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Research Article

Closed Relative Trajectories for Formation Flying with Single-Input Control

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We study the problem of formation shape control under the constraints on the thrust direction. Formations composed of small satellites are usually subject to serious limitations for power consumption, mass, and volume of the attitude and orbit control system (AOCS). If the purpose of the formation flying mission does not require precise tracking of a given relative trajectory, AOCS of satellites may be substantially simplified; however, the capacity of AOCS to ensure a bounded or even periodic relative motion has to be studied first. We consider a formation of two satellites; the deputy one is equipped with a passive attitude control system that provides one-axis stabilization and a propulsion system that consists of one or two thrusters oriented along the stabilized axis. The relative motion of the satellites is modeled by the Schweighart-Sedwick linear equations taking into account the effect of J_2 perturbations. We prove that both in the case of passive magnetic attitude stabilization and spin stabilization for all initial relative positions and velocities of satellites there exists a control guaranteeing their periodic relative motion.

1. Introduction

Nowadays, design of formation flying missions is one of the main directions of modern space system development. Many studies have been carried out, and a number of books on dynamics of such distributed systems have been published (see, e.g., [1, 2]).

One of the main problems to be solved in design of a formation flying mission is that of maintenance of the required spatial configuration of satellites. The straightforward approach is to correct orbits using one or several thrusters. Usually no constraints are imposed on the thrust direction. However, for a nano- or picosatellite formation subject to severe restrictions on mass, volume, and energy resources, the number of thrusters is limited, and the available control systems rarely provide three-axis orientation. Therefore, the thrust direction cannot be arbitrary changed.

Consider a two-satellite formation the aim of which is to perform measurements or observations, at several points of the orbit. Suppose that the deputy satellite is equipped with a propulsion system with its thrust axis fixed in the body of satellite. The thrust can be directed in both ways or in only one, depending on the propulsion system employed. (As the simplest example of such a system, one can suggest a cold gas thruster.) A number of simple and lightweight attitude control systems are available that can stabilize motion of the thrust axis. Thus one can formulate the problem of orbital control assuming the thrust axis orientation to be known at any moment in time. In control theory, the above control is referred to as single-input control. The principal question is whether the above AOCSs suffice to provide the required formation shape at least at some points of the orbit.

The cases of successively implemented single-input control are known since the early days of space exploration. One of them occurred by accident as a result of hull depressurization during one of the first Veneras, Soviet Venus probe missions. The spacecraft was spinning in a sun-oriented mode, and so the average jet force of the leaking air happened to be directed towards the Sun, resulting unexpectedly in the proper orbital correction [3].

Development of modern miniature satellites, such as Cubesats, motivates research on single-input control to simplify satellite control system. Applications of the single-input control concept to the problem of formation maintenance have been considered for several missions. For example, the microsatellite Magion-2 launched in 1989 was equipped with a passive one-axis magnetic attitude control system and a propulsion system with a thrust vector along the oriented axis. The aim was to keep it at 10 km distance from the chief satellite [4]; however, due to the thruster failure, formation maintenance was not possible.

Much research is focused on compensation of the relative drift of satellites caused by the J_2 harmonic of the Earth's gravitational potential. In [5] this problem is studied assuming the deputy satellite to be equipped with a passive magnetic attitude control system and two thrusters installed along the axis of the magnet. The use of solar radiation pressure to solve this problem is studied in [6] (see also [7]).

Another approach to the decoupling of the attitude and orbital control in formation is presented in [8]. The authors interpret a formation as a quasirigid body. It is shown that control of such a formation can be effectively separated into a control torque that maintains the attitude and control forces that maintain the rigidity of a formation. The respective control strategy is based on the Lyapunov controller synthesis [9].

In this paper, we analyze the general problem of compensation of J_2 perturbations for the deputy satellite in two-satellite formation. The chief satellite is assumed to move passively. We study the Schweighart-Sedwick linear equations [10], that is, the modification of the Hill-Clohessy-Wiltshire equations of relative motion. This modification well describes the effect of J_2 perturbations and has been successfully used to study many problems of relative dynamics, such as formation keeping and rendezvous (see, e.g., [11–14]).

We consider two different types of single-input control:

(1) bilateral control oriented along a vector fixed in the inertial space (the case of spin stabilization);

(2) bilateral and unilateral control oriented along the vector of local geomagnetic field (the case of passive magnetic stabilization).

We prove that for any initial conditions there exists a control that provides a periodic relative motion of chief and deputy satellites with a period T between 1 and 2 orbital periods. This means that the maximum distance between satellites does not become very large. Though the shape of relative trajectory is not controlled, the existence of bounded short-period relative motion suffices to perform the required measurements in many nano- and picosatellite formation missions.

Throughout this paper, the set of real numbers is denoted by R and the N-dimensional space of vectors with components in R by R^N. We denote by $\langle a, b\rangle$ the usual scalar product in R^N and by $\|\cdot\|$ the Euclidean norm. The transposition of a matrix A is denoted by A^T.

2. Existence and Stabilization of Closed Trajectories for a Single-Input Control System

Consider a linear single-input control system

$$\dot{\eta}(t) = A\eta(t) + a(t) + w(t)b(t), \quad \eta(t) \in R^n,\ w(t) \in R, \tag{2.1}$$

where A is a $(N \times N)$-matrix, $a : R \to R^N$ and $b : R \to R^N$ are given continuous functions, and $w(t)$ is a control. The control $w(t)$ may be subjected to the constraint

$$w(t) \geq 0. \tag{2.2}$$

The set of admissible controls $w(\cdot)$ is denoted by $\mathcal{W}$ and consists of locally integrable functions. The general solution to (2.1) is given by the Cauchy formula

$$\eta(t) = e^{tA}\eta_0 + \int_0^t e^{(t-s)A}(a(s) + w(s)b(s))ds. \tag{2.3}$$

We say that system (2.1) has a T-closed trajectory $\eta(\cdot)$ satisfying $\eta(0) = \eta_0$, if and only if there exists an admissible control $w_{\eta_0}(\cdot)$ such that

$$\eta_0 = e^{TA}\eta_0 + \int_0^T e^{(T-t)A}\left(a(t) + w_{\eta_0}(t)b(t)\right)dt. \tag{2.4}$$

Put

$$\mathcal{K}_T = \left\{\int_0^T e^{(T-t)A}w(t)b(t)dt \mid w(\cdot) \in \mathcal{W}\right\}. \tag{2.5}$$

If $w(t) \in R$, then $\mathcal{K}_T$ is a subspace. In the case $w(t) \geq 0$, the set $\mathcal{K}_T$ is a convex cone. If $\mathcal{K}_T = R^N$, then for any $\eta(0) = \eta_0$ there exists an admissible control $w_{\eta_0}(\cdot)$ satisfying (2.4). Moreover, for any initial point η_1 and any terminal point η_2 there exists an admissible control w_{η_1,η_2} such that

$$\eta_2 = e^{TA}\eta_1 + \int_0^T e^{(T-t)A}\left(a(t) + w_{\eta_1,\eta_2}(t)b(t)\right)dt, \tag{2.6}$$

that is, the system is controllable.

The established controllability allows one to correct closed trajectories, that is, if there is a deviation in the initial condition of the closed trajectory, it can be compensated for by an appropriate choice of control.

To verify the controllability condition $\mathcal{K}_T = R^N$ in the case of unconstrained control, we use the following direct consequence of the Pontryagin Maximum Principle.

Theorem 2.1. *Assume that there is no nontrivial solution of the equation*

$$\dot{p}(t) = -A^T p(t) \tag{2.7}$$

satisfying

$$\langle p(t), b(t)\rangle = 0, \qquad t \in [0,T], \tag{2.8}$$

then the equality $\mathcal{K}_T = R^N$ holds.

In the case of controls subject to constraint (2.2), the situation is more involved. Later on we consider only the τ-periodic functions $b(\cdot)$. This assumption is satisfied in all applications considered here and significantly simplifies the study.

The following two propositions are well known to the specialists in the control theory. However, to make the presentation self-contained, we include their short proofs in the Appendix.

First of all, note that the periodicity condition $b(t+\tau) = b(t)$ implies that the cones $\mathcal{K}_{M\tau}$, $M = 1, 2, \ldots$, form a monotonously increasing sequence.

Lemma 2.2. *Assume that $b(t)$ is τ-periodic. Let M be a positive integer. Then the inclusion $\mathcal{K}_{M\tau} \subset \mathcal{K}_{(M+1)\tau}$ holds.*

The next theorem is also a consequence of the Pontryagin Maximum Principle and contains sufficient conditions of controllability for system (2.1) when the control satisfies condition (2.2).

Theorem 2.3. *Assume that there is no nontrivial solution to the differential equation*

$$\dot{p}(t) = -A^T p(t) \tag{2.9}$$

satisfying

$$\langle p(t), b(t)\rangle \geq 0, \qquad t \geq 0. \tag{2.10}$$

Then the equality $\mathcal{K}_\infty = \bigcup_M \mathcal{K}_{M\tau} = R^N$ *holds.*

Since the sequence of convex cones $\mathcal{K}_{M\tau}$ is monotonous, the equality $\bigcup_{M>0} \mathcal{K}_{M\tau} = R^N$ implies the existence of a positive integer M such that

$$\mathcal{K}_{M\tau} = R^N. \tag{2.11}$$

Indeed, let points ξ_k, $k = 1, \ldots, N+1$, be the vertices of a simplex Ξ containing the origin as an interior point. Then any point $\xi \in R^N$ can be represented as $\xi = \sum_k \lambda_k \xi_k$ with $\lambda_k \geq 0$. For any $k = 1, \ldots, N+1$, there exist a positive integer M_k and an admissible control $u_k(\cdot)$ satisfying

$$\xi_k = \int_0^{M_k\tau} e^{(M\tau - t)A} b(t) u_k(t)\, dt. \tag{2.12}$$

So from Lemma 2.2 we see that any vertex ξ_k can be represented in the form

$$\xi_k = \int_0^{M\tau} e^{(M\tau - t)A} b(t) w_k(t)\, dt, \tag{2.13}$$

where $M = \max\{M_k \mid k = 1, \ldots, N+1\}$ and $w_k(\cdot)$ is an admissible control. This implies the equality

$$\xi = \int_0^{M\tau} e^{(M\tau - t)A} b(t) \sum_k \lambda_k w_k(t)\, dt, \tag{2.14}$$

arriving at (2.11).

Let $\eta_0 \in R^N$. Condition (2.11) leads to the existence of a control $w_0(\cdot)$ such that

$$\eta_0 - e^{M\tau A}\eta_0 - \int_0^{M\tau} e^{(M\tau - t)A} a(t)\, dt = \int_0^{M\tau} e^{(M\tau - t)A} b(t) w_0(t)\, dt. \tag{2.15}$$

Therefore, the control $w_0(\cdot)$ corresponds to a closed trajectory of (2.1) satisfying $\eta(0) = \eta_0$.

The above results permit one also to compensate for the errors caused by the model or measurements not requiring considerable computational efforts. Under condition (2.11) it is easy to develop an algorithm that reaches the point η_0 even if the initial point η_0' is different from η_0. Note that this algorithm does not require solving the integral equation (2.6).

Consider a simplex Σ containing η_0 in its interior. Let $\{\eta_1, \ldots, \eta_{N+1}\}$ be the vertices of Σ. Condition (2.11) implies the existence of admissible controls $w_k(\cdot)$, $k = 1, \ldots, N+1$, satisfying the equalities

$$\eta_0 - e^{M\tau A}\eta_k - \int_0^{M\tau} e^{(M\tau - t)A} a(t)dt = \int_0^{M\tau} e^{(M\tau - t)A} b(t) w_k(t)dt, \qquad k = 1, \ldots, N+1. \tag{2.16}$$

If $\eta_0' \in \Sigma$, there exist nonnegative numbers λ_k, $k = 1, \ldots, N+1$, such that

$$\eta_0' = \sum_{k=1}^{N+1} \lambda_k \eta_k, \qquad \sum_{k=1}^{N+1} \lambda_k = 1, \tag{2.17}$$

and so the control

$$w(t) = \sum_{k=1}^{N+1} \lambda_k w_k(t) \tag{2.18}$$

drives system (2.1) to the point η_0. Thus, if the controls $w_k(\cdot)$, $k = 1, \ldots, N+1$, are known, it suffices to find nonnegative numbers λ_k, $k = 1, \ldots, N+1$, satisfying (2.17) in order to reach the point η_0 from η_0'.

3. Equations of Relative Motion with Single-Input Control

To take into account the influence of the J_2-harmonic on relative motion of two satellites with close near-circular orbits, the following modification of the Hill-Clohessy-Wiltshire equations has been introduced by Schweighart and Sedwick [10]:

$$\begin{aligned} \ddot{x} + 2nc\dot{z} &= w(t)e_x(t), \\ \ddot{y} + q^2 y &= 2lq\cos(qt + \phi) + w(t)e_y(t), \\ \ddot{z} - 2nc\dot{x} - \left(5c^2 - 2\right)n^2 z &= w(t)e_z(t). \end{aligned} \tag{3.1}$$

The linearization is done with respect to the circular reference orbit with the mean motion n. Here x, y, and z are coordinates in the respective orbital reference frame $Oxyz$. The axes are chosen in the following way: Oz indicates the radial direction outwards from the Earth, Ox is directed along the velocity of the point O, and y is normal to the orbital plane. The coefficients c, q, l, and ϕ are properly defined constants (see the appendix, Proof of Lemma 3.1).

The direction of the control acceleration $w(t)$ is defined by the vector function

$$e(t) = \left(e_x(t), e_y(t), e_z(t)\right)^T. \tag{3.2}$$

Using the notations

$$\begin{gathered}
\eta = (x, y, z, \dot{x}, \dot{y}, \dot{z})^T, \\
a(t) = (0, 0, 0, 0, 2lq\cos(qt + \phi), 0)^T, \\
b(t) = (0, 0, 0, e_x(t), e_y(t), e_z(t))^T, \\
A = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -2nc \\ 0 & -q^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & (5c^2 - 2)n^2 & 2nc & 0 & 0 \end{pmatrix},
\end{gathered} \tag{3.3}$$

we obtain a system of type (2.1). System (2.7) that describes the evolution of the vector $p = (p_1, p_2, p_3, p_4, p_5, p_6)^T$ takes the form

$$\begin{aligned}
\dot{p}_1 &= 0, \\
\dot{p}_2 &= q^2 p_5, \\
\dot{p}_3 &= -\left(5c^2 - 2\right) n^2 p_6, \\
\dot{p}_4 &= -p_1 - 2ncp_6, \\
\dot{p}_5 &= -p_2, \\
\dot{p}_6 &= -p_3 + 2ncp_4.
\end{aligned} \tag{3.4}$$

Its general solution is given by

$$\begin{aligned}
p_1(t) &= p_1^0, \\
p_2(t) &= A_2 \cos(qt + \phi_2), \\
p_3(t) &= p_3^0 - A_6 \frac{(5c^2 - 2)n}{\sqrt{2 - c^2}} \sin\left(\sqrt{2 - c^2} nt + \phi_6\right) + 2nc \frac{5c^2 - 2}{2 - c^2} p_1^0 t, \\
p_4(t) &= \frac{p_3^0}{2nc} - A_6 \frac{2c}{\sqrt{2 - c^2}} \sin\left(\sqrt{2 - c^2} nt + \phi_6\right) + \frac{5c^2 - 2}{2 - c^2} p_1^0 t, \\
p_5(t) &= -\frac{A_2}{q} \sin(qt + \phi_2), \\
p_6(t) &= A_6 \cos\left(\sqrt{2 - c^2} nt + \phi_6\right) - \frac{2cp_1^0}{(2 - c^2)n},
\end{aligned} \tag{3.5}$$

where p_1^0, p_3^0, A_2, A_6, ϕ_2, and ϕ_6 are constants. Conditions (2.8) and (2.9) are equivalent to the conditions

$$p_4(t)e_x(t) + p_5(t)e_y(t) + p_6(t)e_z(t) = 0, \qquad t \in [0, T], \tag{3.6}$$

$$p_4(t)e_x(t) + p_5(t)e_y(t) + p_6(t)e_z(t) \geq 0, \qquad t \in [0, T], \tag{3.7}$$

respectively. According to Theorem 2.3, to prove the controllability of the Schweighart-Sedwick system with single-input control it suffices to show that there are no nontrivial functions $(p_1(t), \ldots, p_6(t))$ satisfying (3.6) or (3.7).

Denote the radius and the inclination of the reference circular orbit by r_{ref} and i_{ref}, respectively. Assume that the chief satellite moves passively in an orbit with inclination i_1. The orbit inclination of the deputy satellite is denoted by i_2. Set

$$\omega_0 = nc, \qquad \omega_1 = q, \qquad \omega_2 = \left(\sqrt{2-c^2} - c\right)n, \qquad \omega_3 = \left(\sqrt{2-c^2} + c\right)n. \tag{3.8}$$

The following lemma proved in the Appendix is crucial for the analysis of the Schweighart-Sedwick system controllability.

Lemma 3.1. *If* $2i_{\text{ref}} \neq \arccos(-1/3)$, *then* $\omega_j \neq 0$, $j = 0, 1, 2, 3$, *and* $\omega_2 < \omega_0 < \omega_1 < \omega_3$.

Below we assume that the main condition of this lemma is satisfied and consider two systems with single-input control relevant for practical applications.

4. Bilateral Control Oriented along the Geomagnetic Field

Consider first a formation with the deputy satellite equipped with a passive magnetic attitude control system (PMACS) and has two thrusters installed along its axis of orientation (i.e., axis of permanent magnet included in PMACS) in opposite directions. We also assume that at any moment in time this axis coincides with the direction of geomagnetic field described by the direct dipole model:

$$\begin{aligned} e_x(t) &= \frac{\cos\theta(t)\sin i_2}{\sqrt{1+3\sin^2\theta(t)\sin^2 i_2}}, \\ e_y(t) &= \frac{\cos i_2}{\sqrt{1+3\sin^2\theta(t)\sin^2 i_2}}, \\ e_z(t) &= \frac{-2\sin\theta(t)\sin i_2}{\sqrt{1+3\sin^2\theta(t)\sin^2 i_2}}. \end{aligned} \tag{4.1}$$

The argument of latitude is given by $\theta(t) = nct$.

Under some nonrestrictive conditions the system is controllable in any time interval $[0, T]$; for example, one can take $T = 2\pi/(nc)$.

Theorem 4.1. *Let* $T > 0$. *If* $\sin 2i_2 \neq 0$, *then there exists a* T*-closed trajectory of system* (2.1). *Moreover, an error in the initial conditions can be compensated for.*

The proof of this theorem can be found in the appendix.

5. Bilateral Control Oriented along a Fixed Vector in the Inertial space

Spin-stabilized satellites represent another interesting possibility for orbit correction by single-input control. Once rapidly rotated about an axis, the spacecraft keeps spinning around this direction in the inertial space in the absence of perturbing torques.

Assume that the deputy satellite possesses a spherically symmetrical mass distribution, is spin stabilized, and has two thrusters oriented in opposite directions along its spin axis fixed in the inertial space. Suppose that λ is the angle between this axis and the vector pointing to the vernal equinox direction, and ε is the inclination of the plane containing these vectors with respect to the Earth's equator. Then in the Earth-centered inertial reference frame the spin axis direction has the components $(\cos\lambda, \sin\lambda\cos\varepsilon, \sin\lambda\sin\varepsilon)^T$. In the $Oxyz$ reference frame the expressions are

$$\begin{aligned} e_x(t) &= \sigma_z \sin\theta(t) - \sigma_x \cos\theta(t), \\ e_y &= -\sigma_y, \\ e_z(t) &= -\sigma_x \sin\theta(t) - \sigma_z \cos\theta(t). \end{aligned} \tag{5.1}$$

Here the vector $\sigma = (\sigma_x, \sigma_y, \sigma_z)^T$ defines the direction of spin axis in the ascending node of the orbit via the inclination i_2 and the right ascension Ω_2:

$$\begin{aligned} \sigma_x &= \cos\Omega_2 \cos i_2 \sin\lambda \cos\varepsilon - \sin\Omega_2 \cos i_2 \cos\lambda + \sin i_2 \sin\lambda \sin\varepsilon, \\ \sigma_y &= -\cos\Omega_2 \sin i_2 \sin\lambda \cos\varepsilon + \sin\Omega_2 \sin i_2 \cos\lambda + \cos i_2 \sin\lambda \sin\varepsilon, \\ \sigma_z &= \cos\Omega_2 \cos\lambda + \sin\Omega_2 \sin\lambda \cos\varepsilon. \end{aligned} \tag{5.2}$$

We set $\theta(t) = nct$. As in the case of the satellite oriented along the local geomagnetic field, under some nonrestrictive conditions the system is controllable in any time interval $[0, T]$, for example, for $T = 2\pi/(nc)$.

Theorem 5.1. *Let $T > 0$. If $\sigma_x^2 + \sigma_z^2 \neq 0$ and $\sigma_y \neq 0$, then there exist a T-closed trajectory of system* (2.1). *Moreover, an error in the initial conditions can be compensated for.*

See the appendix for the proof.

6. Unilateral Control Oriented along the Geomagnetic Field

Now assume that the control $w(t)$ has to satisfy the nonnegativity condition (2.2). Set $\tau = 2\pi/(nc)$. In this case we have the following result.

Theorem 6.1. *If $\sin 2i_2 \neq 0$, there is a positive integer $M > 0$ and a $M\tau$-closed trajectory of system* (2.1). *Moreover, an error in the initial conditions can be compensated for.*

The theorem is proved in the Appendix.

Note that a similar result can be proved for the case of the satellite oriented along a fixed vector in the inertial space. However, this result is of quite limited practical importance. Indeed, while in the case of magnetic orientation $M = 2$ (see the numerical example in the next section), in the case of the satellite oriented along a fixed vector in the inertial space, the

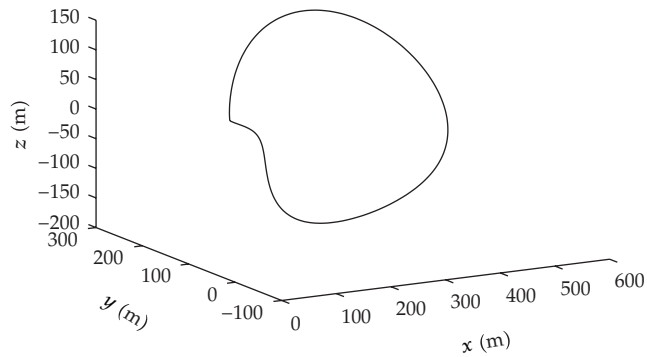

Figure 1: Closed trajectory of linearized system (LT): bilateral control with magnetic ACS.

value of M is very large, and so is the distance between the chief and deputy satellites. In this case the linearized equations cease to describe adequately the system dynamics and so the generated periodic trajectories are of merely academic interest.

7. Numerical Results

The aim of the following numerical simulations is to verify the analytical results listed above and to compare the trajectories of initial and linearized systems in the presence of the control. On solving the integral equation (2.4) numerically, we substitute the obtained control into the Gauss variational equations for the deputy satellite and propagate them in time. For the passively flying chief satellite the propagation can be done directly. Then, subtracting one motion from another, we convert the result to the $Oxyz$ reference frame. Only the J_2 perturbing effect is taken into account. Indeed, for time intervals of several orbital periods the influence of atmospheric drag and solar radiation pressure on relative motion of identical satellites in close orbits is negligible (3 to 4 orders smaller than J_2 perturbations).

Integral equation (2.4) has many solutions; we use the minimal one in the sense of L_2-norm. This criterion can be interpreted as that of minimal energy consumption for low-thrust constant-power engines (see, e.g., [15]).

Below, we compare the trajectories of the linearized Schweighart-Sedwick system (LT) and the trajectories obtained by integration of the nonlinear equations of motion (NT). Figure 1 shows a T-closed LT with the numerically obtained bilateral single-input control oriented along the geomagnetic field. This trajectory has a length $T = \tau = 2\pi(nc)^{-1}$ and corresponds to the following initial conditions: $x_0 = 70.71\,\mathrm{m}$; $y_0 = 70.71\,\mathrm{m}$; $z_0 = 35.36\,\mathrm{m}$; $\dot{x}_0 = 76.25\,\mathrm{mm/s}$; $\dot{y}_0 = 76.32\,\mathrm{mm/s}$; $\dot{z}_0 = -38.07\,\mathrm{mm/s}$. The radius of the circular reference orbit is $r_{\mathrm{ref}} = 7000\,\mathrm{km}$; the inclination of the chief satellite i_1 is the same as the reference inclination $i_{\mathrm{ref}} = 35\,\mathrm{deg}$. The projections of LT on xy and xz planes are demonstrated in Figures 2 and 3. As we see, the shape of trajectories is rather complex. Modelling errors of LT and the corresponding control are shown in Figures 4 and 5, respectively.

Figure 4 shows that the difference between the LT and NT obtained with the same control is not significant. The difference appears because the in-plane drift is not completely eliminated. It is caused by the errors of linearization in the Schweighart-Sedwick model. In the case of free flight, these errors can be compensated for by a proper choice of initial conditions, which should be done numerically (see [10]). We obtain a similar situation with

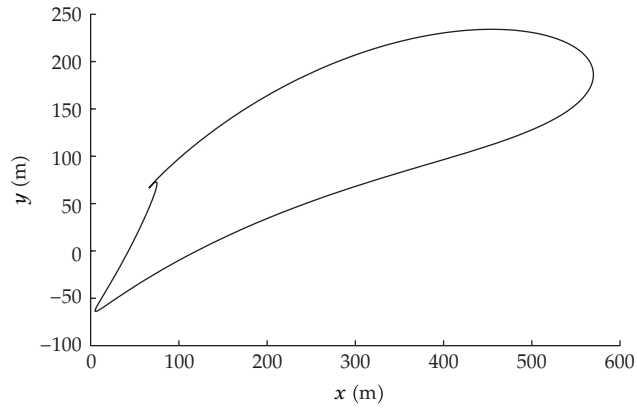

Figure 2: Projection on xy plane.

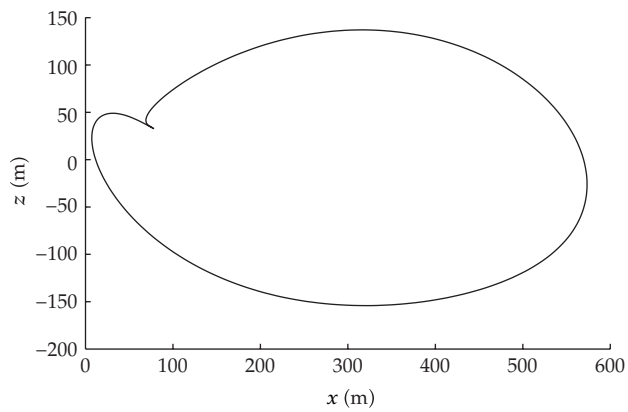

Figure 3: Projection on xz plane.

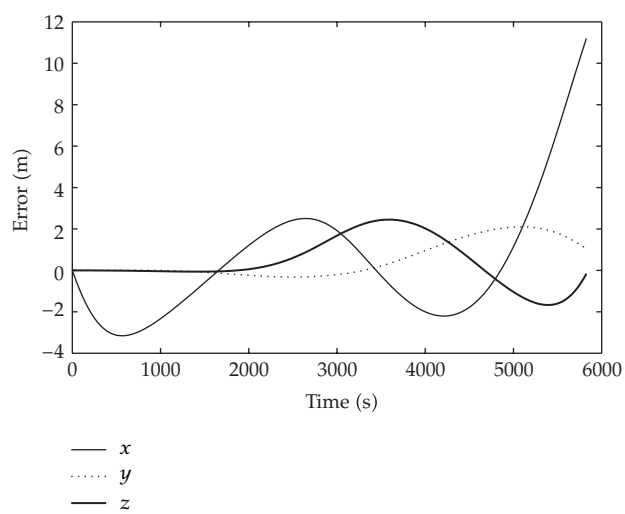

Figure 4: Coordinate-wise modelling error.

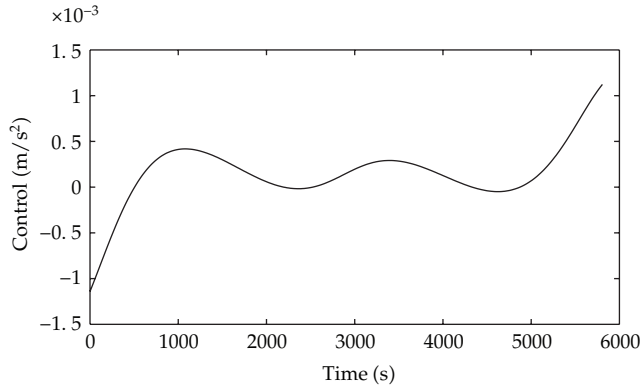

Figure 5: Control.

Table 1: Results of simplex experiment.

Number of vertex	1	2	3	4	5	6	7
x_0, m	70.71	−141.42	70.71	70.71	70.71	70.71	70.71
y_0, m	70.71	70.71	−141.42	70.71	70.71	70.71	70.71
z_0, m	35.36	35.36	35.36	−70.71	35.36	35.36	35.36
$\dot{x}_0$, mm/s	76.25	76.25	76.25	76.25	−152.51	76.25	76.25
$\dot{y}_0$, mm/s	76.32	76.32	76.32	76.32	76.32	−152.64	76.32
$\dot{z}_0$, mm/s	−38.07	−38.07	−38.07	−38.07	−38.07	−38.07	76.15
T/τ	2	2	2	2	2	2	2
$\|w\|_1$, m/s	0.16	0.16	0.14	0.18	0.08	0.15	0.16
$\|w\|_2^2$, 10^{-4} m^2/s^3	6.83	6.84	5.35	7.93	1.64	5.59	6.84

the controlled flight, but now the control has to be corrected; for example, it can be used as the first approximation in an iteration procedure (such as Newton's method) applied to the Gauss system. A Newton-type method suitable to solve control problems with nonnegativity constraints can be found in [16]. The control problem for nonlinear system is to be described in a future paper.

Now proceed with the case of unilateral single-input control oriented along the geomagnetic field. The construction described in Section 2 is fulfilled numerically. We show that it is possible to construct a 2τ-closed trajectory for all vertices of a simplex containing the origin in its interior. Therefore a 2τ-closed trajectory exists for any initial point. The results of the "simplex" experiment are summarized in Table 1. The last two rows of this table contain the values of L_1-norm

$$\|w\|_1 = \int_0^{2\tau} |w(t)| dt \tag{7.1}$$

and of squared L_2-norm

$$\|w\|_2^2 = \int_0^{2\tau} |w(t)|^2 dt, \tag{7.2}$$

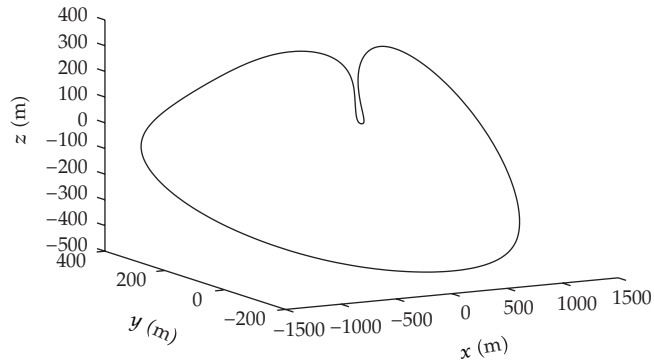

Figure 6: Closed LT: unilateral control with magnetic ACS.

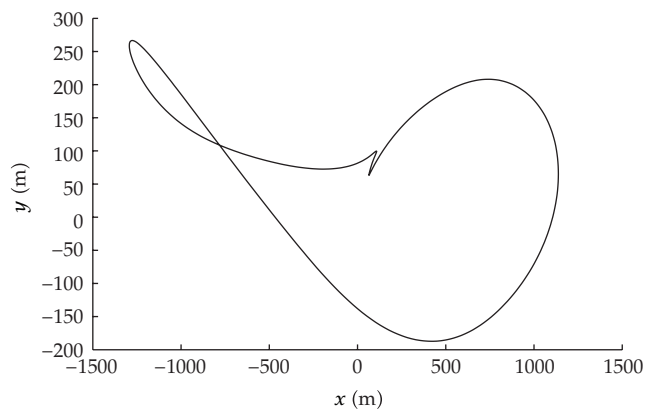

Figure 7: Projection on xy plane.

of the controls $w(t)$ providing closed trajectories for the simplex vertices $1, \ldots, 7$. 2τ-closed LT along with their projections on xy and xz planes, the coordinate-wise errors, and the corresponding nonnegative LT-control are shown in Figures 6, 7, 8, 9, 10. Since the time interval becomes twice as long, the error of linearization results in larger modelling errors due to considerable along-track drift.

For the bilateral control oriented along a fixed vector in the inertial space, the results are qualitatively similar to the case of bilateral magnetic control (see Figures 11, 12, 13). We use the same initial conditions, and the fixed vector is defined by the following angles: $\lambda = 45\,\text{deg}$, $\varepsilon = 23.45\,\text{deg}$. This choice of ε may correspond to stabilization of the spacecraft axis in the Sun direction.

8. Conclusions

We consider the problem of formation maintenance under constraints on the thrust vector directions. The formation consists of two satellites; the deputy satellite is equipped with one or two thrusters oriented along a given axis. We assume that the orientation of this axis is kept by an available passive ACS and is known at any instant of time. A possibility to obtain a periodic relative motion of the chief and deputy satellites is demonstrated for several types of

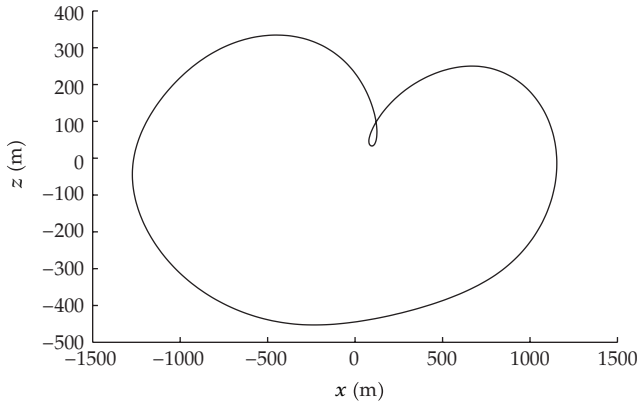

Figure 8: Projection on xz plane.

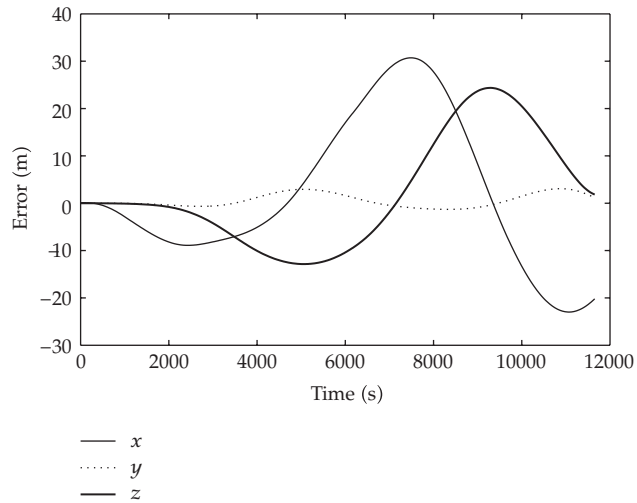

Figure 9: Coordinate-wise modeling error.

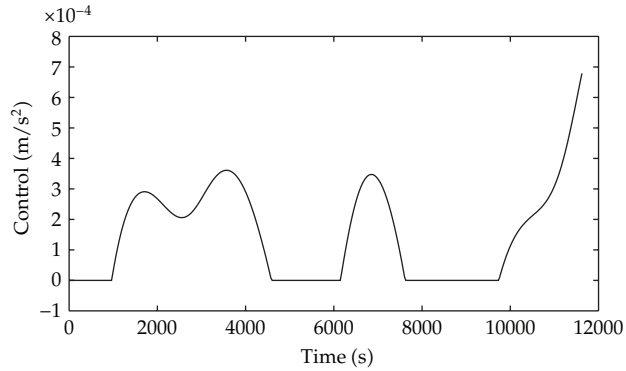

Figure 10: Control.

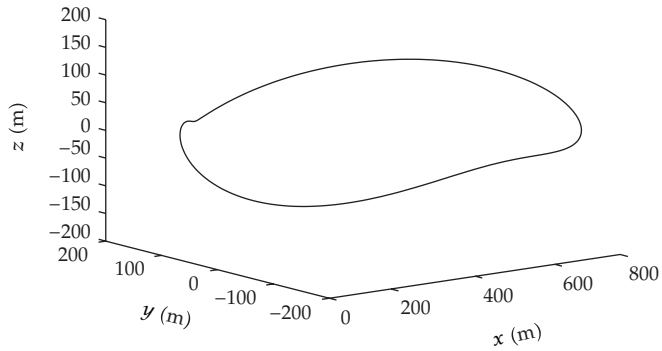

Figure 11: Closed LT: bilateral control along axis fixed in absolute space.

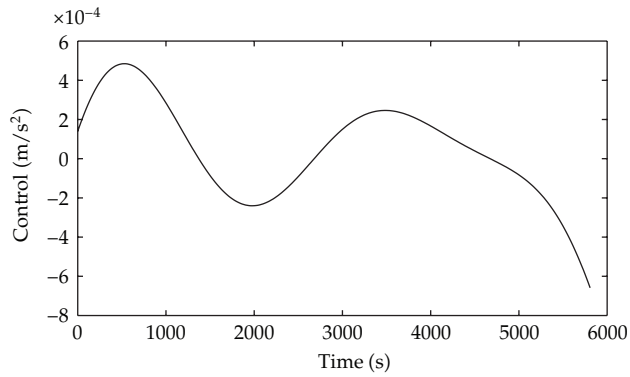

Figure 12: Control.

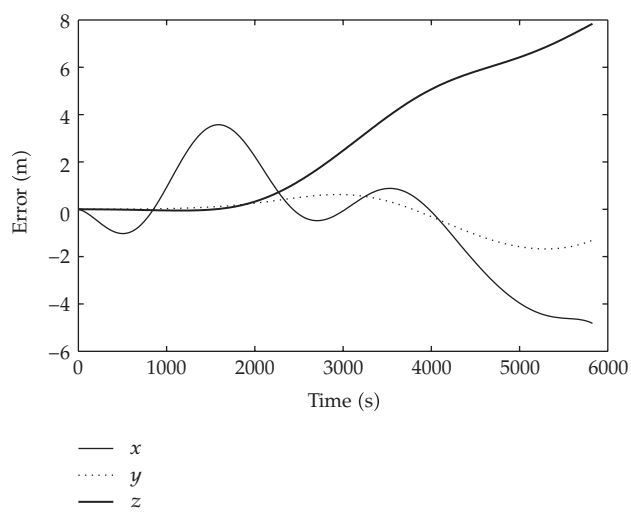

Figure 13: Coordinate-wise modelling error.

single-input control. In each case sufficient controllability conditions are deduced. In general, these conditions can be formulated as follows: the vector of control direction should have nonzero components both in the orbital plane and along the normal to the orbit. For the unilateral control oriented along the geomagnetic field, the existence of a closed trajectory of relative motion with double period is established for arbitrary initial conditions. A single-input control numerically obtained for the system of Schweighart-Sedwick equations suffices to guarantee almost closed trajectories. We also prove that the inaccuracy caused by the errors of the Schweighart-Sedwick model can be corrected.

Appendix

Proof of Lemma 2.2. Let $z \in \mathcal{K}_{M\tau}$. Then an admissible control $u(\cdot)$ exists such that

$$z = \int_0^{M\tau} e^{(M\tau - t)A} b(t) u(t) dt. \tag{A.1}$$

Set

$$w(s) = \begin{cases} 0, & s \in [0, \tau), \\ u(s-\tau), & s \in [\tau, (M+1)\tau]. \end{cases} \tag{A.2}$$

Then we have

$$\begin{aligned} z &= \int_{-\tau}^{M\tau} e^{(M\tau - t)A} b(t) w(\tau + t) dt \\ &= \int_0^{M\tau+\tau} e^{(M\tau+\tau-s)A} b(s-\tau) w(s) ds \\ &= \int_0^{(M+1)\tau} e^{((M+1)\tau - s)A} b(s) w(s) ds. \end{aligned} \tag{A.3}$$

Thus we get $\mathcal{K}_{M\tau} \subset \mathcal{K}_{(M+1)\tau}$. □

Proof of Theorem 2.3. Suppose that $\mathcal{K}_\infty \neq R^N$. From Lemma 2.2 we see that $\mathcal{K}_\infty$ is a convex cone. So there exists a vector $p_\infty \neq 0$ satisfying $\langle x, p_\infty \rangle \geq 0$, for all $x \in \mathcal{K}_\infty$. Therefore, we have $\langle x, p_\infty \rangle \geq 0$, for all $x \in \mathcal{K}_{M\tau}$ and any positive integer M. Consider the functions

$$p_M(t) = \frac{\exp\left(A^T(M\tau - t)\right) p_\infty}{\left\| \exp\left(A^T M\tau\right) p_\infty \right\|}, \quad M = 1, 2, \ldots. \tag{A.4}$$

From the Pontryagin maximum principle we have $\langle p_M(t), b(t) \rangle = 0$, $t \in [0, M\tau]$, if $w(t) \in R$, and $\langle p_M(t), b(t) \rangle \geq 0$, $t \in [0, M\tau]$, if $w(t) \geq 0$. Consider the sequence $p_M(0)$. Without loss of generality it converges to a vector p_0 satisfying $\|p_0\| = 1$. Thus we have

$$\lim_{M \to \infty} p_M(t) = p_0(t) = \exp\left(-A^T t\right) p_0, \quad t \in [0, M\tau]. \tag{A.5}$$

Obviously the solution $p_0(\cdot)$ to (2.7) is nontrivial and satisfies (2.8) if $w(t) \in R$, and (2.9) if $w(t) \geq 0$, a contradiction. □

Proof of Lemma 3.1. By definition,

$$q = nc + \frac{3nJ_2R_\oplus^2}{2r_{\text{ref}}^2}\left(\cos^2 i_2 - \frac{(\cos i_1 - \cos i_2)(\cot i_1 \sin i_2 \cos \Delta\Omega_0 - \cos i_2)}{\sin^2 \Delta\Omega_0 + (\cot i_1 \sin i_2 - \cos i_2 \cos \Delta\Omega_0)^2}\right),$$

$$\Delta\Omega_0 = \frac{y_0}{r_{\text{ref}} \sin i_{\text{ref}}}, \qquad y_0 = y(0), \qquad c = \sqrt{1+s}, \qquad s = \frac{3J_2R_\oplus^2}{8r_{\text{ref}}^2}(1 + 3\cos 2i_{\text{ref}}), \tag{A.6}$$

where $R_\oplus$ is the Earth's radius, $J_2 \approx 10^{-3}$ is the second zonal harmonic. Since the orbits of satellites are close, the difference $i_2 - i_1$ is small. Taking into account $\sin 2i_1 \neq 0$ and $\sin 2i_2 \neq 0$, we have

$$q \approx nc + \frac{3nJ_2R_\oplus^2}{2r_{\text{ref}}^2}\cos^2 i_2. \tag{A.7}$$

Since $2i_{\text{ref}} \neq \arccos(-1/3)$, one can see that $c \neq 1$. At the same time $|c - 1| \ll 1$. Thus, all the frequencies ω_j, $j = 0, 1, 2, 3$, are nonzero and pairwise different: $\omega_2 < \omega_0 < \omega_1 < \omega_3$. □

Proof of Theorem 4.1. Indeed, in this case condition (3.6) reads

$$\kappa(t) = \tilde{g}t \cos \omega_0 t + \sum_{k=0}^{3}\left(g_k \cos \omega_k t + h_k \sin \omega_k t\right) \equiv 0. \tag{A.8}$$

From Lemma 3.1 we see that all the frequencies are pairwise different, and therefore the coefficients of the quasipolynomial $\kappa(t)$ equal zero. Since

$$\begin{aligned}
\tilde{g} &= \frac{5c^2 - 2}{2 - c^2} p_1^0 \sin i_2, \\
g_0 &= \frac{p_3^0}{2nc} \sin i_2, \\
h_0 &= \frac{4cp_1^0}{(2 - c^2)n} \sin i_2, \\
g_1 &= -\frac{A_2}{q} \cos \phi_2 \cos i_2, \\
h_1 &= -\frac{A_2}{q} \sin \phi_2 \cos i_2,
\end{aligned}$$

$$
\begin{aligned}
g_2 &= A_6\left(1-\frac{c}{\sqrt{2-c^2}}\right)\sin i_2 \sin\phi_6,\\
h_2 &= A_6\left(1-\frac{c}{\sqrt{2-c^2}}\right)\sin i_2 \cos\phi_6,\\
g_3 &= -A_6\left(1+\frac{c}{\sqrt{2-c^2}}\right)\sin i_2 \sin\phi_6,\\
h_3 &= -A_6\left(1+\frac{c}{\sqrt{2-c^2}}\right)\sin i_2 \cos\phi_6
\end{aligned}
\tag{A.9}
$$

from the condition $\sin 2i_2 \neq 0$, we obtain $p(t) \equiv 0$. Hence we have $\mathcal{K}_T = R^N$, $T > 0$. This implies the existence of a T-closed trajectory for any initial point. □

Proof of Theorem 5.1. The proof is almost identical to that of the previous theorem. Indeed, condition (3.6) of Theorem 2.1 takes the form

$$
\kappa(t) = \tilde{g}t\cos\omega_0 t + \tilde{h}t\sin\omega_0 t + \sum_{k=0}^{3}\left(g_k\cos\omega_k t + h_k\sin\omega_k t\right) \equiv 0. \tag{A.10}
$$

As in the previous proof, one can apply Lemma 3.1 and see that all the frequencies are pairwise different. Consequently the coefficients of the quasipolynomial $\kappa(t)$ equal zero. Since

$$
\begin{aligned}
\tilde{g} &= -\frac{5c^2-2}{2-c^2}p_1^0\sigma_x,\\
\tilde{h} &= \frac{5c^2-2}{2-c^2}p_1^0\sigma_z,\\
g_0 &= -\frac{p_3^0}{2nc}\sigma_x + \frac{2cp_1^0}{(2-c^2)n}\sigma_z,\\
h_0 &= \frac{p_3^0}{2nc}\sigma_z + \frac{2cp_1^0}{(2-c^2)n}\sigma_x,\\
g_1 &= \frac{A_2}{q}\sigma_y\sin\phi_2,\\
h_1 &= \frac{A_2}{q}\sigma_y\cos\phi_2,\\
g_2 &= -A_6\left(\frac{1}{2}+\frac{c}{\sqrt{2-c^2}}\right)\left(\sigma_z\cos\phi_6 - \sigma_x\sin\phi_6\right),\\
h_2 &= A_6\left(\frac{1}{2}+\frac{c}{\sqrt{2-c^2}}\right)\left(\sigma_x\cos\phi_6 + \sigma_z\sin\phi_6\right),\\
g_3 &= -A_6\left(\frac{1}{2}-\frac{c}{\sqrt{2-c^2}}\right)\left(\sigma_z\cos\phi_6 + \sigma_x\sin\phi_6\right),\\
h_3 &= A_6\left(\frac{1}{2}-\frac{c}{\sqrt{2-c^2}}\right)\left(\sigma_z\sin\phi_6 - \sigma_x\cos\phi_6\right),
\end{aligned}
\tag{A.11}
$$

from the conditions $\sigma_x^2 + \sigma_z^2 \neq 0$, $\sigma_y \neq 0$, we obtain $p(t) \equiv 0$. Thus we have $\mathcal{K}_T = R^N$, $T > 0$. This implies the existence of a T-closed trajectory for any initial point. □

Proof of Theorem 6.1. Assume that $\bigcup_M \mathcal{K}_{M\tau} \neq R^N$. Then, by Theorem 2.3, there exists a nontrivial solution to (2.7) satisfying

$$p_4(t)e_x(t) + p_5(t)e_y(t) + p_6(t)e_z(t) \geq 0, \qquad t \in [0, \infty). \tag{A.12}$$

Condition (A.12) takes the form

$$\kappa(t) = \tilde{g}t \cos \omega_0 t + \sum_{k=0}^{3} \left(g_k \cos \omega_k t + h_k \sin \omega_k t\right) \geq 0, \qquad t \in [0, \infty), \tag{A.13}$$

where the coefficients are those defined in the proof of Theorem 4.1. Now we show that the coefficients of the quasipolynomial $\kappa(t)$ are equal to zero. Indeed, we have

$$0 \leq \frac{\kappa(t)}{t} = \tilde{g} \cos \omega_0 t + O\left(\frac{1}{t}\right), \qquad t \longrightarrow \infty. \tag{A.14}$$

So $\tilde{g} = 0$, and we obtain

$$\kappa(t) = \sum_{k=0}^{3} \left(g_k \cos \omega_k t + h_k \sin \omega_k t\right) \geq 0. \tag{A.15}$$

From Lemma 3.1 we have $\omega_2 < \omega_0 < \omega_1 < \omega_3$ and $\omega_j \neq 0$, $j = 0, \ldots, 3$. Multiplying $\kappa(t)$ by $1 \pm \cos \omega_j t$, $j = 0, 1, 2, 3$, we get

$$0 \leq \lim_{\tau \to \infty} \frac{1}{T} \int_0^T \kappa(t)\left(1 \pm \cos \omega_j t\right) dt = \pm \frac{g_j}{2}. \tag{A.16}$$

Multiplying $\kappa(t)$ by $1 \pm \sin \omega_j t$, $j = 0, 1, 2, 3$, we obtain

$$0 \leq \lim_{\tau \to \infty} \frac{1}{T} \int_0^T \kappa(t)\left(1 \pm \sin \omega_j t\right) dt = \pm \frac{h_j}{2}. \tag{A.17}$$

Therefore all of the coefficients of $\kappa(t)$ are equal to zero. As in the proof of Theorem 4.1, we have $p(t) \equiv 0$, a contradiction. □

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