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Research Article

New Jacobi Elliptic Function Solutions for the Kudryashov-Sinelshchikov Equation Using Improved F-Expansion Method

Yinghui He

Department of Mathematics, Honghe University, Mengzi, Yunnan 661100, China

Correspondence should be addressed to Yinghui He; heyinghui07@163.com

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Based on the F-expansion method with a new subequation, an improved F-expansion method is introduced. As illustrative examples, some new exact solutions expressed by the Jacobi elliptic function of the Kudryashov-Sinelshchikov equation are obtained. When the modulus m of the Jacobi elliptic function is driven to the limits 1 and 0, some exact solutions expressed by hyperbolic function and trigonometric function can also be obtained. The method is straightforward and concise and is promising and powerful for other nonlinear evolution equations in mathematical physics.

1. Introduction

It has recently become more interesting to obtain exact solutions of nonlinear partial differential equations. These equations are mathematical models of complex physical phenomena that arise in engineering, applied mathematics, chemistry, biology, mechanics, physics, and so forth. Thus, the investigation of the traveling wave solutions to nonlinear evolution equations (NLEEs) plays an important role in mathematical physics. A lot of physical models have supported a wide variety of solitary wave solutions.

In 2010, Kudryashov and Sinelshchikov [1] introduced the following equation:

$$\begin{aligned} &u_t + \gamma u u_x + u_{xxx} \\ &\quad - \varepsilon(u u_{xx})_x - \kappa u_x u_{xx} - \nu u_{xx} - \delta(u\, u_x)_x = 0, \end{aligned} \tag{1}$$

where γ, ε, κ, ν, and δ are real parameters. Equation (1) describes the pressure waves in the liquid with gas bubbles taking into account the heat transfer and viscosity. When $\varepsilon = \kappa = \delta = 0$ and $\varepsilon = \kappa = \nu = \delta = 0$, (1) becomes the BKdV equation and the KdV equation, respectively. So, (1) can be considered as the generalization of KdV equation. Therefore, the study to (1) is more meaningful than KdV equation and BKdV equation. We call this equation the Kudryashov-Sinelshchikov equation.

Equation (1) was studied by many researchers in various methods. In the case of $\nu = \delta = 0$, it was studied by Ryabov, using a modification of the truncated expansion method [2], by Randrüüt in a more straightforward manner [3], by Li et al., using the bifurcation method of dynamical systems [4–6], by Nadjafikhah and Shirvani-Sh, using the Lie symmetry method [7], and by He, using G'/G-expansion method [8]. In the case of $\nu \neq 0$, $\delta \neq 0$, (1) was studied by Efimova using the modified simplest equation method [9], by Mirzazadeh and Eslami, using first integral method [10]. And they obtained some results when β took special values.

We noticed that the Jacobi elliptic function solutions of (1) are only reported in [8] with special case $\beta = -3$ and $\beta = -4$. Our aim is to find some new solutions expressed by the Jacobi elliptic function making use of improved F-expansion method.

The organization of the paper is as follows: in Section 2, a brief description of the improved F-expansion for finding traveling wave solutions of nonlinear equations is given. In Sections 3 and 4, we will study, respectively, the Kudryashov-Sinelshchikov equation with the situation $\nu = \delta = 0$ and

Table 1: Relations between values of r, p, q, and corresponding $F(\xi)$ in $F'^2 = qF^3 + pF^2 + rF$.

Case	r	p	q	$F(\xi)$
1	4	$-4(m^2+1)$	$4m^2$	$\mathrm{sn}^2(\xi)$
2	$4(1-m^2)$	$4(2m^2-1)$	$-4m^2$	$\mathrm{cn}^2(\xi)$
3	$-4(1-m^2)$	$4(2-m^2)$	-4	$\mathrm{dn}^2(\xi)$
4	m^2	$2(1+m^2)$	m^2	$(\mathrm{sn}(\xi) \pm i\mathrm{cn}(\xi))^2$
5	$-(1-m^2)^2$	$2(m^2-2)$	-1	$(m\mathrm{cn}(\xi) \pm \mathrm{dn}(\xi))^2$
6	$1-m^2$	$2(1+m^2)$	$1-m^2$	$\left(\frac{\mathrm{cn}(\xi)}{1+\mathrm{sn}(\xi)}\right)^2$
7	m^2-1	$2(1+m^2)$	m^2-1	$\left(\frac{\mathrm{dn}(\xi)}{1+m\mathrm{sn}(\xi)}\right)^2$
8	m^2	$2(m^2-1)$	m^2	$\left(\frac{m\mathrm{sn}(\xi)}{1+\mathrm{dn}(\xi)}\right)^2$
9	1	$2(1-2m^2)$	1	$\left(\frac{\mathrm{sn}(\xi)}{1+\mathrm{cn}(\xi)}\right)^2$

$\nu \neq 0$, $\delta \neq 0$ by the improved F-expansion methods. Finally conclusions are given in Section 5.

2. Description of the Improved Methods

Based on F-expansion method [11–13], the main procedures of the improved F-expansion method are as follows.

Step 1. Consider a general nonlinear PDE in the form

$$F(u, u_x, u_t, u_{xx}, u_{xt}, \ldots) = 0. \tag{2}$$

Using $u(x,t) = U(\xi)$, $\xi = x - ct$, we can rewrite (2) as the following nonlinear ODE:

$$F(U, U', U'', \ldots) = 0, \tag{3}$$

where the prime denotes differentiation with respect to ξ.

Step 2. Suppose that the solution of ODE (3) can be written as follows:

$$U(\xi) = A_0 + \sum_{i=1}^{n} \left(A_i F^i(\xi) + B_i F^{-i}(\xi)\right) \tag{4}$$

or

$$U(\xi) = A_0 + \sum_{i=1}^{n} \left(A_i F^i(\xi) + B_i F^{i-1} F'(\xi)\right), \tag{5}$$

where A_i, B_i $(i = 1, 2, \ldots, n)$ are constants to be determined later and n is a positive integer that is given by the homogeneous balance principle. And $F(\xi)$ satisfies the following equation

$$\left(F'(\xi)\right)^2 = qF^3(\xi) + pF^2(\xi) + rF(\xi), \tag{6}$$

where r, p, and q are constant.

Step 3. Substituting (4) or (5) along with (6) into (3) and then setting all the coefficients of $F^j(\xi)F'^k(\xi)$ $(j = 1, 2, \ldots, k = 0, 1)$ of the resulting system to zero yield a set of overdetermined nonlinear algebraic equations for A_0, A_i, and B_i $(i = 1, 2, \ldots n)$.

Step 4. Assuming that the constants A_0, A_i, and B_i $(i = 1, 2, \ldots n)$ can be obtained by solving the algebraic equations in Step 3, then by substituting these constants and the solutions of (6) that can be found in Table 1 into (4), we can obtain the explicit solutions of (2) immediately.

3. Exact Solutions of the Kudryashov-Sinelshchikov Equation in the Case of $\delta = \nu = 0$

Using scale transformation

$$x = x', \qquad t = t', \qquad u = \frac{u'}{\varepsilon}, \tag{7}$$

we can write the Kudryashov-Sinelshchikov equation (1) in the form

$$\begin{aligned} &u_t + \alpha u u_x + u_{xxx} \\ &\quad - (u u_{xx})_x - \beta u_x u_{xx} - \nu u_{xx} - \mu (u\, u_x)_x = 0, \end{aligned} \tag{8}$$

where $\alpha = \gamma/\varepsilon$, $\beta = \kappa/\varepsilon$, and $\mu = \delta/\varepsilon$ (primes are omitted). When $\delta = \nu = 0$, (8) becomes

$$u_t + \alpha u u_x + u_{xxx} - (u u_{xx})_x - \beta u_x u_{xx} = 0. \tag{9}$$

Let

$$u(x,t) = 1 - \varphi(\xi), \qquad \xi = x - ct, \tag{10}$$

where c is the wave speed. Under this transformation, (9) can be reduced to the following ordinary differential equation (ODE):

$$c\varphi' - \alpha(1-\varphi)\varphi' - \varphi''' + \left((1-\varphi)\varphi''\right)' - \beta\varphi'\varphi'' = 0. \tag{11}$$

Integrating (11) once with respect to ξ and setting the constant of integration to R, we have

$$\frac{1}{2}\alpha\varphi^2 + (c-\alpha)\varphi - \varphi\varphi'' - \frac{1}{2}\beta\left(\varphi'\right)^2 + R = 0. \tag{12}$$

Balancing $\varphi\varphi''$ with $(\varphi')^2$ in (12) we find that $n+n+2=2(n+1)$, so n is an arbitrary positive integer. For simplicity, we take $n=2$. Suppose that (12) owns the solutions in the form

$$\varphi(\xi)=A_0+A_1F(\xi)+A_2F^2(\xi)+\frac{B_1}{F(\xi)}+\frac{B_2}{F^2(\xi)}. \quad (13)$$

Substituting (13) and (6) into (12) and then setting all the coefficients of F^k $(k=-5,\ldots,5)$ of the resulting system to zero, we can obtain the following results:

$$\begin{gathered} A_1=A_2=B_2=0,\\ c=\frac{2A_0B_1p-B_1^2q+3rA_0-B_1p-3rA_0^2}{B_1},\\ \alpha=\frac{3rA_0-B_1p}{B_1},\qquad \beta=-3, \end{gathered} \quad (14)$$

$$\begin{gathered} A_2=B_1=B_2=0,\\ c=\frac{2A_0A_1p-A_1p+3qA_0-A_1^2r-3qA_0^2}{A_1},\\ \alpha=\frac{3qA_0-A_1p}{A_1},\qquad \beta=-3, \end{gathered} \quad (15)$$

$$\begin{gathered} A_2=0,\qquad B_1=\frac{A_1r}{q},\qquad B_2=0,\\ c=\frac{4A_1^2r+2A_0A_1p-A_1p+3qA_0-3qA_0^2}{A_1},\\ \alpha=\frac{3qA_0-A_1p}{A_1},\qquad \beta=-3, \end{gathered} \quad (16)$$

$$\begin{gathered} A_0=\frac{A_2(2wp-r)}{3q},\qquad A_1=2wA_2,\qquad B_1=B_2=0,\\ c=-\frac{-18q^2w+2A_2p^2w-6wA_2rq+6pq-A_2pr}{3q},\\ \alpha=-2p+6wq,\qquad \beta=-\frac{5}{2}, \end{gathered} \quad (17)$$

where $w=(p\pm\sqrt{p^2-3rq})/3q$, $q\neq 0$. Consider the following:

$$\begin{gathered} A_0=\frac{2A_2(2wp+5r)}{3q},\qquad A_1=4A_2w,\\ B_1=\frac{4A_2wr}{q},\qquad B_2=\frac{A_2r^2}{q^2},\\ c=-\left(2\left(-18wq^2+2A_2p^2w+24A_2wrq\right.\right.\\ \left.\left.+3pq-16A_2pr\right)\right)\times(3q)^{-1},\\ \alpha=-2p+12wq,\qquad \beta=-\frac{5}{2}, \end{gathered} \quad (18)$$

where $w=(p\pm\sqrt{p^2-3rq})/3q$, $q\neq 0$.

$$\begin{gathered} A_0=\frac{B_2(2w_1p-q)}{3r},\\ A_1=A_2=0,\qquad B_1=2w_1B_2,\\ c=-\frac{-18w_1r^2+2B_2p^2w_1-6w_1B_2qr+6pr-B_2pq}{3r},\\ \alpha=-2p+6w_1r,\qquad \beta=-\frac{5}{2}, \end{gathered} \quad (19)$$

where $w_1=(p\pm\sqrt{p^2-3rq})/3r$, $r\neq 0$.

Substituting (14)–(19) into (13) with (10), we obtain, respectively, the following formal solution of (9):

$$u(x,t)=1-A_0-\frac{B_1}{F(\xi)}, \quad (20)$$

where $\xi=x-((2A_0B_1p-B_1^2q+3rA_0-B_1p-3rA_0^2)/B_1)t$, $\alpha=(3rA_0-B_1p)/B_1$, and $\beta=-3$. Consider the following:

$$u(x,t)=1-A_0-A_1F(\xi), \quad (21)$$

where $\xi=x-((2A_0A_1p-A_1p+3qA_0-A_1^2r-3qA_0^2)/A_1)t$, $\alpha=(3qA_0-A_1p)/A_1$, and $\beta=-3$. Consider the following:

$$u(x,t)=1-A_0-A_1F(\xi)-\frac{A_1r}{qF(\xi)}, \quad (22)$$

where $\xi=x-((4A_1^2r+2A_0A_1p-A_1p+3qA_0-3qA_0^2)/A_1)t$, $\alpha=(3qA_0-A_1p)/A_1$, and $\beta=-3$. Consider the following:

$$u(x,t)=1-\frac{A_2(2wp-r)}{3q}-2wA_2F(\xi)-A_2F^2(\xi), \quad (23)$$

where $\xi=x+((-18q^2w+2A_2p^2w-6wA_2rq+6pq-A_2pr)/3q)t$, $\alpha=-2p+6wq$, and $\beta=-5/2$. Consider the following:

$$\begin{gathered} u(x,t)=1-\frac{2A_2(2wp+5r)}{3q}-4wA_2F(\xi)\\ -A_2(F(\xi))^2-\frac{4wA_2r}{qF(\xi)}-\frac{A_2r^2}{q^2F^2(\xi)}, \end{gathered} \quad (24)$$

where $\xi=x+(2(-18wq^2+2A_2p^2w+24A_2wrq+3pq-16A_2pr)/3q)t$, $\alpha=-2p+12wq$, and $\beta=-5/2$. Consider the following:

$$u(x,t)=1-\frac{B_2w_1}{3r}-\frac{2B_2w_1}{F(\xi)}-\frac{B_2}{F^2(\xi)}, \quad (25)$$

where $\xi=x+((-18w_1r^2+2B_2p^2w_1-6w_1B_2qr+6pr-B_2pq)/3r)t$, $\alpha=-2p+6w_1r$, and $\beta=-5/2$.

Combining (20)–(25) with Table 1, many exact solutions of (9) can be obtained. For simplicity, we just give out case 1 of Table 1, the other cases can be discussed similarly.

When $r = 4$, $p = -4(m^2+1)$, and $q = 4m^2$, the solution of elliptic Equation (6) is $F(\xi) = \mathrm{sn}^2(\xi, m)$. Substituting it into (20)–(24), we can obtain the following solutions of (9).

From (20), one has

$$u_1(x,t) = 1 - A_0 - B_1 \mathrm{ns}^2(\xi, m), \tag{26}$$

where $\xi = x + (4(2A_0B_1m^2 + 2A_0B_1 + B_1^2m^2 - 3A_0 - B_1m^2 - B_1 + 3A_0^2)/B_1)t$, $\alpha = (4(3A_0 + B_1m^2 + B_1)/B_1)$, and $\beta = -3$.

When $m \to 1$ and $\mathrm{sn}(\xi, m) \to \tanh(\xi)$, (26) becomes

$$u(x,t) = 1 - A_0 - B_1 \coth^2(\xi), \tag{27}$$

where $\xi = x + (4(4A_0B_1 - 3A_0 - 2B_1 + B_1^2 + 3A_0^2)/B_1)t$, $\alpha = 4(3A_0 + 2B_1)/B_1$, and $\beta = -3$.

When $m \to 0$ and $\mathrm{sn}(\xi, m) \to \sin(\xi)$, (26) becomes

$$u(x,t) = 1 - A_0 - B_1 \sec^2(\xi), \tag{28}$$

where $\xi = x + (4(2A_0B_1 - 3A_0 - B_1 + 3A_0^2)/B_1)t$, $\alpha = (4(3A_0 + B_1)/B_1)$, and $\beta = -3$.

From (21), we have

$$u(x,t) = 1 - A_0 - A_1 \mathrm{sn}^2(\xi, m), \tag{29}$$

where $\xi = x + 4((2A_0A_1m^2 + 2A_0A_1 + 3m^2A_0^2 + A_1^2 - A_1m^2 - A_1 - 3m^2A_0)/A_1)t$, $\alpha = 4((A_1m^2 + A_1 + 3m^2A_0)/A_1)$, and $\beta = -3$.

When $m \to 1$, (29) becomes a hyperbolic function solution,

$$u(x,t) = 1 - A_0 - B_1 \tanh^2(\xi), \tag{30}$$

where $\xi = x + (4(4A_0B_1 - 3A_0 - 2B_1 + B_1^2 + 3A_0^2)/B_1)t$, $\alpha = (4(3A_0 + 2B_1)/B_1)$, and $\beta = -3$.

When $m \to 0$, (29) becomes a trigonometric function solution,

$$u(x,t) = 1 - A_0 - B_1 \sin^2(\xi), \tag{31}$$

where $\xi = x + 4(2A_0 + A_1 - 1)t$, $\alpha = 4$, and $\beta = -3$.

From (22), we have

$$u(x,t) = 1 - A_0 - A_1 \mathrm{sn}^2(\xi, m) - \frac{A_1 r}{q} \mathrm{ns}^2(\xi, m), \tag{32}$$

where $\xi = x + (4(2A_0A_1m^2 + 2A_0A_1 + 3m^2A_0^2 - 4A_1^2 - A_1m^2 - A_1 - 3m^2A_0)/A_1)t$, $\alpha = (4(A_1m^2 + A_1 + 3m^2A_0)/A_1)$, and $\beta = -3$.

When $m \to 1$, (32) becomes a hyperbolic function solution,

$$u(x,t) = 1 - A_0 - A_1 \tanh^2(\xi) - \frac{A_1 r}{q} \coth^2(\xi), \tag{33}$$

where $\xi = x + (4(4A_0A_1 + 3A_0^2 - 4A_1^2 - 2A_1 - 3A_0)/A_1)t$, $\alpha = (4(2A_1 + 3A_0)/A_1)$, and $\beta = -3$.

From (23), we have

$$\begin{aligned} u(x,t) = 1 &+ \frac{A_2\left(2w\left(m^2+1\right)+1\right)}{3m^2} \\ &- 2wA_2 \mathrm{sn}^2(\xi, m) - A_2 \mathrm{sn}^4(\xi, m), \end{aligned} \tag{34}$$

where $w = (-m^2 - 1 \pm \sqrt{m^4 - m^2 + 1})/3m^2$, $\xi = x + (4(-2wA_2m^2 + 2wA_2m^4 + 2wA_2 - 18wm^4 + A_2m^2 + A_2 - 6m^4 - 6m^2)/3m^2)t$, $\alpha = 8(3wm^2 + m^2 + 1)$, and $\beta = -5/2$.

When $m \to 1$, (34) becomes a hyperbolic function solution,

$$u(x,t) = 1 - \frac{1}{12}A_2w - 2wA_2 \tanh^2(\xi) - A_2 \tanh^4(\xi, x), \tag{35}$$

where $w = (-2 \pm 1)/3$, $\xi = 8(3w - (1/3)A_2w + 2 - (1/3)A_1)t$, $\alpha = 16 + 24w$, and $\beta = -5/2$.

From (24), we have

$$\begin{aligned} u(x,t) = 1 &+ \frac{2A_2\left(2w\left(m^2+1\right)-5\right)}{3m^2} \\ &- 4wA_2 \mathrm{sn}^2(\xi, m) - A_2 \mathrm{sn}^4(\xi, m) \\ &- \frac{4wA_2}{m^2} \mathrm{ns}^2(\xi, m) - \frac{A_2}{m^4} \mathrm{ns}^4(\xi, m), \end{aligned} \tag{36}$$

where $w = (-m^2 - 1 \pm \sqrt{m^4 + 14m^2 + 1})/6m^2$, $\xi = x + (8(-18wm^4 + 2wA_2m^4 + 28wA_2m^2 + 2wA_2 - 3m^4 - 3m^2 + 16A_2m^2 + 16A_2)/3m^2)t$, $\alpha = 8(m^2 + 1 + 6wm^2)$, and $\beta = -5/2$.

When $m \to 1$, (36) becomes a hyperbolic function solution,

$$\begin{aligned} u(x,t) = 1 &+ \frac{2}{3}A_2(4w - 5) - 4wA_2 \tanh^2(\xi) \\ &- A_2 \tanh^4(\xi) - 4wA_2 \coth^2(\xi) - A_2 \coth^4(\xi), \end{aligned} \tag{37}$$

where $w = (-1 \pm 2)/3$, $\xi = x - (48w - (256/3)wA_2 + 16 - (256/3)A_2)\,t$, $\alpha = 16 + 48w$, and $\beta = -5/2$.

From (25), we have

$$\begin{aligned} u(x,t) = 1 &+ \frac{1}{3}B_2\left(2w\left(m^2+1\right)+m^2\right) \\ &- 2wB_2 \mathrm{ns}^2(\xi, m) - B_2 \mathrm{ns}^4(\xi, m), \end{aligned} \tag{38}$$

where $w = (-m^2 - 1 \pm \sqrt{m^4 - m^2 + 1})/3$, $\xi = x + ((8/3)wB_2m^4 + (8/3)wB_2m^2 - (8/3)wB_2 + 24w + 8m^2 + 8 - (4/3)B_2m^4 - (4/3)B_2m^2)t$, $\alpha = 8(3w + m^2 + 1)$, and $\beta = -5/2$.

When $m \to 1$, (38) becomes a hyperbolic function solution,

$$\begin{aligned} u(x,t) = 1 &+ \frac{1}{3}B_2(4w + 1) \\ &- 2wB_2 \coth^2(\xi) - B_2 \coth^4(\xi), \end{aligned} \tag{39}$$

where $w = (-2 \pm 1)/3$, $\xi = x - (24w - (8/3)wB_2 + 16 - (8/3)B_2)t$, $\alpha = 16 + 24w$, and $\beta = -5/2$.

When $m \to 0$, (38) becomes a triangle function solution,

$$u(x,t) = 1 + \frac{2}{3}wB_2 - 2wB_2 \csc^2(\xi) - B_2 \csc^4(\xi), \tag{40}$$

where $w = (-1 \pm 1)/3$, $\xi = x - (24w + 8 - (8/3)B_2w)t$, $\alpha = 24w + 8$, and $\beta = -5/2$.

We notice that when $\beta = -3$, some Jacobi elliptic function solutions have been given in [8]. Part of our results may be the same with them. However, when $\beta = -(5/2)$, the Jacobi elliptic function solutions of (9) have not been reported in the related literatures, so we believe that our solutions (34), (36), and (38) are new.

4. Exact Solutions of the Kudryashov-Sinelshchikov Equation in the Case of $\delta \neq 0, \mu \neq 0$

In this case, by similar process, (1) can be changed into the following PDE:

$$\begin{aligned} &\frac{1}{2}\alpha\varphi^2 + (c-\alpha)\varphi - \varphi\varphi'' \\ &\quad - \frac{1}{2}\beta\left(\varphi'\right)^2 + \nu\varphi' + \mu(1-\varphi)\varphi' + R = 0, \end{aligned} \tag{41}$$

where $\varphi' = d\varphi/d\xi$, $\xi = x - ct$, and R is an integral constant.

Suppose that (41) owns the solutions in the form

$$\begin{aligned} \varphi(\xi) &= A_0 + A_1 F(\xi) + A_2 F^2(\xi) \\ &\quad + B_1 F(\xi)' + B_2 F(\xi) F(\xi)'. \end{aligned} \tag{42}$$

Substituting (42) and (6) into (41) and then setting all the coefficients of $F^j(\xi)F'^k(\xi)$ $(j = 0, 1, \ldots, 6, k = 0, 1)$ of the resulting system to zero, we can obtain the following results:

$$\begin{gathered} A_0 = 0, \qquad A_1 = wB_1, \qquad A_2 = B_2 = 0, \\ c = -\frac{5}{3}wB_1 r - \frac{4}{3}p, \qquad \alpha = -\frac{4}{3}p, \\ \beta = -\frac{8}{3}, \qquad \lambda = \frac{2}{3}B_1 r + \frac{1}{3}w, \qquad \mu = -\frac{1}{3}w, \end{gathered} \tag{43}$$

where $w = \pm\sqrt{p}$. Consider the following:

$$\begin{gathered} A_0 = \frac{B_2 w\left(2pw^2 - 2p^2 + 3rq\right)}{9q^2}, \\ A_1 = \frac{2B_2 w(w-p)}{3q}, \\ A_2 = -B_2 w, \qquad B_1 = -\frac{B_2\left(w^2 - p\right)}{3q}, \\ c = -w\left(14w^2B_2p^2 - 42w^2B_2rq + 63B_2rpq \right. \\ \left. +108wq^2 - 14B_2p^3\right) \times \left(45q^2\right)^{-1}, \\ \alpha = -\frac{12}{5}w^2, \qquad \beta = -\frac{12}{5}, \\ \lambda = -\left(2w^2B_2p^2 - 2B_2p^3 + 9B_2rpq \right. \\ \left. -6w^2B_2rq + 9wq^2\right) \times \left(45q^2\right)^{-1}, \\ \mu = \frac{1}{5}w \end{gathered} \tag{44}$$

where $w = \pm\sqrt[4]{p^2 - 3rq}$.

Substituting (43) and (44) along with (10) into (42), we obtain, respectively, the following formal solution of (8):

$$u(x,t) = 1 - wB_1F(\xi) - B_1F'(\xi), \tag{45}$$

where $w = \pm\sqrt{p}$, $\xi = x + ((5/3)wB_1r + (4/3)p)t$, $\alpha = -(4/3)p$, $\beta = -8/3$, $\lambda = (2/3)B_1r + (1/3)w$, and $\mu = -(1/3)w$. Consider the following:

$$\begin{aligned} u(x,t) &= 1 - \frac{B_2 w\left(2pw^2 - 2p^2 + 3rq\right)}{9q^2} \\ &\quad - \frac{2B_2 w\left(w^2 - p\right)F(\xi)}{3q} + wB_2F^2(\xi) \\ &\quad + \frac{B_2\left(w^2 - p\right)F'(\xi)}{3q} - B_2F(\xi)F'(\xi), \end{aligned} \tag{46}$$

where $w = \pm\sqrt[4]{p^2 - 3rq}$, $\xi = x + (w(14w^2B_2p^2 - 42w^2B_2rq + 63B_2rpq + 108wq^2 - 14B_2p^3)/45q^2)t$, $\alpha = -(12/5)w^2$, $\beta = -12/5$, $\lambda = -((2w^2B_2p^2 - 2B_2p^3 + 9B_2rpq - 6w^2B_2rq + 9wq^2)/45q^2)$, and $\mu = (1/5)w$.

Combining (45) and (46) with Table 1, many exact solutions of (8) can be obtained. For simplicity, we just give out case 3 of Table 1, and the other cases can be discussed similarly.

When $r = -4(1 - m^2)$, $p = 4(2 - m^2)$, $q = -4$, and the solution of elliptic Equation (6) is $F(\xi) = \mathrm{dn}^2(\xi, m)$. Substituting it into (45) and (46), we can obtain the following Jacobi elliptic function solutions of (8):

$$\begin{aligned} u(x,t) &= 1 - B_1\left(w\mathrm{dn}^2(\xi,m) \right. \\ &\quad \left. -2m^2\mathrm{sn}(\xi,m)\,\mathrm{cn}(\xi,m)\,\mathrm{dn}(\xi,m)\right), \end{aligned} \tag{47}$$

where $w = \pm 2\sqrt{2 - m^2}$, $\xi = x - ((20/3)wB_1 - (20/3)wB_1m^2 - (32/3) + (16/3)m^2)t$, $\alpha = (16/3)(m^2 - 2)$, $\beta = -8/3$, $\nu = (1/3)(-8B_1 + 8B_1m^2 + w)$, $\mu = -(1/3)w$, and

$$\begin{aligned} u(x,t) &= 1 - \frac{B_2 w\left(2pw^2 - 2p^2 + 3rq\right)}{9q^2} \\ &\quad - \frac{2B_2 w\left(w^2 - p\right)\mathrm{dn}^2(\xi,m)}{3q} + wB_2\mathrm{dn}^4(\xi,m) \\ &\quad - \frac{2m^2B_2\left(w^2 - p\right)\mathrm{dn}(\xi,m)\,\mathrm{cn}(\xi,m)\,\mathrm{sn}(\xi,m)}{3q} \\ &\quad + 2m^2B_2\,\mathrm{dn}^3(\xi,m)\,\mathrm{cn}(\xi,m)\,\mathrm{sn}(\xi,m), \end{aligned} \tag{48}$$

where $w = \pm 2\sqrt[4]{1 - m^2 - m^4}$, $\xi = x + (2w/45)(7w^2B_2 - 7w^2B_2m^2 + 28B_2 - 42B_2m^2 - 42B_2m^4 + 7w^2B_2m^4 + 28B_2m^6 + 54w)t$, $\alpha = -(12/5)w^2$, $\beta = -(12/5)$, $\nu = -(2/45)w^2B_2 + (2/45)w^2B_2m^2 - (2/45)w^2B_2m^4 - (8/45)B_2 + (4/15)B_2m^2 + (4/15)B_2m^4 - (8/45)B_2m^6 - (1/5)w$, and $\mu = (1/5)w$.

Similarly, when $m \to 1$ we can obtain hyperbolic function solutions of (8). Here, we omit them.

We indicate that these results with $\beta = -12/5$ and $\beta = -8/3$ are new.

5. Conclusions

The Jacobi elliptic function solutions of (1) are only reported in [8] with special case of $\beta = -3$ and $\beta = -4$. In the present work, we successfully obtained some new Jacobi elliptic function solutions of the Kudryashov-Sinelshchikov equation with $\beta = -3$, $\beta = -5/2$, $\beta = -8/3$, and $\beta = -12/5$ using the improved F-expansion method. When the modulus m of the Jacobi elliptic function is driven to the limits 1 and 0, some exact solutions expressed by hyperbolic function and trigonometric function can also be obtained. All the results we obtained have been verified. The related results are enriched.

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