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Research Article

Improved Stability Analysis for Neural Networks with Time-Varying Delay

Yongming Li, Junkang Tian, Jinzhou Zhao, Liehui Zhang, and Tiejun Li

State Key Laboratory of Oil and Gas Reservoir Geology and Exploitation, Southwest Petroleum University, Chengdu, Sichuan 610500, China

Correspondence should be addressed to Junkang Tian, junkangtian@163.com

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This paper concerned the problem of delay-dependent asymptotic stability for neural networks with time-varying delay. A new class of Lyapunov functional dividing the interval delay is constructed to derive some new delay-dependent stability criteria. The obtained criteria are less conservative because free-weighting matrices method, a convex optimization approach, and a mixed dividing delay interval approach are considered. Finally, numerical examples are given to illustrate the effectiveness of the proposed method.

1. Introduction

In the past few decades, delayed neural networks have been investigated extensively because of their successful applications in various scientific areas, such as pattern recognition, image processing, associative memories, and parallel computation. It is well known that time delay is frequently encountered in neural networks, and it is often a major cause of instability and oscillation. Thus, the stability analysis of delayed neural networks has been widely considered by many research results, delay-independent ones [1–3], and delay-dependent ones [4–37]. Generally speaking, the delay-dependent stability criteria are less conservative than delay-independent ones when the time delay is small. Therefore, much attention has been paid to develop delay-dependent derived in [6] by considering some useful terms and using the free-weighting matrices method. By the fact that the neuron activation functions are sector bounded and nondecreasing, [7] presents an improved method, named the delay-slope-dependent method, for stability analysis of neural networks with time-varying delays. The method includes more information on the slope of neuron activation functions

and fewer matrix variables in the constructing Lyapunov functionals. Then some new delay-dependent stability criteria with less conservatism are obtained. Recently, some new Lyapunov functionals based on the idea of decomposing the delay were introduced to investigate the stability of neural networks with time-invariant delay [10–12] and time-varying delay [13–16], which significantly reduced the conservativeness of the derived stability criteria. In [13], different from some previous results, the delay interval $[0, d(t)]$ is divided into some variable subintervals by employing weighting delays. Thus, some new delay-dependent stability criteria for neural networks with time varying delay are derived by applying the weighting-delay method, which are less conservative than the existing results. However, when the delay is time-varying, the information of subinterval is not considered sufficiently. For example, the time-varying delay $\tau(t)$ satisfies $0 \leqslant \tau(t) \leqslant h$. When the delay interval $[0, \tau(t)]$ is divided into some subintervals, the delay interval $[0, h]$ is also divided into some subintervals, in essence. But in the construction of Lyapunov functional in [15], this important information is ignored, which is a major source of conservativeness. Furthermore, the purpose of reducing conservatism is still limited due to the existence of multiple coefficients and the impact of subintervals with uniform size. Thus, it is still a quite difficult task to divide interval $[0, \tau(t)]$ in a more reasonable manner, so that the functional with the augmented matrix can easily be constructed to obtain less conservative stability results, which motivates our present study.

In this paper, the problem of delay-dependent asymptotic stability criterion for neural networks with time-varying delay has been considered. A new class of Lyapunov functional is constructed to derive some new delay-dependent stability criteria. The obtained criteria are less conservative because a mixed dividing delay interval approach is considered. Finally, the numerical examples are given to indicate significant improvements over some existing results.

2. Problem Formulation

Consider the following neural networks with time-varying delay:

$$\dot{x}(t) = -Cx(t) + Ag(x(t)) + Bg(x(t - \tau(t))) + \mu, \tag{2.1}$$

where $x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in \mathcal{R}^n$ is the neuron state vector, $g(x(\cdot)) = [g_1(x_1(\cdot)), g_2(x_2(\cdot)), \ldots, g_n(x_n(\cdot))]^T \in \mathcal{R}^n$ denotes the neuron activation function, and $\mu = (\mu_1, \mu_2, \ldots, \mu_n)^T \in \mathcal{R}^n$ is a constant input vector. $A, B \in \mathcal{R}^{n\times n}$ are the connection weight matrix and the delayed connection weight matrix, respectively. $C = \operatorname{diag}(C_1, C_2, \ldots, C_n)$ with $C_i > 0, i = 1, 2, \ldots, n$. $\tau(t)$ is a time-varying continuous function that satisfies $0 \leqslant \tau(t) \leqslant h, \dot{\tau}(t) \leqslant u$, where h and u are constants. In addition, it is assumed that each neuron activation function in (2.1), $g_i(\cdot),\ i = 1, 2, \ldots, n$, is bounded and satisfies the following condition:

$$0 \leqslant \frac{g_i(x) - g_i(y)}{x - y} \leqslant k_i, \quad \forall x, y \in R,\ x \neq y,\ i = 1, 2, \ldots, n, \tag{2.2}$$

where $k_i,\ i = 1, 2, \ldots, n$ are positive constants.

Assuming that $x^* = [x_1^*, x_2^*, \ldots, x_n^*]^T$ is the equilibrium point of (2.1) whose uniqueness has been given in [31] and using the transformation $z(\cdot) = x(\cdot) - x^*$, system (2.1) can be converted to the following system:

$$\dot{z}(t) = -Cz(t) + Af(z(t)) + Bf(z(t-\tau(t))), \tag{2.3}$$

where $z(t) = [z_1(t), z_2(t), \ldots, z_n(t)]^T$, $f(z(\cdot)) = [f_1(z_1(\cdot)), f_2(z_2(\cdot)), \ldots, f_n(z_n(\cdot))]^T$, and $f_i(z_i(\cdot)) = g_i(z_i(\cdot)+x_i^*)-g_i(x_i^*)$, $i = 1, 2, \ldots, n$. According to the inequality (2.2), one can obtain that

$$0 \leqslant \frac{f_i(z_i(t))}{z_i(t)} \leqslant k_i, \qquad f_i(0) = 0, \ i = 1, 2, \ldots, n. \tag{2.4}$$

Thus, under this assumption, the following inequality holds for any diagonal matrix $Q > 0$,

$$z^T(t)KQKz(t) - f^T(z(t))Qf(z(t)) \geqslant 0, \tag{2.5}$$

where $K = \operatorname{diag}(k_1, k_2, \ldots, k_n)$.

Lemma 2.1 (see [38]). *For any constant matrix $Z \in \mathcal{R}^{n\times n}$, $Z = Z^T > 0$, scalars $h_2 > h_1 > 0$, such that the following integrations are well defined, then*

$$-(h_2 - h_1)\int_{t-h_2}^{t-h_1} x^T(s)Zx(s)ds \leqslant -\int_{t-h_2}^{t-h_1} x^T(s)ds Z \int_{t-h_2}^{t-h_1} x(s)ds. \tag{2.6}$$

3. Main Results

In this section, a new Lyapunov functional is constructed, and a new delay-dependent stability criterion is obtained.

Theorem 3.1. *For given scalars $K = \operatorname{diag}(k_1, k_2, \ldots, k_n)$, $h \geqslant 0, u$, and $0 < \alpha < 1$, the system (2.3) is globally asymptotically stable if there exist symmetric positive matrices $P = [P_{ij}]_{3\times 3}, Q_i$ $(i = 1, 2, \ldots, 6)$, R_i $(i = 1, 2, \ldots, 6)$, positive diagonal matrices $T_1, T_2, Q, \Delta = \operatorname{diag}(\delta_1, \delta_2, \ldots, \delta_n), \Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, and any matrices $P_1, P_2, N_i, M_i, L_i, H_i, Z_i, S_i, U_i, V_i, W_i$ $(i = 1, 2, 3)$ with appropriate dimensions, such that the following LMIs hold:*

$$E_1 = \begin{bmatrix} E & \alpha hN & \dfrac{\alpha^2 h^2}{2}H & \dfrac{(1-\alpha^2)h^2}{2}M \\ * & -\alpha hR_3 & 0 & 0 \\ * & * & -\dfrac{\alpha^2 h^2}{2}R_5 & 0 \\ * & * & * & -\dfrac{(1-\alpha^2)h^2}{2}R_6 \end{bmatrix} < 0, \tag{3.1}$$

$$E_2 = \begin{bmatrix} E & (1-\alpha)\alpha hL & \alpha^2 hS & \frac{\alpha^2 h^2}{2}H & \frac{(1-\alpha^2)h^2}{2}M \\ * & -(1-\alpha)\alpha hR_3 & 0 & 0 & 0 \\ * & * & -\alpha^2 hR_3 & 0 & 0 \\ * & * & * & -\frac{\alpha^2 h^2}{2}R_5 & 0 \\ * & * & * & * & -\frac{(1-\alpha^2)h^2}{2}R_6 \end{bmatrix} < 0, \tag{3.2}$$

$$\Phi_1 = \begin{bmatrix} \Phi & (1-\alpha)\alpha hU & (1-\alpha)hW & \frac{\alpha^2 h^2}{2}H & \frac{(1-\alpha^2)h^2}{2}M \\ * & -(1-\alpha)\alpha hR_3 & 0 & 0 & 0 \\ * & * & -(1-\alpha)hR_4 & 0 & 0 \\ * & * & * & -\frac{\alpha^2 h^2}{2}R_5 & 0 \\ * & * & * & * & -\frac{(1-\alpha^2)h^2}{2}R_6 \end{bmatrix} < 0, \tag{3.3}$$

$$\Phi_2 = \begin{bmatrix} \Phi & (1-\alpha)\alpha hV & (1-\alpha)hZ & \frac{\alpha^2 h^2}{2}H & \frac{(1-\alpha^2)h^2}{2}M \\ * & -(1-\alpha)\alpha hR_3 & 0 & 0 & 0 \\ * & * & -(1-\alpha)hR_4 & 0 & 0 \\ * & * & * & -\frac{\alpha^2 h^2}{2}R_5 & 0 \\ * & * & * & * & -\frac{(1-\alpha^2)h^2}{2}R_6 \end{bmatrix} < 0, \tag{3.4}$$

where

$$E = \begin{pmatrix} E_{11} & E_{12} & E_{13} & E_{14} & 0 & -P_{13} & E_{17} & P_1A + KT_1 & P_1B & P_{22} - H_1 & P_{23} - M_1 \\ * & E_{22} & E_{23} & -N_2 & 0 & 0 & 0 & 0 & 0 & -H_2 & -M_2 \\ * & * & E_{33} & -N_3 & 0 & 0 & 0 & 0 & KT_2 & -H_3 & -M_3 \\ * & * & * & E_{44} & 0 & \frac{R_4}{(1-\alpha)h} & 0 & 0 & 0 & -P_{22} + P_{23}^T & -P_{23} + P_{33} \\ * & * & * & * & -Q_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & E_{66} & 0 & 0 & 0 & -P_{23}^T & -P_{33} \\ * & * & * & * & * & * & E_{77} & P_2A + \Lambda - \Delta & P_2B & P_{12} & P_{13} \\ * & * & * & * & * & * & * & Q_6 - 2T_1 - Q & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & E_{99} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -R_1 & 0 \\ * & * & * & * & * & * & * & * & * & * & -R_2 \end{pmatrix},$$

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & V_1+\frac{R_3}{\alpha^2 h} & -P_{13}-W_1 & \Phi_{17} & P_1A+KT_1 & P_1B & P_{22}-H_1 & P_{23}-M_1 \\ * & \Phi_{22} & \Phi_{23} & -U_2+Z_2 & V_2 & -W_2 & 0 & 0 & 0 & -H_2 & -M_2 \\ * & * & \Phi_{33} & -U_3+Z_3 & V_3 & -W_3 & 0 & 0 & KT_2 & -H_3 & -M_3 \\ * & * & * & -Q_3 & 0 & 0 & 0 & 0 & 0 & -P_{22}+P_{23}^T & -P_{23}+P_{33} \\ * & * & * & * & -Q_5-\frac{R_3}{\alpha^2 h} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Q_4 & 0 & 0 & 0 & -P_{23}^T & -P_{33} \\ * & * & * & * & * & * & \Phi_{77} & P_2A+\Lambda-\Delta & P_2B & P_{12} & P_{13} \\ * & * & * & * & * & * & * & Q_6-2T_1-Q & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & \Phi_{99} & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -R_1 & 0 \\ * & * & * & * & * & * & * & * & * & * & -R_2 \end{pmatrix}$$

$$\begin{aligned}
E_{11} &= P_{12} + P_{12}^T + Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + \alpha^2 h^2 R_1 + (1-\alpha)^2 h^2 R_2 + S_1 + S_1^T - P_1 C - C P_1^T \\
&\quad + \alpha h\left(H_1 + H_1^T\right) + (1-\alpha) h\left(M_1 + M_1^T\right) + KQK, \\
E_{12} &= L_1 + S_2^T - S_1 + \alpha h H_2^T + (1-\alpha) h M_2^T, \\
E_{13} &= N_1 - L_1 + S_3^T + \alpha h H_3^T + (1-\alpha) h M_3^T, \\
E_{14} &= -P_{12} + P_{13} - N_1, \\
E_{17} &= P_{11} + K\Delta - P_1 - C P_2^T, \\
E_{22} &= -(1-\alpha u) Q_1 + L_2 + L_2^T - S_2 - S_2^T, \\
E_{23} &= -L_2 + L_3^T - S_3^T + N_2, \\
E_{33} &= -(1-u) Q_2 + N_3 + N_3^T - L_3 - L_3^T - (1-u) KQK, \\
E_{44} &= -Q_3 - \frac{R_4}{(1-\alpha)h}, \\
E_{66} &= -Q_4 - \frac{R_4}{(1-\alpha)h}, \\
E_{77} &= \alpha h R_3 + (1-\alpha) h R_4 + \frac{\alpha^2 h^2}{2} R_5 + \frac{(1-\alpha^2) h^2}{2} R_6 - P_2 - P_2^T, \\
E_{99} &= -(1-u) Q_6 - 2T_2 + (1-u) Q,
\end{aligned}$$

$$\begin{aligned}
\Phi_{11} &= P_{12} + P_{12}^T + Q_1 + Q_2 + Q_3 + Q_4 + Q_5 + \alpha^2 h^2 R_1 + (1-\alpha)^2 h^2 R_2 - P_1 C - C P_1^T \\
&\quad + \alpha h\left(H_1 + H_1^T\right) + (1-\alpha) h\left(M_1 + M_1^T\right) - \frac{R_3}{\alpha^2 h} + KQK, \\
\Phi_{12} &= U_1 - V_1 + \alpha h H_2^T + (1-\alpha) h M_2^T, \\
\Phi_{13} &= W_1 - Z_1 + \alpha h H_3^T + (1-\alpha) h M_3^T,
\end{aligned}$$

$$
\begin{aligned}
\Phi_{14} &= -P_{12} + P_{13} - U_1 + Z_1,\\
\Phi_{17} &= P_{11} + K\Delta - P_1 - CP_2^T,\\
\Phi_{22} &= -(1-\alpha u)Q_1 + U_2 + U_2^T - V_2 - V_2^T,\\
\Phi_{23} &= U_3^T - V_3^T + W_2 - Z_2,\\
\Phi_{33} &= -(1-u)Q_2 + W_3 + W_3^T - Z_3 - Z_3^T - (1-u)KQK,\\
\Phi_{77} &= \alpha h R_3 + (1-\alpha)hR_4 + \frac{\alpha^2 h^2}{2}R_5 + \frac{(1-\alpha^2)h^2}{2}R_6 - P_2 - P_2^T,\\
\Phi_{99} &= -(1-u)Q_6 - 2T_2 + (1-u)Q,\\
N &= \begin{pmatrix} N_1^T & N_2^T & N_3^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T,\\
L &= \begin{pmatrix} L_1^T & L_2^T & L_3^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T,\\
S &= \begin{pmatrix} S_1^T & S_2^T & S_3^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T,\\
H &= \begin{pmatrix} H_1^T & H_2^T & H_3^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T,\\
M &= \begin{pmatrix} M_1^T & M_2^T & M_3^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T,\\
U &= \begin{pmatrix} U_1^T & U_2^T & V_3^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T,\\
V &= \begin{pmatrix} V_1^T & V_2^T & V_3^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T,\\
W &= \begin{pmatrix} W_1^T & W_2^T & W_3^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T,\\
Z &= \begin{pmatrix} Z_1^T & Z_2^T & Z_3^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T.
\end{aligned}
\tag{3.5}
$$

Proof. Construct a new class of Lyapunov functional candidate as follows:

$$
V(z(t)) = \sum_{i=1}^{6} V_i(z(t)), \tag{3.6}
$$

where

$$
\begin{aligned}
V_1(z(t)) &= \xi^T(t)P\xi(t) + 2\sum_{i=1}^{n}\lambda_i \int_0^{z_i(t)} f_i(s)ds + 2\sum_{i=1}^{n}\delta_i \int_0^{z_i(t)} \left(k_i s - f_i(s)\right)ds,\\
V_2(z(t)) &= \int_{t-\alpha\tau(t)}^{t} z^T(s)Q_1 z(s)ds + \int_{t-\tau(t)}^{t} z^T(s)Q_2 z(s)ds + \int_{t-\alpha h}^{t} z^T(s)Q_3 z(s)ds\\
&\quad + \int_{t-h}^{t} z^T(s)Q_4 z(s)ds + \int_{t-\alpha^2 h}^{t} z^T(s)Q_5 z(s)ds + \int_{t-\tau(t)}^{t} f^T(z(s))Q_6 f(z(s))ds,
\end{aligned}
$$

$$
\begin{aligned}
V_3(z(t)) &= \alpha h \int_{-\alpha h}^{0} \int_{t+\theta}^{t} z^T(s) R_1 z(s) ds\, d\theta + (1-\alpha) h \int_{-h}^{-\alpha h} \int_{t+\theta}^{t} z^T(s) R_2 z(s) ds\, d\theta, \\
V_4(z(t)) &= \int_{-\alpha h}^{0} \int_{t+\theta}^{t} \dot{z}^T(s) R_3 \dot{z}(s) ds\, d\theta + \int_{-h}^{-\alpha h} \int_{t+\theta}^{t} \dot{z}^T(s) R_4 \dot{z}(s) ds\, d\theta, \\
V_5(z(t)) &= \int_{-\alpha h}^{0} \int_{\theta}^{0} \int_{t+\lambda}^{t} \dot{z}^T(s) R_5 \dot{z}(s) ds\, d\lambda\, d\theta + \int_{-h}^{-\alpha h} \int_{\theta}^{0} \int_{t+\lambda}^{t} \dot{z}^T(s) R_6 \dot{z}(s) ds\, d\lambda\, d\theta, \\
V_6(z(t)) &= \int_{t-\tau(t)}^{t} \left[z^T(s) KQKz(s) - f^T(z(s)) Q f(z(s)) \right] ds,
\end{aligned}
\tag{3.7}
$$

where

$$
\xi^T(t) = \left[z^T(t) \int_{t-\alpha h}^{t} z^T(s) ds \int_{t-h}^{t-\alpha h} z^T(s) ds \right], \quad 0 < \alpha < 1. \tag{3.8}
$$

□

Remark 3.2. Since the term $2\sum_{i=1}^{n} \delta_i \int_0^{z_i(t)} (k_i s - f_i(s)) ds$ in $V_1(z(t))$ and $V_6(z(t)) = \int_{t-\tau(t)}^{t} [z^T(s) KQKz(s) - f^T(z(s)) Q f(z(s))] ds$ is taken into account, it is clear that the Lyapunov functional candidate in this paper is more general than that in [5, 6, 8, 9]. So the stability criteria in this paper may be more applicable.

The time derivative of $V(z(t))$ along the trajectory of system (2.3) is given by

$$
\dot{V}(z(t)) = \sum_{i=1}^{6} \dot{V}_i(z(t)), \tag{3.9}
$$

where

$$
\begin{aligned}
\dot{V}_1(z(t)) &= 2\xi^T(t) P \begin{bmatrix} \dot{z}(t) \\ z(t) - z(t-\alpha h) \\ z(t-\alpha h) - z(t-h) \end{bmatrix} + 2 f^T(z(t)) \Lambda \dot{z}(t) + 2(Kz(t)) - f(z(t))^T \Delta \dot{z}(t), \\
\dot{V}_2(z(t)) &= z^T(t)(Q_1 + Q_2 + Q_3 + Q_4 + Q_5) z(t) + f^T(z(t)) Q_6 f(z(t)) \\
&\quad - (1-\alpha u) z^T(t - \alpha\tau(t)) Q_1 z(t - \alpha\tau(t)) - (1-u) z^T(t-\tau(t)) Q_2 z(t-\tau(t)) \\
&\quad - z^T(t-\alpha h) Q_3 z(t-\alpha h) - z^T(t-h) Q_4 z(t-h) - z^T\left(t - \alpha^2 h\right) Q_5 z\left(t - \alpha^2 h\right) \\
&\quad - (1-u) f^T(z(t-\tau(t))) Q_6 f(z(t-\tau(t))).
\end{aligned}
\tag{3.10}
$$

Using Lemma 2.1, one can obtain that

$$\begin{aligned}
\dot{V}_3(z(t)) &= \alpha^2 h^2 z^T(t) R_1 z(t) + (1-\alpha)^2 h^2 z^T(t) R_2 z(t) - \alpha h \int_{t-\alpha h}^{t} z^T(s) R_1 z(s) ds \\
&\quad - (1-\alpha) h \int_{t-h}^{t-\alpha h} z^T(s) R_2 z(s) ds \\
&\leqslant \alpha^2 h^2 z^T(t) R_1 z(t) + (1-\alpha)^2 h^2 z^T(t) R_2 z(t) \\
&\quad - \left(\int_{t-\alpha h}^{t} z(s) ds\right)^T R_1 \left(\int_{t-\alpha h}^{t} z(s) ds\right) - \left(\int_{t-h}^{t-\alpha h} z(s) ds\right)^T R_2 \left(\int_{t-h}^{t-\alpha h} z(s) ds\right), \\
\dot{V}_4(z(t)) &= \dot{z}^T(t)(\alpha h R_3 + (1-\alpha) h R_4)\dot{z}(t) - \int_{t-\alpha h}^{t} \dot{z}^T(s) R_3 \dot{z}(s) ds - \int_{t-h}^{t-\alpha h} \dot{z}^T(s) R_4 \dot{z}(s) ds, \\
\dot{V}_5(z(t)) &= \dot{z}^T(t)\left(\frac{\alpha^2 h^2}{2} R_5 + \frac{(1-\alpha^2) h^2}{2} R_6\right)\dot{z}(t) - \int_{-\alpha h}^{0} \int_{t+\theta}^{t} \dot{z}^T(s) R_5 \dot{z}(s) ds\, d\theta \\
&\quad - \int_{-h}^{-\alpha h} \int_{t+\theta}^{t} \dot{z}^T(s) R_6 \dot{z}(s) ds\, d\theta, \\
\dot{V}_6(z(t)) &\leqslant z^T(t) K Q K z(t) - f^T(z(t)) Q f(z(t)) - (1-u) z^T(t-\tau(t)) K Q K z(t-\tau(t)) \\
&\quad + (1-u) f^T(z(t-\tau(t))) Q f(z(t-\tau(t))).
\end{aligned} \tag{3.11}$$

(1) For the case of $0 \leqslant \tau(t) \leqslant \alpha h$, then it gets

$$\begin{aligned}
-\int_{t-\alpha h}^{t} \dot{z}^T(s) R_3 \dot{z}(s) ds &= -\int_{t-\alpha h}^{t-\tau(t)} \dot{z}^T(s) R_3 \dot{z}(s) ds - \int_{t-\tau(t)}^{t-\alpha\tau(t)} \dot{z}^T(s) R_3 \dot{z}(s) ds \\
&\quad - \int_{t-\alpha\tau(t)}^{t} \dot{z}^T(s) R_3 \dot{z}(s) ds.
\end{aligned} \tag{3.12}$$

Similar to [8], the following equalities hold:

$$2\zeta^T(t) N \left[z(t-\tau(t)) - z(t-\alpha h) - \int_{t-\alpha h}^{t-\tau(t)} \dot{z}(s) ds\right] = 0, \tag{3.13}$$

$$2\zeta^T(t) L \left[z(t-\alpha\tau(t)) - z(t-\tau(t)) - \int_{t-\tau(t)}^{t-\alpha\tau(t)} \dot{z}(s) ds\right] = 0, \tag{3.14}$$

$$2\zeta^T(t) S \left[z(t) - z(t-\alpha\tau(t)) - \int_{t-\alpha\tau(t)}^{t} \dot{z}(s) ds\right] = 0, \tag{3.15}$$

$$2\left[z^T(t) P_1 + \dot{z}^T(t) P_2\right]\left[-\dot{z}(t) - C z(t) + A f(z(t)) + B f(z(t-\tau(t)))\right] = 0, \tag{3.16}$$

$$2\zeta^T(t)H\left[\alpha hz(t)-\int_{t-\alpha h}^{t}z(s)ds-\int_{-\alpha h}^{0}\int_{t+\theta}^{t}\dot{z}(s)ds\,d\theta\right]=0, \tag{3.17}$$

$$2\zeta^T(t)M\left[(1-\alpha)hz(t)-\int_{t-h}^{t-\alpha h}z(s)ds-\int_{-h}^{-\alpha h}\int_{t+\theta}^{t}\dot{z}(s)ds\,d\theta\right]=0, \tag{3.18}$$

$$\begin{aligned}\zeta^T(t)=\Big[&z^T(t)\quad z^T(t-\alpha\tau(t))\quad z^T(t-\tau(t))\quad z^T(t-\alpha h)\quad z^T\left(t-\alpha^2h\right)\quad z^T(t-h)\quad \dot{z}^T(t)\\&f^T(z(t))\quad f^T(z(t-\tau(t)))\quad \int_{t-\alpha h}^{t}z^T(s)ds\quad \int_{t-h}^{t-\alpha h}z^T(s)ds\Big].\end{aligned} \tag{3.19}$$

It is easy to see that

$$-2\zeta^T(t)N\int_{t-\alpha h}^{t-\tau(t)}\dot{z}(s)ds\leqslant(\alpha h-\tau(t))\zeta^T(t)NR_3^{-1}N^T\zeta(t)+\int_{t-\alpha h}^{t-\tau(t)}\dot{z}^T(s)R_3\dot{z}(s)ds, \tag{3.20}$$

$$-2\zeta^T(t)L\int_{t-\tau(t)}^{t-\alpha\tau(t)}\dot{z}(s)ds\leqslant(1-\alpha)\tau(t)\zeta^T(t)LR_3^{-1}L^T\zeta(t)+\int_{t-\tau(t)}^{t-\alpha\tau(t)}\dot{z}^T(s)R_3\dot{z}(s)ds, \tag{3.21}$$

$$-2\zeta^TS\int_{t-\alpha\tau(t)}^{t}\dot{z}(s)ds\leqslant\alpha\tau(t)\zeta^T(t)SR_3^{-1}S^T\zeta(t)+\int_{t-\alpha\tau(t)}^{t}\dot{z}^T(s)R_3\dot{z}(s)ds, \tag{3.22}$$

$$-2\zeta^TH\int_{-\alpha h}^{0}\int_{t+\theta}^{t}\dot{z}(s)ds\,d\theta\leqslant\frac{\alpha^2h^2}{2}\zeta^T(t)HR_5^{-1}H^T\zeta(t)+\int_{-\alpha h}^{0}\int_{t+\theta}^{t}\dot{z}^T(s)R_5\dot{z}(s)ds\,d\theta, \tag{3.23}$$

$$-2\zeta^TM\int_{-h}^{-\alpha h}\int_{t+\theta}^{t}\dot{z}(s)ds\,d\theta\leqslant\frac{(1-\alpha^2)h^2}{2}\zeta^T(t)MR_6^{-1}M^T\zeta(t)+\int_{-h}^{-\alpha h}\int_{t+\theta}^{t}\dot{z}^T(s)R_6\dot{z}(s)ds\,d\theta. \tag{3.24}$$

Using Lemma 2.1, it is easy to obtain that

$$-\int_{t-h}^{t-\alpha h}\dot{z}^T(s)R_4\dot{z}(s)ds\leqslant-[z(t-\alpha h)-z(t-h)]^T\frac{R_4}{(1-\alpha)h}[z(t-\alpha h)-z(t-h)]. \tag{3.25}$$

Furthermore, there exist positive diagonal matrices T_1, T_2, such that the following inequalities hold based on (2.4):

$$-2f^T(z(t))T_1f(z(t))+2z^T(t)KT_1f(z(t))\geqslant0, \tag{3.26}$$

$$-2f^T(z(t-\tau(t)))T_2f(z(t-\tau(t)))+2z^T(t-\tau(t))KT_2f(z(t-\tau(t)))\geqslant0. \tag{3.27}$$

From (3.10)–(3.27), one can obtain that

$$\dot{V}(z(t)) \leqslant \zeta^T(t)\Sigma_1\zeta(t), \tag{3.28}$$

where

$$\begin{aligned}\Sigma_1 = {} & E + (\alpha h - \tau(t))NR_3^{-1}N^T + (1-\alpha)\tau(t)LR_3^{-1}L^T + \alpha\tau(t)SR_3^{-1}S^T \\ & + \frac{\alpha^2h^2}{2}HR_5^{-1}H^T + \frac{(1-\alpha^2)h^2}{2}MR_6^{-1}M^T.\end{aligned} \tag{3.29}$$

Note that $0 \leqslant \tau(t) \leqslant \alpha h$, $(\alpha h - \tau(t))NR_3^{-1}N^T + (1-\alpha)\tau(t)LR_3^{-1}L^T + \alpha\tau(t)SR_3^{-1}S^T$ can be seen as the convex combination of $NR_3^{-1}N^T, LR_3^{-1}L^T$, and $SR_3^{-1}S^T$ on $\tau(t)$. Therefore, $\Sigma_1 < 0$ holds if and only if

$$E + \alpha hNR_3^{-1}N^T + \frac{\alpha^2h^2}{2}HR_5^{-1}H^T + \frac{(1-\alpha^2)h^2}{2}MR_6^{-1}M^T < 0, \tag{3.30}$$

$$E + (1-\alpha)\alpha hLR_3^{-1}L^T + \alpha^2hSR_3^{-1}S^T + \frac{\alpha^2h^2}{2}HR_5^{-1}H^T + \frac{(1-\alpha^2)h^2}{2}MR_6^{-1}M^T < 0. \tag{3.31}$$

Applying the Schur complement, the inequalities (3.30) and (3.31) are equivalent to the LMI (3.1) and (3.2), respectively.

(2) When $\alpha h \leqslant \tau(t) \leqslant h$, then it gets

$$\begin{aligned}-\int_{t-\alpha h}^{t}\dot{z}^T(s)R_3\dot{z}(s)ds &= -\int_{t-\alpha h}^{t-\alpha\tau(t)}\dot{z}^T(s)R_3\dot{z}(s)ds - \int_{t-\alpha\tau(t)}^{t-\alpha^2h}\dot{z}^T(s)R_3\dot{z}(s)ds \\ &\quad - \int_{t-\alpha^2h}^{t}\dot{z}^T(s)R_3\dot{z}(s)ds, \\ -\int_{t-h}^{t-\alpha h}\dot{z}^T(s)R_4\dot{z}(s)ds &= -\int_{t-h}^{t-\tau(t)}\dot{z}^T(s)R_4\dot{z}(s)ds - \int_{t-\tau(t)}^{t-\alpha h}\dot{z}^T(s)R_4\dot{z}(s)ds.\end{aligned} \tag{3.32}$$

Similar to [8], the following equalities hold:

$$\begin{aligned}2\zeta^T(t)U\left[z(t-\alpha\tau(t)) - z(t-\alpha h) - \int_{t-\alpha h}^{t-\alpha\tau(t)}\dot{z}(s)ds\right] &= 0, \\ 2\zeta^T(t)V\left[z\left(t-\alpha^2h\right) - z(t-\alpha\tau(t)) - \int_{t-\alpha\tau(t)}^{t-\alpha^2h}\dot{z}(s)ds\right] &= 0, \\ 2\zeta^T(t)W\left[z(t-\tau(t)) - z(t-h) - \int_{t-h}^{t-\tau(t)}\dot{z}(s)ds\right] &= 0, \\ 2\zeta^T(t)Z\left[z(t-\alpha h) - z(t-\tau(t)) - \int_{t-\tau(t)}^{t-\alpha h}\dot{z}(s)ds\right] &= 0.\end{aligned} \tag{3.33}$$

It is easy to obtain that

$$\begin{aligned}
-2\zeta^T(t)U\int_{t-\alpha h}^{t-\alpha\tau(t)}\dot{z}(s)\mathrm{d}s &\leqslant \alpha(h-\tau(t))\zeta^T(t)UR_3^{-1}U^T\zeta(t)+\int_{t-\alpha h}^{t-\alpha\tau(t)}\dot{z}^T(s)R_3\dot{z}(s)ds,\\
-2\zeta^T(t)V\int_{t-\alpha\tau(t)}^{t-\alpha^2 h}\dot{z}(s)\mathrm{d}s &\leqslant \alpha(\tau(t)-\alpha h)\zeta^T(t)VR_3^{-1}V^T\zeta(t)+\int_{t-\alpha\tau(t)}^{t-\alpha^2 h}\dot{z}^T(s)R_3\dot{z}(s)ds,\\
-2\zeta^T(t)W\int_{t-h}^{t-\tau(t)}\dot{z}(s)\mathrm{d}s &\leqslant (h-\tau(t))\zeta^T(t)WR_4^{-1}W^T\zeta(t)+\int_{t-h}^{t-\tau(t)}\dot{z}^T(s)R_4\dot{z}(s)ds,\\
-2\zeta^T(t)Z\int_{t-\tau(t)}^{t-\alpha h}\dot{z}(s)\mathrm{d}s &\leqslant (\tau(t)-\alpha h)\zeta^T(t)ZR_4^{-1}Z^T\zeta(t)+\int_{t-\tau(t)}^{t-\alpha h}\dot{z}^T(s)R_4\dot{z}(s)ds.
\end{aligned} \tag{3.34}$$

Using Lemma 2.1, one can obtain that

$$-\int_{t-\alpha^2 h}^{t}\dot{z}^T(s)R_4\dot{z}(s)\mathrm{d}s \leqslant -\frac{1}{\alpha^2 h}\left[z(t)-z\left(t-\alpha^2 h\right)\right]^T R_3\left[z(t)-z\left(t-\alpha^2 h\right)\right]. \tag{3.35}$$

From (3.10)–(3.11), (3)–(3.18), (3.23), (3.24), (3.26), (3.27) and (3.33)–(3.35), one can obtain that

$$\dot{V}(z(t)) \leqslant \zeta^T(t)\Sigma_2\zeta(t), \tag{3.36}$$

where

$$\begin{aligned}
\Sigma_2 = \Phi &+ \alpha(h-\tau(t))UR_3^{-1}U^T+\alpha(\tau(t)-\alpha h)VR_3^{-1}V^T+(h-\tau(t))WR_4^{-1}W^T\\
&+(\tau(t)-\alpha h)ZR_4^{-1}Z^T+\frac{\alpha^2h^2}{2}HR_5^{-1}H^T+\frac{(1-\alpha^2)h^2}{2}MR_6^{-1}M^T.
\end{aligned} \tag{3.37}$$

Note that $\alpha h \leqslant \tau(t) \leqslant h$, $\alpha(h-\tau(t))UR_3^{-1}U^T+\alpha(\tau(t)-\alpha h)VR_3^{-1}V^T+(h-\tau(t))WR_4^{-1}W^T+(\tau(t)-\alpha h)ZR_4^{-1}Z^T$ can be seen as the convex combination of $UR_3^{-1}U^T, VR_3^{-1}V^T, WR_4^{-1}W^T$, and $ZR_4^{-1}Z^T$ on $\tau(t)$. Therefore, $\Sigma_2 < 0$ holds if and only if

$$\Phi+\alpha(1-\alpha)hUR_3^{-1}U^T+(1-\alpha)hWR_4^{-1}W^T+\frac{\alpha^2h^2}{2}HR_5^{-1}H^T+\frac{(1-\alpha^2)h^2}{2}MR_6^{-1}M^T<0, \tag{3.38}$$

$$\Phi+\alpha(1-\alpha)hVR_3^{-1}V^T+(1-\alpha)hZR_4^{-1}Z^T+\frac{\alpha^2h^2}{2}HR_5^{-1}H^T+\frac{(1-\alpha^2)h^2}{2}MR_6^{-1}M^T<0. \tag{3.39}$$

Applying the Schur complement, the inequalities (3.38) and (3.39) are equivalent to the LMI (3.3) and (3.4), respectively. Therefore, if the LMIs (3.1)–(3.4) are satisfied, then the system (2.3) is guaranteed to be asymptotically stable for $0 \leqslant \tau(t) \leqslant h$.

Remark 3.3. It is well known that the delay-dividing approach can reduce the conservatism notably. But some previous literature only uses single method to divide the delay interval $[0,h]$. Unlike [10, 25], the new Lyapunov functional in our paper which not only divides the delay interval $[0,h]$ into two ones $[0,\alpha h]$ and $[\alpha h,h]$ but also divides the delay interval $[0,h]$ into three ones $[0,\alpha\tau(t)]$, $[\alpha\tau(t),\tau(t)]$, and $[\tau(t),h]$ is proposed. Each segment has a different positive matrix, which has the potential to yield less conservative results.

Remark 3.4. In this paper, by taking the states $z(t-\alpha h), z(t-\tau(t)), z(t-\alpha^2 h), z(t-h)$, and $z(t-\alpha\tau(t))$ as augmented variables, the stability in Theorem 3.1 utilizes more information on state variables. And in deriving upper bounds of integral terms in $\dot{V}_4(z(t))$, different free-weighting matrices are introduced in two different intervals $0 \leqslant \tau(t) \leqslant \alpha h$ and $\alpha h \leqslant \tau(t) \leqslant h$. These methods mentioned above may lead to obtain an improved feasible region for delay-dependent stability criteria.

Remark 3.5. In (3.28), $E + (\alpha h - \tau(t))NR_3^{-1}N^T + (1-\alpha)\tau(t)LR_3^{-1}L^T + \alpha\tau(t)SR_3^{-1}S^T + (\alpha^2h^2/2)HR_5^{-1}H^T + ((1-\alpha^2)h^2/2)MR_6^{-1}M^T$ is not simply guaranteed by $E + \alpha hNR_3^{-1}N^T + (1-\alpha)\alpha hLR_3^{-1}L^T + \alpha^2 hSR_3^{-1}S^T + (\alpha^2h^2/2)HR_5^{-1}H^T + ((1-\alpha^2)h^2/2)MR_6^{-1}M^T$ but is evaluated by the LMIs (3.1) and (3.2), which can help reduce much more conservatism than the results in [8].

Remark 3.6. In many cases, u is unknown. For this situation, a rate-independent criterion for a delay satisfying $0 \le \tau(t) \le h$ is derived as follows by setting $Q_1 = 0$, $Q_2 = 0$, $Q_6 = 0$, and $Q = 0$ in the proof of Theorem 3.1.

Corollary 3.7. *For given scalars $K = \mathrm{diag}(k_1, k_2, \ldots, k_n)$, $h \geqslant 0$, and $0 < \alpha < 1$, the system* (2.3) *is globally asymptotically stable if there exist symmetric positive matrices $P = [P_{ij}]_{3\times 3}$, Q_3, Q_4, Q_5, R_i $(i = 1,2,\ldots,6)$, positive diagonal matrices T_1, T_2, $\Lambda = \mathrm{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$, $\Delta = \mathrm{diag}(\delta_1,\delta_2,\ldots,\delta_n)$, and any matrices P_1, P_2, N_i, M_i, L_i, H_i, Z_i, S_i, U_i, V_i, W_i $(i = 1,2,3)$ with appropriate dimensions, such that the following LMIs hold:*

$$\overline{E}_1 = \begin{bmatrix} \overline{E} & \alpha hN & \dfrac{\alpha^2h^2}{2}H & \dfrac{(1-\alpha^2)h^2}{2}M \\ * & -\alpha hR_3 & 0 & 0 \\ * & * & -\dfrac{\alpha^2h^2}{2}R_5 & 0 \\ * & * & * & -\dfrac{(1-\alpha^2)h^2}{2}R_6 \end{bmatrix} < 0,$$

$$\overline{E}_2 = \begin{bmatrix} \overline{E} & (1-\alpha)\alpha hL & \alpha^2 hS & \dfrac{\alpha^2h^2}{2}H & \dfrac{(1-\alpha^2)h^2}{2}M \\ * & -(1-\alpha)\alpha hR_3 & 0 & 0 & 0 \\ * & * & -\alpha^2 hR_3 & 0 & 0 \\ * & * & * & -\dfrac{\alpha^2h^2}{2}R_5 & 0 \\ * & * & * & * & -\dfrac{(1-\alpha^2)h^2}{2}R_6 \end{bmatrix} < 0,$$

$$\overline{\Phi}_1 = \begin{bmatrix} \overline{\Phi} & (1-\alpha)\alpha h U & (1-\alpha)hW & \frac{\alpha^2 h^2}{2}H & \frac{(1-\alpha^2)h^2}{2}M \\ * & -(1-\alpha)\alpha h R_3 & 0 & 0 & 0 \\ * & * & -(1-\alpha)hR_4 & 0 & 0 \\ * & * & * & -\frac{\alpha^2 h^2}{2}R_5 & 0 \\ * & * & * & * & -\frac{(1-\alpha^2)h^2}{2}R_6 \end{bmatrix} < 0,$$

$$\overline{\Phi}_2 = \begin{bmatrix} \overline{\Phi} & (1-\alpha)\alpha h V & (1-\alpha)hZ & \frac{\alpha^2 h^2}{2}H & \frac{(1-\alpha^2)h^2}{2}M \\ * & -(1-\alpha)\alpha h R_3 & 0 & 0 & 0 \\ * & * & -(1-\alpha)hR_4 & 0 & 0 \\ * & * & * & -\frac{\alpha^2 h^2}{2}R_5 & 0 \\ * & * & * & * & -\frac{(1-\alpha^2)h^2}{2}R_6 \end{bmatrix} < 0, \tag{3.40}$$

where

$$\overline{E} = \begin{pmatrix} \overline{E}_{11} & E_{12} & E_{13} & E_{14} & 0 & -P_{13} & E_{17} & P_1A+KT_1 & P_1B & P_{22}-H_1 & P_{23}-M_1 \\ * & \overline{E}_{22} & E_{23} & -N_2 & 0 & 0 & 0 & 0 & 0 & -H_2 & -M_2 \\ * & * & \overline{E}_{33} & -N_3 & 0 & 0 & 0 & 0 & KT_2 & -H_3 & -M_3 \\ * & * & * & E_{44} & 0 & \frac{R_4}{(1-\alpha)h} & 0 & 0 & 0 & -P_{22}+P_{23}^T & -P_{23}+P_{33} \\ * & * & * & * & -Q_5 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & E_{66} & 0 & 0 & 0 & -P_{23}^T & -P_{33} \\ * & * & * & * & * & * & E_{77} & P_2A+\Lambda-\Delta & P_2B & P_{12} & P_{13} \\ * & * & * & * & * & * & * & -2T_1 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -2T_2 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -R_1 & 0 \\ * & * & * & * & * & * & * & * & * & * & -R_2 \end{pmatrix},$$

$$\overline{\Phi} = \begin{pmatrix} \overline{\Phi}_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & V_1+\frac{R_3}{\alpha^2 h} & -P_{13}-W_1 & \Phi_{17} & P_1A+KT_1 & P_1B & P_{22}-H_1 & P_{23}-M_1 \\ * & \overline{\Phi}_{22} & \Phi_{23} & -U_2+Z_2 & V_2 & -W_2 & 0 & 0 & 0 & -H_2 & -M_2 \\ * & * & \overline{\Phi}_{33} & -U_3+Z_3 & V_3 & -W_3 & 0 & 0 & KT_2 & -H_3 & -M_3 \\ * & * & * & -Q_3 & 0 & 0 & 0 & 0 & 0 & -P_{22}+P_{23}^T & -P_{23}+P_{33} \\ * & * & * & * & -Q_5-\frac{R_3}{\alpha^2 h} & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & * & -Q_4 & 0 & 0 & 0 & -P_{23}^T & -P_{33} \\ * & * & * & * & * & * & \Phi_{77} & P_2A+\Lambda-\Delta & P_2B & P_{12} & P_{13} \\ * & * & * & * & * & * & * & -2T_1 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & * & -2T_2 & 0 & 0 \\ * & * & * & * & * & * & * & * & * & -R_1 & 0 \\ * & * & * & * & * & * & * & * & * & * & -R_2 \end{pmatrix}$$

$$\begin{aligned}
\overline{E}_{11} &= P_{12} + P_{12}^T + Q_3 + Q_4 + Q_5 + \alpha^2 h^2 R_1 + (1-\alpha)^2 h^2 R_2 + S_1 + S_1^T - P_1 C - C P_1^T \\
&\quad + \alpha h\left(H_1 + H_1^T\right) + (1-\alpha) h\left(M_1 + M_1^T\right), \\
\overline{E}_{22} &= L_2 + L_2^T - S_2 - S_2^T, \\
\overline{E}_{33} &= N_3 + N_3^T - L_3 - L_3^T, \\
\overline{\Phi}_{11} &= P_{12} + P_{12}^T + Q_3 + Q_4 + Q_5 + \alpha^2 h^2 R_1 + (1-\alpha)^2 h^2 R_2 - P_1 C - C P_1^T \\
&\quad + \alpha h\left(H_1 + H_1^T\right) + (1-\alpha) h\left(M_1 + M_1^T\right) - \frac{R_3}{\alpha^2 h}, \\
\overline{\Phi}_{22} &= U_2 + U_2^T - V_2 - V_2^T, \\
\overline{\Phi}_{33} &= W_3 + W_3^T - Z_3 - Z_3^T.
\end{aligned} \tag{3.41}$$

The other E_{ij}, Φ_{ij} are defined in Theorem 3.1.

4. Numerical Examples

Example 4.1. Consider the stability of system (2.3) with time-varying delay and

$$\begin{gathered}
C = \operatorname{diag}(2,2), \qquad A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.88 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \sigma_1 = 0.4, \\
\sigma_2 = 0.8, \qquad \gamma_1 = 0, \qquad \gamma_2 = 0.
\end{gathered} \tag{4.1}$$

Our purpose is to estimate the allowable upper bounds delay h under different u such that the system (2.3) is globally asymptotically stable. According to Table 1, this example shows that the stability criterion in this paper gives much less conservative results than those in the literature. By using the Matlab LMI toolbox, we solve LMIs (3.1)–(3.4) for the case $\alpha = 0.4$, $u = 0.8$, and $h = 2.9144$ and obtain

$$P_{11} = \begin{bmatrix} 0.0005 & -0.0002 \\ -0.0002 & 0.0018 \end{bmatrix}, \qquad P_{12} = 1.0e^{-003} \times \begin{bmatrix} -0.0363 & 0.0081 \\ 0.1663 & -0.1945 \end{bmatrix},$$

$$P_{13} = 1.0e^{-006} \times \begin{bmatrix} -0.1002 & 0.0252 \\ 0.3993 & -0.2498 \end{bmatrix}, \qquad P_{22} = 1.0e^{-004} \times \begin{bmatrix} 0.1678 & -0.2152 \\ -0.2152 & 0.5047 \end{bmatrix},$$

$$P_{23} = 1.0e^{-006} \times \begin{bmatrix} 0.0221 & -0.0836 \\ 0.0468 & -0.3688 \end{bmatrix}, \qquad P_{33} = 1.0e^{-006} \times \begin{bmatrix} 0.2614 & -0.2146 \\ -0.2146 & 0.5418 \end{bmatrix},$$

$$Q_1 = 1.0e^{-006} \times \begin{bmatrix} 0.2368 & -0.2454 \\ -0.2454 & 0.4008 \end{bmatrix}, \qquad Q_2 = 1.0e^{-005} \times \begin{bmatrix} 0.0174 & -0.0087 \\ -0.0087 & 0.1976 \end{bmatrix},$$

$$Q_3 = 1.0e^{-003} \times \begin{bmatrix} 0.0526 & 0.0296 \\ 0.0296 & 0.2541 \end{bmatrix}, \qquad Q_4 = 1.0e^{-004} \times \begin{bmatrix} 0.9971 & 0.5110 \\ 0.5110 & 0.4639 \end{bmatrix},$$

$$Q_5 = 1.0e^{-005} \times \begin{bmatrix} 0.0232 & -0.0561 \\ -0.0561 & 0.1584 \end{bmatrix}, \qquad Q_6 = \begin{bmatrix} 0.0021 & 0.0035 \\ 0.0035 & 0.0057 \end{bmatrix},$$

$$R_1 = 1.0e^{-004} \times \begin{bmatrix} 0.4523 & -0.6020 \\ -0.6020 & 0.8760 \end{bmatrix}, \qquad R_2 = 1.0e^{-006} \times \begin{bmatrix} 0.1049 & -0.2159 \\ -0.2159 & 0.5299 \end{bmatrix},$$

$$R_3 = 1.0e^{-004} \times \begin{bmatrix} 0.7915 & 0.3650 \\ 0.3650 & 0.4110 \end{bmatrix}, \qquad R_4 = 1.0e^{-004} \times \begin{bmatrix} 0.9709 & 0.5071 \\ 0.5071 & 0.4212 \end{bmatrix},$$

$$R_5 = 1.0e^{-003} \times \begin{bmatrix} 0.1083 & -0.1872 \\ -0.1872 & 0.3776 \end{bmatrix}, \qquad R_6 = 1.0e^{-006} \times \begin{bmatrix} 0.0698 & -0.1266 \\ -0.1266 & 0.2910 \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} 0.0000 & 0 \\ 0 & 0.0014 \end{bmatrix}, \qquad \Delta = 1.0e^{-003} \times \begin{bmatrix} 0.1686 & 0 \\ 0 & 0.0027 \end{bmatrix},$$

$$T_1 = \begin{bmatrix} 0.0037 & 0 \\ 0 & 0.0039 \end{bmatrix}, \qquad T_2 = \begin{bmatrix} 0.0009 & 0 \\ 0 & 0.0010 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0.0000 & 0 \\ 0 & 0.0038 \end{bmatrix}. \tag{4.2}$$

Therefore, it follows from Theorem 3.1 that the system (2.3) with given parameters is globally asymptotically stable.

Example 4.2. Consider the stability of system (2.3) with time-varying delay and

$$\begin{gathered} C = \operatorname{diag}(1.5, 0.7), \qquad A = \begin{bmatrix} 0.0503 & 0.0454 \\ 0.0987 & 0.2075 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.2381 & 0.9320 \\ 0.0388 & 0.5062 \end{bmatrix}, \\ \sigma_1 = 0.3, \qquad \sigma_2 = 0.8, \qquad \gamma_1 = 0, \qquad \gamma_2 = 0. \end{gathered} \tag{4.3}$$

Table 2 gives the comparison results on the maximum delay bound allowed via the methods in recent paper and our new study. According to Table 2, this example shows that

Table 1: Allowable upper bound of h for different u (Example 4.1).

u	$u = 0.8$	$u = 0.9$	Unknown u
[4, 5]	1.2281	0.8636	0.8298
[6]	1.6831	1.1493	1.0880
[7]	1.7347	1.1662	—
[8]	2.3534	1.6050	1.5103
[13] ($\rho = 0.8$)	2.5406	1.7273	—
[14] ($m = 2$)	2.2495	1.5966	1.4902
[15] ($m = 2$)	2.1150	1.4286	1.3126
[15] ($m = 3$)	2.3838	1.6229	1.4740
This paper ($\alpha = 0.4$)	2.9144	1.9095	1.7437
This paper ($\alpha = 0.5$)	2.9063	1.9443	1.7769

Table 2: Allowable upper bound of h for different u (Example 4.2).

u	$u = 0.4$	$u = 0.45$	$u = 0.5$	$u = 0.55$
[6]	3.9972	3.2760	3.0594	2.9814
[9]	4.3814	3.6008	3.3377	3.2350
This paper ($\alpha = 0.4$)	4.7444	3.9482	3.7000	3.5667
This paper ($\alpha = 0.5$)	4.6587	3.9069	3.6496	3.5276

the stability criterion in this paper can lead to less conservative results. By using the Matlab LMI toolbox, we solve LMIs (3.1)–(3.4) for the case $\alpha = 0.4$, $u = 0.4$, and $h = 4.7444$ and obtain

$$P_{11} = \begin{bmatrix} 1.0206 & -0.7528 \\ -0.7528 & 1.2378 \end{bmatrix}, \qquad P_{12} = 1.0e^{-003} \times \begin{bmatrix} -0.0045 & 0.0417 \\ -0.0652 & -0.0137 \end{bmatrix},$$

$$P_{13} = 1.0e^{-003} \times \begin{bmatrix} -0.2467 & 0.6404 \\ -0.1839 & -0.3921 \end{bmatrix}, \qquad P_{22} = \begin{bmatrix} 0.0073 & -0.0020 \\ -0.0020 & 0.0022 \end{bmatrix},$$

$$P_{23} = 1.0e^{-004} \times \begin{bmatrix} -0.0086 & 0.4616 \\ -0.4108 & 0.1085 \end{bmatrix}, \qquad P_{33} = 1.0e^{-003} \times \begin{bmatrix} 0.1904 & -0.1292 \\ -0.1292 & 0.1059 \end{bmatrix},$$

$$Q_1 = \begin{bmatrix} 0.0013 & -0.0012 \\ -0.0012 & 0.0012 \end{bmatrix}, \qquad Q_2 = \begin{bmatrix} 0.0258 & 0.0566 \\ 0.0566 & 0.1268 \end{bmatrix}, \qquad Q_3 = \begin{bmatrix} 0.0653 & 0.0051 \\ 0.0051 & 0.0050 \end{bmatrix},$$

$$Q_4 = \begin{bmatrix} 0.0114 & -0.0300 \\ -0.0300 & 0.0839 \end{bmatrix}, \qquad Q_5 = \begin{bmatrix} 0.0014 & -0.0009 \\ -0.0009 & 0.0006 \end{bmatrix}, \qquad Q_6 = \begin{bmatrix} 0.0340 & 0.0300 \\ 0.0300 & 0.0310 \end{bmatrix},$$

$$R_1 = \begin{bmatrix} 0.0090 & -0.0066 \\ -0.0066 & 0.0062 \end{bmatrix}, \qquad R_2 = 1.0e^{-003} \times \begin{bmatrix} 0.1943 & -0.1361 \\ -0.1361 & 0.1036 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 0.2485 & -0.3846 \\ -0.3846 & 0.7348 \end{bmatrix}, \qquad R_4 = \begin{bmatrix} 0.0802 & -0.2197 \\ -0.2197 & 0.6122 \end{bmatrix},$$

$$R_5 = \begin{bmatrix} 0.1553 & -0.0821 \\ -0.0821 & 0.0709 \end{bmatrix}, \qquad R_6 = 1.0e^{-003} \times \begin{bmatrix} 0.7651 & -0.4906 \\ -0.4906 & 0.5825 \end{bmatrix},$$

$$\Lambda = \begin{bmatrix} 0.0318 & 0 \\ 0 & 0.0344 \end{bmatrix}, \qquad \Delta = 1.0e^{-003} \times \begin{bmatrix} 0.0797 & 0 \\ 0 & 0.3593 \end{bmatrix},$$

$$T_1 = \begin{bmatrix} 0.0136 & 0 \\ 0 & 0.1968 \end{bmatrix}, \qquad T_2 = \begin{bmatrix} 0.3056 & 0 \\ 0 & 0.3423 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0.4567 & 0 \\ 0 & 0.0141 \end{bmatrix}. \tag{4.4}$$

Therefore, it follows from Theorem 3.1 that the system (2.3) with given parameters is globally asymptotically stable.

5. Conclusions

In this paper, a new delay-dependent asymptotic stability criterion for neural networks with time-varying delay has been proposed. A new class of Lyapunov functional has been introduced to derive some less conservative delay-dependent stability criteria by using the free-weighting matrices method and the technique of dealing with some integral terms. Finally, numerical examples have been given to illustrate the effectiveness of the proposed method.

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