M = M^T > 0$, scalar $\gamma > 0$, vector function $\omega : [0,\gamma] \to \mathbb{R}^n$ such that the integrations are well defined, the following inequality holds:

$$
\left[ \int_0^\gamma w(s) ds \right]^T M \left[ \int_0^\gamma w(s) ds \right] \leq \gamma \int_0^\gamma w^T(s) M w(s) ds.
$$

(2.17)

Lemma 2.5 (see [26]). For given matrices $H$, $E$, and $F(t)$ with $F^T(t) F(t) \leq I$ and scalar $\varepsilon > 0$, the following inequality holds:

$$
HF(t)E + F^T(t)H^T \leq \varepsilon HH^T + \varepsilon^{-1} E^T E.
$$

(2.18)

3. Main Results

Section 3.1 provides an $H_\infty$ performance condition for the filtering error system (2.13). Design of $H_\infty$ filter for the system (2.8) with no uncertainty will be developed in Section 3.2, and robust $H_\infty$ filter design for the uncertain system (2.1) will be developed in Section 3.3.

3.1. Performance Analysis of $H_\infty$ Filter

Theorem 3.1. Consider the system in (2.8). For a specified filter (2.12) and constants $\tau_1$ and $\tau_2$, the filtering error system (2.13) is globally asymptotically stable with performance $\gamma$ if there exist real matrices $P > 0$, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $R_1 > 0$, and $R_2 > 0$, such that the following LMIs are satisfied:

$$
\begin{bmatrix}
\Sigma_{11} & P\bar{A}^d & H^TR_1 & 0 & PB & \bar{A}^T H^T R & \bar{L}^T \\
* & \Sigma_{22} & 2R_2 & R_2 & 0 & 0 & 0 \\
* & * & -Q_1 - R_1 - 2R_2 & 0 & 0 & 0 & 0 \\
* & * & * & -Q_2 - 2R_2 & 0 & 0 & 0 \\
* & * & * & * & -\gamma^2 I & \bar{B}^T \bar{R} & 0 \\
* & * & * & * & * & -\bar{R} & 0 \\
* & * & * & * & * & * & -I
\end{bmatrix} < 0,
$$

(3.1)

$$
\begin{bmatrix}
\Sigma_{11} & P\bar{A}^d & H^TR_1 & 0 & PB & \bar{A}^T H^T R & \bar{L}^T \\
* & \Sigma_{22} & R_2 & 2R_2 & 0 & 0 & 0 \\
* & * & -Q_1 - R_1 - R_2 & 0 & 0 & 0 & 0 \\
* & * & * & -Q_2 - 2R_2 & 0 & 0 & 0 \\
* & * & * & * & -\gamma^2 I & \bar{B}^T \bar{R} & 0 \\
* & * & * & * & * & -\bar{R} & 0 \\
* & * & * & * & * & * & -I
\end{bmatrix} < 0,
$$

(3.2)
where

\[ \Sigma_{22} = -(1 - \mu) Q_3 - 3 R_2, \quad R = \tau_2^2 R_1 + \tau_{21}^2 R_2, \quad \tau_{21} = \tau_2 - \tau_1. \]  

**Proof.** Consider the Lyapunov-Krasovskii functional candidate as follows:

\[ V(x_t) = \frac{1}{2} P Q(t) + \int_{t-\tau_1}^{t} x^T(s) Q_1 x(s) ds + \int_{t-\tau_2}^{t} x^T(s) Q_2 x(s) ds + \int_{t-\tau_1}^{t} x^T(s) Q_3 x(s) ds 
+ \int_{-\tau_1}^{0} \int_{t+\theta}^{t} \tau_1 \dot{x}^T(s) R_1 \dot{x}(s) ds d\theta + \int_{-\tau_2}^{0} \int_{t+\theta}^{t} \tau_{21} \dot{x}^T(s) R_2 \dot{x}(s) ds d\theta. \]  

Calculating the time derivative of along the trajectory of system (3.4), one has

\[ \dot{V}(x_t) \leq 2 x^T(t) [A \frac{1}{2} x(t - \tau(t)) + P B \omega(t)] + \sum_{i=1}^{3} x^T(t) Q_i x(t) 
- \sum_{i=1}^{2} x^T(t - \tau_i) Q_i x(t - \tau_i) + \eta^T(t) [H A 0 0 0 B]^T R [H A 0 0 0 B] \eta(t) 
- \int_{t-\tau_1}^{t} \tau_1 \dot{x}^T(s) R_1 \dot{x}(s) ds - \int_{t-\tau_2}^{t} \tau_{21} \dot{x}^T(s) R_2 \dot{x}(s) ds, \]  

where

\[ \eta^T(t) = \left[ \zeta^T(t) x^T(t - \tau(t)) x^T(t - \tau_1) x^T(t - \tau_2) \omega^T(t) \right]. \]  

By using Lemma 2.3, we have that

\[ -\int_{t-\tau_1}^{t} \tau_1 \dot{x}^T(s) R_1 \dot{x}(s) ds \leq \begin{bmatrix} \zeta(t) \\ x(t - \tau_1) \end{bmatrix}^T \begin{bmatrix} -H^T R_1 H & H^T R_1 \\ R_1 H & -R_1 \end{bmatrix} \begin{bmatrix} \zeta(t) \\ x(t - \tau_1) \end{bmatrix}. \]  

On the other hand,
\[- \int_{t-	au_2}^{t-	au_1} \tau_2 \dot{x}^T(s) R_2 \dot{x}(s) ds = - \int_{t-	au_2}^{t-	au(t)} \tau_2 \dot{x}^T(s) R_2 \dot{x}(s) ds - \int_{t-	au(t)}^{t-	au_1} \tau_2 \dot{x}^T(s) R_2 \dot{x}(s) ds \]

\[= - \int_{t-	au_2}^{t-	au(t)} (\tau_2 - \tau(t)) \dot{x}^T(s) R_2 \dot{x}(s) ds \]

\[- \int_{t-	au(t)}^{t-	au_1} (\tau(t) - \tau_1) \dot{x}^T(s) R_2 \dot{x}(s) ds \]

\[- \int_{t-	au(t)}^{t-	au_1} (\tau(t) - \tau_1) \dot{x}^T(s) R_2 \dot{x}(s) ds \]

\[\leq - \beta \int_{t-	au_2}^{t-	au(t)} (\tau_2 - \tau(t)) \dot{x}^T(s) R_2 \dot{x}(s) ds, \]

\[- \int_{t-	au(t)}^{t-	au_1} (\tau(t) - \tau_1) \dot{x}^T(s) R_2 \dot{x}(s) ds \]

\[\leq - (1 - \beta) \int_{t-	au(t)}^{t-	au_1} \tau_1 \dot{x}^T(s) R_2 \dot{x}(s) ds. \]

Set

\[\beta = \frac{(\tau(t) - \tau_1)}{\tau_2}. \]

Then

\[- \int_{t-	au_2}^{t-	au(t)} (\tau(t) - \tau_1) \dot{x}^T(s) R_2 \dot{x}(s) ds = - \beta \int_{t-	au_2}^{t-	au(t)} \tau_2 \dot{x}^T(s) R_2 \dot{x}(s) ds \]

\[\leq - \beta \int_{t-	au_2}^{t-	au(t)} (\tau_2 - \tau(t)) \dot{x}^T(s) R_2 \dot{x}(s) ds, \]

\[- \int_{t-	au(t)}^{t-	au_1} (\tau_2 - \tau(t)) \dot{x}^T(s) R_2 \dot{x}(s) ds = - (1 - \beta) \int_{t-	au(t)}^{t-	au_1} \tau_2 \dot{x}^T(s) R_2 \dot{x}(s) ds. \]

Combining (3.8)–(3.10) and by Lemma 2.4, we have

\[- \int_{t-	au_2}^{t-	au_1} \tau_2 \dot{x}^T(s) R_2 \dot{x}(s) ds \leq - \begin{bmatrix} x(t - \tau(t)) \\ x(t - \tau_1) \end{bmatrix}^T \begin{bmatrix} R_2 & -R_2 \\ -R_2 & R_2 \end{bmatrix} \begin{bmatrix} x(t - \tau(t)) \\ x(t - \tau_1) \end{bmatrix} - \beta \begin{bmatrix} x(t - \tau(t)) \\ x(t - \tau_1) \end{bmatrix}^T \begin{bmatrix} R_2 & -R_2 \\ -R_2 & R_2 \end{bmatrix} \begin{bmatrix} x(t - \tau(t)) \\ x(t - \tau_1) \end{bmatrix} - (1 - \beta) \begin{bmatrix} x(t - \tau(t)) \\ x(t - \tau_1) \end{bmatrix}^T \begin{bmatrix} R_2 & -R_2 \\ -R_2 & R_2 \end{bmatrix} \begin{bmatrix} x(t - \tau(t)) \\ x(t - \tau_1) \end{bmatrix}. \]
Combining (3.5), (3.7), and (3.11) yields

\[ \dot{V}(x_i) \leq \eta^T(t) \left( (1-\beta) \Phi_1 + \beta \Phi_2 + [HA \ 0 \ 0 \ 0 B]^T R [HA \ 0 \ 0 \ 0 B] \right) \eta(t), \]

where

\[
\Phi_1 = \begin{bmatrix} \Sigma_{11} & P \overline{A}_d & H^T R_1 & 0 & P \overline{B} \\ * & \Sigma_{22} & 2R_2 & R_2 & 0 \\ * & * & -Q_1 - R_1 - 2R_2 & 0 & 0 \\ * & * & * & -Q_2 - R_2 & 0 \\ * & * & * & * & 0 \end{bmatrix},
\]

\[
\Phi_2 = \begin{bmatrix} \Sigma_{11} & P \overline{A}_d & H^T R_1 & 0 & P \overline{B} \\ * & \Sigma_{22} & R_2 & 2R_2 & 0 \\ * & * & -Q_1 - R_1 - R_2 & 0 & 0 \\ * & * & * & -Q_2 - 2R_2 & 0 \\ * & * & * & * & 0 \end{bmatrix}.
\]

Under the zero-initial condition, one can obtain that \( V(x_i)|_{t=0} = 0 \) and \( V(x_i) \geq 0 \). Define

\[ J(t) = \int_0^\infty \left[ z^T(t)z(t) - \gamma^2 w^T(t)w(t) \right] dt \]

then, for any nonzero \( w(t) \),

\[ J(t) \leq \int_0^\infty \left[ z^T(t)z(t) - \gamma^2 w^T(t)w(t) \right] dt + V(x_i)|_{t=\infty} - V(x_i)|_{t=0} \]

\[ = \int_0^\infty \left[ z^T(t)z(t) - \gamma^2 w^T(t)w(t) + \dot{V}(x_i) \right] dt \]

\[ \leq \int_0^\infty \left[ \eta^T(t) \left( (1-\beta) \overline{\Phi}_1 + \beta \overline{\Phi}_2 + [HA \ 0 \ 0 \ 0 B]^T R [HA \ 0 \ 0 \ 0 B] \right) \eta(t) \right] dt, \]

where

\[
\overline{\Phi}_1 = \begin{bmatrix} \Sigma_{11} + L^T \overline{P} \overline{A}_d & H^T R_1 & 0 & P \overline{B} \\ * & \Sigma_{22} & 2R_2 & R_2 & 0 \\ * & * & -Q_1 - R_1 - 2R_2 & 0 & 0 \\ * & * & * & -Q_2 - R_2 & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix},
\]

\[
\overline{\Phi}_2 = \begin{bmatrix} \Sigma_{11} + L^T \overline{P} \overline{A}_d & H^T R_1 & 0 & P \overline{B} \\ * & \Sigma_{22} & R_2 & 2R_2 & 0 \\ * & * & -Q_1 - R_1 - R_2 & 0 & 0 \\ * & * & * & -Q_2 - 2R_2 & 0 \\ * & * & * & * & -\gamma^2 I \end{bmatrix}.
\]
Due to $0 < \beta < 1$, our elaborate estimation $J(t)$ induces a convex domain of matrices $(1-\beta)\Phi_1 + \beta\Phi_2$, which are negative definite if and only if $\Phi_1 < 0$ and $\Phi_2 < 0$. By using Lemma 2.3, the LMIs (3.1) and (3.2) can guarantee $(1-\beta)\Phi_1 + \beta\Phi_2 + [HA 0 0 0 B]^T R [HA 0 0 0 B] < 0$. Since $w(t) \neq 0$, it implies that $z^T(t)z(t) - \gamma^2 w^T(t)w(t) + V(x_t) < 0$, and thus, $J(t) < 0$. That is, $\|z(t)\|_2 < \gamma\|w(t)\|_2$.

Second, we also can prove that under the condition of Theorem 3.1, the filtering error system (2.13) with $w(t) = 0$ is globally asymptotically stable. This completes the proof. ∎

### 3.2. Design of $H_\infty$ Filter

Now based on the previous result, we are in a position to present the main result in this paper, which offers a new networked-based $H_\infty$ filter design approach for the system (2.8).

**Theorem 3.2.** Consider the system in (2.8). A filter of form (2.12) and constants $\tau_1$ and $\tau_2$, the filtering error system is globally asymptotically stable with performance $\gamma$, if there exist real matrices $P_1 > 0$, $P_3 > 0$, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $R_1 > 0$, $R_2 > 0$ and any matrices $A_f$, $B_f$, and $C_f$ such that the following LMIs are satisfied:

$$Y_1 = \begin{bmatrix}
\Sigma_{11} & \overline{A}_f + A_f^T P_3 & \overline{B}_f GC & R_1 & 0 & P_3 B & A^T R & L^T \\
\ast & \overline{A}_f + A_f^T & \overline{B}_f GC & 0 & 0 & \overline{P}_3 B & 0 & -C_f^T \\
\ast & \ast & \Sigma_{33} & 2R_2 & R_2 & 0 & 0 & 0 \\
\ast & \ast & \ast & \Sigma_{441} & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \Sigma_{531} & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & -\gamma^2 I & B^T R \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & -R \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & -I \\
\end{bmatrix} < 0, \quad (3.17)$$

$$Y_2 = \begin{bmatrix}
\Sigma_{11} & \overline{A}_f + A_f^T P_3 & \overline{B}_f GC & R_1 & 0 & P_3 B & A^T R & L^T \\
\ast & \overline{A}_f + A_f^T & \overline{B}_f GC & 0 & 0 & \overline{P}_3 B & 0 & -C_f^T \\
\ast & \ast & \Sigma_{33} & R_2 & 2R_2 & 0 & 0 & 0 \\
\ast & \ast & \ast & \Sigma_{442} & 0 & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \Sigma_{552} & 0 & 0 & 0 \\
\ast & \ast & \ast & \ast & \ast & \ast & -\gamma^2 I & B^T R \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & -R \\
\ast & \ast & \ast & \ast & \ast & \ast & \ast & -I \\
\end{bmatrix} < 0, \quad (3.18)$$

$$P_3 - P_1 < 0, \quad (3.19)$$

where

$$\Sigma_{11} = PA_1 + A_1^T P + Q_1 + Q_2 + Q_3 - R_1, \quad \Sigma_{33} = -(1 - \mu)Q_3 - 3R_2,$$

$$\Sigma_{441} = -Q_1 - R_1 - 2R_2, \quad \Sigma_{442} = -Q_1 - R_1 - R_2,$$

$$\Sigma_{551} = -Q_2 - R_2, \quad \Sigma_{552} = -Q_2 - 2R_2 \quad (3.20)$$

Moreover, if the previous conditions are satisfied, an acceptable state-space realization of the $H_\infty$ filter

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is given by
\[ A_f = \overline{A}_f \overline{P}_3^{-1}, \quad B_f = \overline{B}_f, \quad C_f = \overline{C}_f \overline{P}_3^{-1}. \] (3.21)

**Proof.** Defining
\[ P = \begin{bmatrix} P_1 & P_1^T \\ P_2 & P_2^T \end{bmatrix}, \quad J = \begin{bmatrix} I & 0 \\ 0 & P_2^T P_3^{-1} \end{bmatrix} \] (3.22)
then \( P_1 > 0 \) and \( P_1 - P_3 > 0 \). Choose the LKF candidate as (3.4), and set
\[ \Delta = \text{diag} \{ J, I, I, I, I, I \}, \quad \overline{A}_f = P_2^T A_f P_2, \quad \overline{B}_f = P_2^T P_3^{-1} B_f, \]
\[ \overline{C}_f = P_2^T P_3^{-T} P_2, \quad \overline{P}_3 = P_2^T P_5^{-T} P_2. \] (3.23)

Pre- and postmultiplying (3.1) and (3.2) by \( \Delta \) and \( \Delta^T \), respectively, we can obtain that (3.1) is equivalent to (3.17), and (3.2) is equivalent to (3.18). Thus, we can conclude from Theorem 3.2 that the error systems are globally asymptotically stable with the \( H_\infty \) attenuation level \( \gamma \). In addition, the filter matrices \( A_f, B_f, \) and \( C_f \) can be constructed from (3.21). This completes the proof. \( \square \)

**Remark 3.3.** When \( \tau_1, \tau_2, \) and \( \mu \) are given, matrix inequalities (3.17) and (3.18) are linear matrix inequalities in matrix variables \( P_1, \overline{P}_3, Q_1, Q_2, Q_3, R_1, R_2, P_2, \overline{A}_f, \overline{B}_f, \) and \( \overline{C}_f \), which can be efficiently solved by the developed interior point algorithm [24]. Meanwhile, it is easy to find the minimal attenuation level \( \gamma \).

**Remark 3.4.** From the proof process of Theorem 3.1, one can clearly see that neither model transformation nor bounding technique for cross terms is used. Therefore, the obtained filter design method is expected less conservative. It is well known that the number of variables has a great influence on the computation burden. The number of variables involved in the LMIs (3.17)–(3.19) is \((9/2)n^2 + (7/2)n + 2n\). However, the numbers of variables in [22] is \(15n^2 + 5n\). With much fewer matrix variables Theorem 3.2 also saves much computation than Theorem 2 in [22].

### 3.3. Robust \( H_\infty \) Filter

On the basis of the result of Theorem 3.2, it is easy to obtain the network-based robust \( H_\infty \) filter design for the uncertain systems (2.1) with uncertainties \( \Delta A(t), \Delta B(t), \) and \( \Delta C(t) \) satisfying (2.2)–(2.3).

**Theorem 3.5.** Consider the uncertain system in (2.1). A filter of form (2.12) and constants \( \tau_1 \) and \( \tau_2 \), the filtering error system is robust globally asymptotically stable with performance \( \gamma \), if there exist
real matrices $P_1 > 0, \overline{P}_3 > 0, Q_1 > 0, Q_2 > 0, Q_3 > 0, R_1 > 0, R_2 > 0$ and any matrices $\overline{A}_f, \overline{B}_f,$ and $\overline{C}_f,$ and scalars $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ such that the following LMIs and (3.19) are satisfied:

\[
\begin{bmatrix}
\Sigma_{11} & \overline{A}_f + A^T \overline{P}_3 & \overline{B}_f \Sigma_{16} & A^T \overline{R} & L^T & P_1M_1 & \overline{B}_f GM_2 \\
* & \overline{A}_f + \overline{A}_f^T & \overline{B}_f \Sigma_{444} & 0 & 0 & -\overline{C}_f^T & \overline{P}_3 \Sigma_{444} & \overline{B}_f GM_2 \\
* & * & 2R_2 & R_2 & 0 & 0 & 0 & 0 \\
* & * & * & \Sigma_{33} & 0 & 0 & 0 & 0 \\
* & * & * & * & \Sigma_{552} & 0 & 0 & 0 \\
* & * & * & * & * & -R & 0 & 0 \\
* & * & * & * & * & * & -I & 0 \\
* & * & * & * & * & * & * & -\varepsilon_1 I \\
\end{bmatrix} < 0,
\]

(3.24)

\[
\begin{bmatrix}
\Sigma_{11} & \overline{A}_f + A^T \overline{P}_3 & \overline{B}_f \Sigma_{16} & A^T \overline{R} & L^T & P_1M_1 & \overline{B}_f GM_2 \\
* & \overline{A}_f + \overline{A}_f^T & \overline{B}_f \Sigma_{444} & 0 & 0 & -\overline{C}_f^T & \overline{P}_3 \Sigma_{444} & \overline{B}_f GM_2 \\
* & * & 2R_2 & R_2 & 0 & 0 & 0 & 0 \\
* & * & * & \Sigma_{33} & 0 & 0 & 0 & 0 \\
* & * & * & * & \Sigma_{552} & 0 & 0 & 0 \\
* & * & * & * & * & -R & 0 & 0 \\
* & * & * & * & * & * & -I & 0 \\
* & * & * & * & * & * & * & -\varepsilon_1 I \\
\end{bmatrix} < 0,
\]

where

\[
\begin{align*}
\hat{\Sigma}_{11} &= PA_1 + A_1^T P + Q_1 + Q_2 + Q_3 - R_1 + \varepsilon_1 N_1^T N_1, \\
\hat{\Sigma}_{16} &= P_1B + \varepsilon_1 N_2^T N_1, \\
\hat{\Sigma}_{33} &= -(1 - \mu)Q_3 - 3R_2 + \varepsilon_1 N_3^T N_3, \\
\hat{\Sigma}_{66} &= -\gamma^2 I + \varepsilon_1 N_2^T N_2.
\end{align*}
\]

Moreover, if the previous conditions are satisfied, an acceptable state-space realization of the $H_\infty$ filter $A_f, B_f,$ and $C_f$ are given by (3.21).

**Proof.** Replace $A, B$ and $C$ in the LMIs (3.17) and (3.18) with $A + \Delta A(t), B + \Delta B(t)$ and $C + \Delta C(t),$ respectively, then the LMIs (3.17) and (3.18) can be rewritten as

\[
\begin{align*}
Y_1 + \chi_1 F_1(t) \chi_2 + \chi_2^T F_1^T(t) \chi_1^T + \chi_3 F_2(t) \chi_4 + \chi_4^T F_2^T(t) \chi_3^T < 0, \\
Y_2 + \chi_1 F_1(t) \chi_2 + \chi_2^T F_1^T(t) \chi_1^T + \chi_3 F_2(t) \chi_4 + \chi_4^T F_2^T(t) \chi_3^T < 0,
\end{align*}
\]

(3.26)
where

\[
\chi_1 = \begin{bmatrix} M_1^T P_1 M_1^T P_3 & 0 & 0 & 0 & 0 & M_1^T R \\ \end{bmatrix}^T,
\]
\[
\chi_2 = [N_1 \ 0 \ 0 \ 0 \ 0 \ N_2 \ 0],
\]
\[
\chi_3 = \begin{bmatrix} M_2^T & GB_f & M_2^T & GB_f & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T,
\]
\[
\chi_4 = [0 \ 0 \ N_3 \ 0 \ 0 \ 0 \ 0].
\]

By Lemma 2.5, there exist scalars \( \epsilon_1 > 0 \) and \( \epsilon_2 > 0 \) such that

\[
\begin{align*}
\Upsilon_1 + \epsilon_1^{-1} \chi_1 \chi_1^T + \epsilon_1 \chi_2 \chi_2^T + \epsilon_2^{-1} \chi_3 \chi_3^T + \epsilon_2 \chi_4 \chi_4^T & < 0, \\
\Upsilon_2 + \epsilon_1^{-1} \chi_1 \chi_1^T + \epsilon_1 \chi_2 \chi_2^T + \epsilon_2^{-1} \chi_3 \chi_3^T + \epsilon_2 \chi_4 \chi_4^T & < 0,
\end{align*}
\]

then by Lemma 2.3, (3.24) follows directly.

Remark 3.6. When the \( \mu \) is unknown, by setting \( Q_3 = 0 \), Theorem 3.2 and Theorem 3.5 reduce to a delay-dependent and rate-independent network-based robust \( H_\infty \) filter design condition for the uncertain systems (2.1) with uncertainties \( \Delta A(t) \), \( \Delta B(t) \), and \( \Delta C(t) \) satisfying (2.2)-(2.3).

4. Numerical Example

In this section, two examples are given to illustrate the effectiveness and benefits of the proposed approach.

Example 4.1. Consider the system (2.8) with [22]

\[
A = \begin{bmatrix} 0.5 & 3 \\ -2 & -5 \end{bmatrix}, \quad B = \begin{bmatrix} -0.5 \\ 0.9 \end{bmatrix}, \quad C = [0 \ 1], \quad L = [1 \ 1].
\]

This example has been considered in [22], and assume that \( \tau(t) \) satisfies \( 0.01 \leq \tau(t) \leq 0.2 \), \( w(t) = 0.1 \sin e^{-0.1t} \) and the sensor has a probabilistic distort, that is, the distort matrix \( G = 0.6 \).

Note that different \( \tau_1 \) and \( \tau_2 \) yield different \( \gamma_{\text{min}} \), to compare with the existing results [22]; we assume that \( \mu \) is unknown, and by setting \( Q_3 = 0 \) in Theorem 3.2, the computation results of \( \gamma_{\text{min}} \) under different \( \tau_1 \) and \( \tau_2 \) are listed in Tables 1 and 2. Minimum index \( \gamma \) for different \( \mu \) and \( \tau_1 = 0 \) is listed in Table 3. From Tables 1–3, it can be seen that the value of \( \gamma_{\text{min}} \) grows for \( \tau_2 \to 20 \) and for given \( \tau_1 \), which tends to be 0.2143.

To get minimum index \( \gamma \), the approach in [22] needs 70 decision variables; however, the number of decision variables involved in Theorem 3.2 is only 29, which sufficiently demonstrates the efficiency of the proposed method. With fewer matrix variables the minimum index \( \gamma \) obtained in this paper are less conservative than those in [22].
Table 1: Minimum index $\gamma$ for unknown $\mu$ and $\tau_1 = 0.01$.

<table>
<thead>
<tr>
<th>$\tau_2$</th>
<th>0.2</th>
<th>0.5</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>[22]</td>
<td>0.2143</td>
<td>0.2143</td>
<td>0.2143</td>
<td>0.2143</td>
<td>0.2143</td>
<td>0.2143</td>
</tr>
<tr>
<td>Theorem 3.2</td>
<td>0.1361</td>
<td>0.1900</td>
<td>0.2067</td>
<td>0.2140</td>
<td>0.2142</td>
<td>0.2143</td>
</tr>
</tbody>
</table>

Table 2: Minimum index $\gamma$ for unknown $\mu$ and $\tau_1 = 0$.

<table>
<thead>
<tr>
<th>$\tau_2$</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>[22]</td>
<td>0.2143</td>
<td>0.2143</td>
<td>0.2143</td>
<td>0.2143</td>
<td>0.2143</td>
<td>0.2143</td>
</tr>
<tr>
<td>Theorem 3.2</td>
<td>0.0884</td>
<td>0.1903</td>
<td>0.2068</td>
<td>0.2140</td>
<td>0.2142</td>
<td>0.2143</td>
</tr>
</tbody>
</table>

The initial conditions $x(t)$ and $x_f(t)$ are $[0.1 - 0.1]^T$ and $[0.2 - 0.5]^T$, respectively, for an appropriate initial interval. For given $\tau_1 = 0.01$, $\tau_2 = 0.2$ with $\gamma_{\min} = 0.1361$, according to Theorem 3.2, we can obtain the desired $H_\infty$ filter parameters as follows:

\[
A_f = \begin{bmatrix} 74.8448 & -136.6475 \\ 146.4634 & -261.6214 \end{bmatrix}, \quad B_f = \begin{bmatrix} -0.9327 \\ -1.3543 \end{bmatrix}, \quad C_f = \begin{bmatrix} 127.2527 & -228.1007 \end{bmatrix}.
\] (4.2)

Next, we apply the filter (2.12) with the filter matrices (3.21) to the system (2.8) and obtain the simulation results as in Figures 1–3. Figure 1 shows the state response $x(t)$ under the initial condition. Figure 2 shows error response $e(t) = z(t) - z_f(t)$. Figure 3 shows the output $z(t)$ and $z_f(t)$. From these simulation results, we can see that the designed $H_\infty$ filter can stabilize the system (2.8) with sensor failures and noise disturbance.

The example conclusively shows that our results are less conservative than the previous ones in [22].

Example 4.2. Consider the uncertain system (2.1) with

\[
A = \begin{bmatrix} -5 & 0 & 1 \\ 0 & -4 & 1 \\ 1 & 0 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ -0.5 \\ 0.1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix},
\] (4.3)

and the uncertainties of the system are of the forms (2.2) and (2.3) with

\[
M_1 = \begin{bmatrix} 0.1 \\ -0.2 \\ 0.1 \end{bmatrix}, \quad N_1 = \begin{bmatrix} 0.1 & 0.1 & 0.1 \end{bmatrix}, \quad N_2 = 1, \quad N_3 = \begin{bmatrix} 0.1 & 0.1 & 0.1 \end{bmatrix}.
\] (4.4)

\[
M_2 = \begin{bmatrix} 0.1 \\ -0.2 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 0.1 & 0.1 & 0.1 \end{bmatrix}.
\]
Table 3: Minimum index $\gamma$ for different $\mu$ and $\tau_1 = 0$.

<table>
<thead>
<tr>
<th>$\tau_2$</th>
<th>0.1</th>
<th>0.5</th>
<th>1</th>
<th>5</th>
<th>10</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu = 0$</td>
<td>0.0797</td>
<td>0.1733</td>
<td>0.1949</td>
<td>0.2122</td>
<td>0.2137</td>
<td>0.2141</td>
</tr>
<tr>
<td>$\mu = 0.5$</td>
<td>0.0884</td>
<td>0.1851</td>
<td>0.2005</td>
<td>0.2125</td>
<td>0.2137</td>
<td>0.2142</td>
</tr>
<tr>
<td>$\mu \geq 1$</td>
<td>0.0884</td>
<td>0.1903</td>
<td>0.2068</td>
<td>0.2140</td>
<td>0.2142</td>
<td>0.2143</td>
</tr>
<tr>
<td>unknown $\mu$</td>
<td>0.0884</td>
<td>0.1903</td>
<td>0.2068</td>
<td>0.2140</td>
<td>0.2142</td>
<td>0.2143</td>
</tr>
</tbody>
</table>

Figure 1: The state response of system $x(t)$ in Example 4.1.

Figure 2: The error response $e(t) = z(t) - z_f(t)$ in Example 4.1.
We assume $\tau(t)$ satisfies $0 \leq \tau(t) \leq 1$, $w(t) = 0.1 \sin e^{-0.1t}$, and the sensor has a probabilistic distort, that is, the distort matrix

$$G = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix}. \quad (4.5)$$

When $\mu$ is unknown, by setting $Q_3 = 0$ in Theorem 3.5, the minimum achievable noise attenuation level is given by $\gamma = 0.24$ and the correspond filter parameters as follows:

$$A_f = \begin{bmatrix} 53.9212 & -73.0807 & 38.4644 \\ 132.0705 & -156.6194 & 87.3469 \\ -321.6563 & 355.4126 & -233.8653 \end{bmatrix}, \quad B_f = \begin{bmatrix} -0.2813 & -0.7511 \\ -0.4865 & -1.5153 \\ -0.4195 & -1.0039 \end{bmatrix}, \quad (4.6)$$

$$C_f = [-192.4663 \ 205.3754 \ -136.1696].$$

When $\mu = 0.5$, by Theorem 3.5, the minimum achievable noise attenuation level is given by $\gamma = 0.24$ and the correspond filter parameters as follows:

$$A_f = \begin{bmatrix} 41.5346 & -62.0934 & 26.4966 \\ 104.1741 & -131.5645 & 66.4086 \\ -218.6961 & 248.5441 & -162.7258 \end{bmatrix}, \quad B_f = \begin{bmatrix} -0.2547 & -0.7358 \\ -0.5284 & -1.6716 \\ -0.5576 & -1.3325 \end{bmatrix}, \quad (4.7)$$

$$C_f = [-132.7246 \ 143.3911 \ -97.2733].$$

The initial conditions $x(t)$ and $x_f(t)$ are $[0.3 - 0.1 - 0.2]^T$ and $[-0.1 - 0.1 - 0.3]^T$, respectively, for an appropriate initial interval. Next, we apply the filter (2.12) with the
filter matrices (3.21) to the uncertain system (2.1) and obtain the simulation results as in Figures 4–6. Figure 4 shows the state response $x(t)$ under the initial condition. Figure 5 shows error response $e(t) = z(t) - z_f(t)$. Figure 6 shows the output $z(t)$ and $z_f(t)$. From these simulation results, we can see that the designed $H_\infty$ filter can stabilize the system (2.1) with sensor failures and noise disturbance.
5. Conclusions

In this paper the network-based robust $H_\infty$ filtering for the uncertain system with sensor failures and noise disturbance has been developed. A new type of Lyapunov-Krasovskii functional has been constructed to derive a less conservative sufficient condition for a linear full-order filter in terms of LMIs, which guarantees a prescribed $H_\infty$ performance index for the filtering error system. Two numerical examples have shown the usefulness and effectiveness of the proposed filter design method. Finally, our future study will focus mainly on the following two issues: (1) to further improve our results by using the delay decomposition LKF. (2) When the noise is stochastic, that is to say, the network-based robust $H_\infty$ filtering for the uncertain system with sensor failures and stochastic noise could be considered.

References


