Hindawi Publishing Corporation
Mathematical Problems in Engineering
Volume 2013, Article ID 157849, 9 pages
http://dx.doi.org/10.1155/2013/157849

Research Article

Existence of Positive Solution for a Third-Order BVP with Advanced Arguments and Stieltjes Integral Boundary Conditions

Jian-Ping Sun, Ping Yan, Ya-Hong Zhao, and Fang-Di Kong

Department of Applied Mathematics, Lanzhou University of Technology, Lanzhou, Gansu 730050, China

Correspondence should be addressed to Jian-Ping Sun; jpsun2012@163.com

Received 20 June 2013; Revised 1 August 2013; Accepted 1 August 2013

Academic Editor: Fazal M. Mahomed

A class of third-order boundary value problems with advanced arguments and Stieltjes integral boundary conditions is discussed. Some existence criteria of at least one positive solution are established. The main tool used is the Guo-Krasnoselskii fixed point theorem.

1. Introduction

Third-order differential equations arise in a variety of different areas of applied mathematics and physics, for example, in the deflection of a curved beam having a constant or varying cross-section, a three-layer beam, electromagnetic waves or gravity-driven flows, and so on [1].

Recently, third-order boundary value problems (BVPs for short) with integral boundary conditions, which cover third-order multipoint BVPs as special cases, have attracted much attention from many authors; see [2–6] and the references therein. In particular, in 2012, by using a fixed point theorem due to Avery and Peterson [7], Jankowski [4] established the existence of at least three nonnegative solutions to the following BVP:

$$\begin{gathered} u'''(t) + h(t) f(t, u(\alpha(t))) = 0, \quad t \in (0,1), \\ u(0) = u''(0) = 0, \qquad u(1) = \beta u(\eta) + \lambda[u], \end{gathered} \tag{1}$$

where λ denoted a linear functional on $C[0,1]$ given by

$$\lambda[u] = \int_0^1 u(t)\, d\Lambda(t) \tag{2}$$

involving a Stieltjes integral with a suitable function Λ of bounded variation. The measure $d\Lambda$ could be a signed one. The situation with a signed measure $d\Lambda$ was first discussed in [8, 9] for second-order differential equations; it was also discussed in [10, 11] for second-order impulsive differential equations. For some other related results, one can refer to [12–14].

Among the boundary conditions in (1), only $u(1)$ is related to a Stieltjes integral. In this paper, we are concerned with the following third-order BVP with advanced arguments and Stieltjes integral boundary conditions:

$$\begin{gathered} u'''(t) + f(t, u(\alpha(t))) = 0, \quad t \in (0,1), \\ u(0) = \gamma u(\eta) + \lambda[u], \qquad u''(0) = 0, \\ u(1) = \beta u(\eta) + \lambda[u]. \end{gathered} \tag{3}$$

Throughout this paper, we always assume that $\alpha : [0,1] \to [0,1]$ is continuous and $\alpha(t) \geq t$ for $t \in [0,1]$, $0 < \eta < 1$, $0 \leq \gamma < \beta < 1$, Λ is a suitable function of bounded variation, and $\lambda[u]$ is defined as in (2). It is important to indicate that it is not assumed that $\lambda[u]$ is positive to all positive u. Some existence criteria of at least one positive solution to the BVP (3) are obtained by using the following well-known Guo-Krasnoselskii fixed point theorem [15, 16].

Theorem 1. *Let E be a Banach space, and let K be a cone in E. Assume that Ω_1 and Ω_2 are bounded open subsets of E such that $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(1) *$\|Tu\| \leq \|u\|$ for $u \in K \cap \partial\Omega_1$ and $\|Tu\| \geq \|u\|$ for $u \in K \cap \partial\Omega_2$ or*

(2) $\|Tu\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_2$.

Then, T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2. Preliminaries

Let $\Delta = 1 - \gamma - (\beta - \gamma)\eta$. Then, $\Delta > 0$.

Lemma 2. *For any $y \in C[0,1]$, the BVP*

$$\begin{gathered} u'''(t) = -y(t), \quad t \in (0,1), \\ u(0) = \gamma u(\eta) + \lambda[u], \qquad u''(0) = 0, \\ u(1) = \beta u(\eta) + \lambda[u] \end{gathered} \tag{4}$$

has the unique solution

$$\begin{aligned} u(t) = {} & \frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta}\lambda[u] \\ & + \frac{\gamma+(\beta-\gamma)t}{\Delta}\int_0^1 k(\eta,s)\,y(s)\,ds \\ & + \int_0^1 k(t,s)\,y(s)\,ds, \quad t \in [0,1], \end{aligned} \tag{5}$$

where

$$k(t,s) = \frac{1}{2}\begin{cases} (1-t)\left(t-s^2\right), & 0 \leq s \leq t \leq 1, \\ t(1-s)^2, & 0 \leq t \leq s \leq 1. \end{cases} \tag{6}$$

Proof. By integrating the differential equation in (4) three times from 0 to t and using the boundary condition $u''(0) = 0$, we get

$$u(t) = u(0) + u'(0)\,t - \frac{1}{2}\int_0^t (t-s)^2 y(s)\,ds, \quad t \in [0,1]. \tag{7}$$

So,

$$u'(0) = u(1) - u(0) + \frac{1}{2}\int_0^1 (1-s)^2 y(s)\,ds. \tag{8}$$

In view of (7), (8), and the boundary conditions $u(0) = \gamma u(\eta) + \lambda[u]$ and $u(1) = \beta u(\eta) + \lambda[u]$, we have

$$\begin{aligned} u(t) = {} & [\gamma + (\beta-\gamma)t]\,u(\eta) + \lambda[u] \\ & + \int_0^1 k(t,s)\,y(s)\,ds, \quad t \in [0,1]. \end{aligned} \tag{9}$$

So,

$$u(\eta) = \frac{1}{\Delta}\lambda[u] + \frac{1}{\Delta}\int_0^1 k(\eta,s)\,y(s)\,ds. \tag{10}$$

Substituting (10) into (9), we get

$$\begin{aligned} u(t) = {} & \frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta}\lambda[u] \\ & + \frac{\gamma+(\beta-\gamma)t}{\Delta}\int_0^1 k(\eta,s)\,y(s)\,ds \\ & + \int_0^1 k(t,s)\,y(s)\,ds, \quad t \in [0,1]. \end{aligned} \tag{11}$$

□

Lemma 3 (see [4]). *Consider that* $0 \leq k(t,s) \leq (1/2)(1+s)(1-s)^2$, $(t,s) \in [0,1] \times [0,1]$.

Throughout, we assume that the following conditions are fulfilled:

(C1) $f \in C([0,1] \times [0,+\infty), [0,+\infty))$,

(C2)

$$\begin{gathered} \int_0^1 d\Lambda(t) \geq 0, \qquad \int_0^1 t\,d\Lambda(t) \geq 0, \\ \kappa(s) = \int_0^1 k(t,s)\,d\Lambda(t) \geq 0, \quad s \in [0,1]. \end{gathered} \tag{12}$$

For convenience, we denote

$$\begin{gathered} \rho = [1-(\beta-\gamma)\eta]\int_0^1 d\Lambda(t) + (\beta-\gamma)\int_0^1 t\,d\Lambda(t), \\ \rho' = \gamma\int_0^1 d\Lambda(t) + (\beta-\gamma)\int_0^1 t\,d\Lambda(t). \end{gathered} \tag{13}$$

Obviously, $\rho, \rho' \geq 0$. In the remainder of this paper, we always assume that $\rho < \Delta$.

Let $C[0,1]$ be equipped with the maximum norm. Then, $C[0,1]$ is a Banach space. Define

$$\begin{aligned} K = \Big\{ & u \in C[0,1] : u(t) \geq 0,\ t \in [0,1], \\ & \min_{t\in[\eta,1]} u(t) \geq \Gamma\|u\|,\ \lambda[u] \geq 0 \Big\}, \end{aligned} \tag{14}$$

where

$$\Gamma = \min\left\{\frac{\beta(1-\eta)}{1-\beta\eta}, \frac{\beta\eta}{1-\gamma(1-\eta)}\right\}. \tag{15}$$

Then, K is a cone in $C[0,1]$.

Now, we define operators T and S on K by

$$\begin{aligned} (Tu)(t) &= \frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta}\lambda[u] + (Fu)(t), \\ & \qquad t \in [0,1], \\ (Su)(t) &= \frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta-\rho}\lambda[Fu] + (Fu)(t), \\ & \qquad t \in [0,1], \end{aligned} \tag{16}$$

where

$$(Fu)(t) = \frac{\gamma + (\beta - \gamma) t}{\Delta} \int_0^1 k(\eta, s) f(s, u(\alpha(s))) ds + \int_0^1 k(t, s) f(s, u(\alpha(s))) ds, \quad t \in [0, 1]. \tag{17}$$

Lemma 4. *Consider that $T, S : K \to K$.*

Proof. Let $u \in K$. Then, it is easy to verify that

$$(Tu)''(t) = -\int_0^t f(s, u(\alpha(s))) ds \le 0, \quad t \in [0, 1], \tag{18}$$

which shows that Tu is concave down on $[0, 1]$. In view of

$$(Fu)(0) = \frac{\gamma}{\Delta} \int_0^1 k(\eta, s) f(s, u(\alpha(s))) ds \ge 0,$$
$$(Fu)(1) = \frac{\beta}{\Delta} \int_0^1 k(\eta, s) f(s, u(\alpha(s))) ds \ge 0, \tag{19}$$

we have

$$(Tu)(0) = \frac{1 - (\beta - \gamma)\eta}{\Delta} \lambda[u] + (Fu)(0) \ge 0,$$
$$(Tu)(1) = \frac{1 + (\beta - \gamma)(1 - \eta)}{\Delta} \lambda[u] + (Fu)(1) \ge 0. \tag{20}$$

So, $(Tu)(t) \ge 0$, $t \in [0, 1]$.

Now, we prove that $\min_{t\in[\eta,1]}(Tu)(t) \ge \Gamma\|Tu\|$. To do it, we consider two cases.

Case 1. Let $(Tu)(\eta) \le (Tu)(1)$. Then $\min_{t\in[\eta,1]}(Tu)(t) = (Tu)(\eta)$, and there exists $\bar{t} \in [\eta, 1]$ such that $\|Tu\| = (Tu)(\bar{t})$. Moreover,

$$\frac{(Tu)(\bar{t}) - (Tu)(0)}{\bar{t} - 0} \le \frac{(Tu)(\eta) - (Tu)(0)}{\eta - 0}. \tag{21}$$

So,

$$\|Tu\| \le \frac{1}{\eta}(Tu)(\eta) - \frac{1 - \eta}{\eta}(Tu)(0), \tag{22}$$

which together with

$$(Tu)(0) = \gamma(Tu)(\eta) + \lambda[u] \tag{23}$$

implies that

$$\|Tu\| \le \frac{1 - \gamma(1 - \eta)}{\eta}(Tu)(\eta); \tag{24}$$

that is,

$$\min_{t\in[\eta,1]}(Tu)(t) \ge \frac{\eta}{1 - \gamma(1 - \eta)}\|Tu\|. \tag{25}$$

Case 2. Let $(Tu)(\eta) > (Tu)(1)$ and $\|Tu\| = (Tu)(\bar{t})$. Note that in this case $\min_{t\in[\eta,1]}(Tu)(t) = (Tu)(1)$.

If $\bar{t} \in [0, \eta]$, then

$$\frac{(Tu)(1) - (Tu)(\bar{t})}{1 - \bar{t}} \ge \frac{(Tu)(1) - (Tu)(\eta)}{1 - \eta}. \tag{26}$$

So,

$$\|Tu\| \le \frac{1}{1 - \eta}(Tu)(\eta) - \frac{\eta}{1 - \eta}(Tu)(1), \tag{27}$$

which together with

$$(Tu)(\eta) = \frac{1}{\beta}((Tu)(1) - \lambda[u]) \tag{28}$$

implies that

$$\|Tu\| \le \frac{1 - \beta\eta}{\beta(1 - \eta)}(Tu)(1); \tag{29}$$

that is,

$$\min_{t\in[\eta,1]}(Tu)(t) \ge \frac{\beta(1 - \eta)}{1 - \beta\eta}\|Tu\|. \tag{30}$$

If $\bar{t} \in (\eta, 1)$, then

$$\frac{(Tu)(\bar{t}) - (Tu)(\eta)}{\bar{t} - \eta} \le \frac{(Tu)(\eta) - (Tu)(0)}{\eta - 0}. \tag{31}$$

So,

$$\|Tu\| \le \frac{1}{\eta}(Tu)(\eta) - \frac{1 - \eta}{\eta}(Tu)(0), \tag{32}$$

which together with (23) and (28) implies that

$$\|Tu\| \le \frac{1 - \gamma(1 - \eta)}{\beta\eta}(Tu)(1); \tag{33}$$

that is,

$$\min_{t\in[\eta,1]}(Tu)(t) \ge \frac{\beta\eta}{1 - \gamma(1 - \eta)}\|Tu\|. \tag{34}$$

It follows from (25), (30), and (34) that

$$\min_{t\in[\eta,1]}(Tu)(t) \ge \Gamma\|Tu\|. \tag{35}$$

Finally, we need to show that $\lambda[Tu] \ge 0$. In view of

$$\begin{aligned}\lambda[Fu] &= \int_0^1 \frac{\gamma + (\beta - \gamma) t}{\Delta} \int_0^1 k(\eta, s) f(s, u(\alpha(s))) ds\, d\Lambda(t) \\ &\quad + \int_0^1 \int_0^1 k(t, s) f(s, u(\alpha(s))) ds\, d\Lambda(t) \\ &= \frac{\rho'}{\Delta} \int_0^1 k(\eta, s) f(s, u(\alpha(s))) ds \\ &\quad + \int_0^1 \kappa(s) f(s, u(\alpha(s))) ds \ge 0,\end{aligned} \tag{36}$$

we have

$$\lambda[Tu] = \frac{\rho}{\Delta}\lambda[u] + \lambda[Fu] \geq 0. \tag{37}$$

This shows that $T : K \to K$. Similarly, we can prove that $S : K \to K$. □

Lemma 5. *The operators T and S have the same fixed points in K.*

Proof. Suppose that $u \in K$ is a fixed point of S. Then,

$$\lambda[u] = \int_0^1 \left(\frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta-\rho}\lambda[Fu] + (Fu)(t) \right) d\Lambda(t) = \frac{\Delta}{\Delta-\rho}\lambda[Fu], \tag{38}$$

which shows that

$$\lambda[Fu] = \frac{\Delta-\rho}{\Delta}\lambda[u]. \tag{39}$$

So,

$$\begin{aligned} u(t) &= (Su)(t) \\ &= \frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta-\rho}\lambda[Fu] + (Fu)(t) \\ &= \frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta}\lambda[u] + (Fu)(t) \\ &= (Tu)(t), \quad t \in [0,1], \end{aligned} \tag{40}$$

which indicates that u is a fixed point of T.

Suppose that $u \in K$ is a fixed point of T. Then,

$$\lambda[u] = \int_0^1 \left(\frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta}\lambda[u] + (Fu)(t) \right) d\Lambda(t) = \frac{\rho}{\Delta}\lambda[u] + \lambda[Fu], \tag{41}$$

which shows that

$$\lambda[u] = \frac{\Delta}{\Delta-\rho}\lambda[Fu]. \tag{42}$$

So,

$$\begin{aligned} u(t) &= (Tu)(t) \\ &= \frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta}\lambda[u] + (Fu)(t) \\ &= \frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta-\rho}\lambda[Fu] + (Fu)(t) \\ &= (Su)(t), \quad t \in [0,1], \end{aligned} \tag{43}$$

which indicates that u is a fixed point of S. □

Lemma 6. *$T, S : K \to K$ is completely continuous.*

Proof. First, by Lemma 4, we know that $T(K) \subset K$.

Next, we show that T is compact. Let $D \subset K$ be a bounded set. Then, there exists $M_1 > 0$ such that $\|u\| \leq M_1$ for any $u \in D$. Since Λ is a function of bounded variation, there exists $M_2 > 0$ such that $v_{\Delta'} = \sum_{i=1}^{n} |\Lambda(t_i) - \Lambda(t_{i-1})| \leq M_2$ for any partition $\Delta' : 0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$. Let

$$M_3 = \sup\{f(t,u) : (t,u) \in [0,1] \times [0,M_1]\}. \tag{44}$$

Then, for any $u \in D$,

$$\begin{aligned} \|Tu\| &= \max_{t\in[0,1]} (Tu)(t) \\ &\leq \frac{1+(\beta-\gamma)(1-\eta)}{\Delta}\lambda[u] \\ &\quad + \frac{\beta}{\Delta}\int_0^1 k(\eta,s) f(s,u(\alpha(s)))\,ds \\ &\quad + \frac{1}{2}\int_0^1 (1+s)(1-s)^2 f(s,u(\alpha(s)))\,ds \\ &\leq \frac{1+(\beta-\gamma)(1-\eta)}{\Delta}M_1M_2 \\ &\quad + \frac{\beta M_3}{\Delta}\int_0^1 k(\eta,s)\,ds + \frac{5}{24}M_3, \end{aligned} \tag{45}$$

which shows that $T(D)$ is uniformly bounded.

On the other hand, for any $\varepsilon > 0$, since $k(t,s)$ is uniformly continuous on $[0,1] \times [0,1]$, there exists $\delta_1(\varepsilon) > 0$ such that for any $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta_1(\varepsilon)$,

$$|k(t_1,s) - k(t_2,s)| < \frac{\varepsilon}{3M_3}, \quad s \in [0,1]. \tag{46}$$

Let $\delta = \min\{\delta_1(\varepsilon),\ \varepsilon\Delta/3(\beta-\gamma)M_1M_2,\ \varepsilon\Delta/3(\beta-\gamma)M_3\int_0^1 k(\eta,s)ds\}$. Then, for any $u \in D$, $t_1, t_2 \in [0,1]$ with $|t_1 - t_2| < \delta$, we have

$$\begin{aligned} &|(Tu)(t_1) - (Tu)(t_2)| \\ &\quad = \left| \frac{(\beta-\gamma)(t_1-t_2)}{\Delta}\lambda[u] + \frac{(\beta-\gamma)(t_1-t_2)}{\Delta} \right. \\ &\qquad \times \int_0^1 k(\eta,s) f(s,u(\alpha(s)))\,ds \\ &\qquad + \int_0^1 (k(t_1,s) - k(t_2,s)) \\ &\qquad \left. \times f(s,u(\alpha(s)))\,ds \right| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{(\beta-\gamma)|t_1-t_2|}{\Delta}\lambda[u]+\frac{(\beta-\gamma)|t_1-t_2|}{\Delta}\\
&\quad\times\int_0^1 k(\eta,s)f(s,u(\alpha(s)))\,ds\\
&\quad+\int_0^1 |k(t_1,s)-k(t_2,s)|f(s,u(\alpha(s)))\,ds\\
&\leq \frac{(\beta-\gamma)|t_1-t_2|M_1M_2}{\Delta}\\
&\quad+\frac{(\beta-\gamma)|t_1-t_2|M_3}{\Delta}\int_0^1 k(\eta,s)\,ds\\
&\quad+M_3\int_0^1 |k(t_1,s)-k(t_2,s)|\,ds<\varepsilon,
\end{aligned}\tag{47}$$

which shows that $T(D)$ is equicontinuous. It follows from the Arzela-Ascoli theorem that $T(D)$ is relatively compact. Thus, we have shown that T is a compact operator.

Finally, we prove that T is continuous. Suppose that u_n, $u\in K$ and $\lim_{n\to\infty}u_n=u$. Then, there exists $M_4>0$ such that $\|u\|\leq M_4$ and $\|u_n\|\leq M_4$ $(n=1,2,\ldots)$. For any $\varepsilon>0$, since $f(s,x)$ is uniformly continuous on $[0,1]\times[0,M_4]$, there exists $\delta>0$ such that for any $x_1,x_2\in[0,M_4]$ with $|x_1-x_2|<\delta$,

$$|f(s,x_1)-f(s,x_2)|<\frac{\varepsilon}{(2\beta/\Delta)\int_0^1 k(\eta,s)\,ds+(5/12)},\quad s\in[0,1].\tag{48}$$

At the same time, since $\lim_{n\to\infty}u_n=u$, there exists positive integer N such that for any $n>N$,

$$\|u_n-u\|<\min\left\{\delta,\frac{\varepsilon\Delta}{2[1+(\beta-\gamma)(1-\eta)]|\Lambda(1)-\Lambda(0)|}\right\}.\tag{49}$$

It follows from (48) and (49) that for any $n>N$,

$$\begin{aligned}
\|Tu_n-Tu\|&=\max_{t\in[0,1]}|(Tu_n)(t)-(Tu)(t)|\\
&\leq\frac{1+(\beta-\gamma)(1-\eta)}{\Delta}|\lambda[u_n]-\lambda[u]|\\
&\quad+\frac{\beta}{\Delta}\int_0^1 k(\eta,s)|f(s,u_n(\alpha(s)))\\
&\qquad-f(s,u(\alpha(s)))|\,ds\\
&\quad+\frac{1}{2}\int_0^1(1+s)(1-s)^2|f(s,u_n(\alpha(s)))\\
&\qquad-f(s,u(\alpha(s)))|\,ds\\
&\leq\frac{1+(\beta-\gamma)(1-\eta)}{\Delta}\|u_n-u\||\Lambda(1)-\Lambda(0)|\\
&\quad+\int_0^1\left(\frac{\beta}{\Delta}k(\eta,s)+\frac{1}{2}(1+s)(1-s)^2\right)\\
&\qquad\times|f(s,u_n(\alpha(s)))-f(s,u(\alpha(s)))|\,ds<\varepsilon,
\end{aligned}\tag{50}$$

which indicates that T is continuous. Therefore, $T:K\to K$ is completely continuous. Similarly, we can prove that $S:K\to K$ is also completely continuous. □

3. Main Results

For convenience, we define

$$\begin{aligned}
f^0&=\limsup_{x\to0^+}\max_{t\in[0,1]}\frac{f(t,x)}{x},\qquad f_0=\liminf_{x\to0^+}\min_{t\in[\eta,1]}\frac{f(t,x)}{x},\\
f^\infty&=\limsup_{x\to+\infty}\max_{t\in[0,1]}\frac{f(t,x)}{x},\qquad f_\infty=\liminf_{x\to+\infty}\min_{t\in[\eta,1]}\frac{f(t,x)}{x},\\
H_1&=\frac{1+(\beta-\gamma)(1-\eta)}{\Delta-\rho}\left[\frac{\rho'}{\Delta}\int_0^1 k(\eta,s)\,ds+\int_0^1\kappa(s)\,ds\right]\\
&\quad+\frac{\beta}{\Delta}\int_0^1 k(\eta,s)\,ds+\frac{5}{24},\\
H_2&=\Gamma\left\{\frac{1}{\Delta-\rho}\left[\frac{\rho'}{\Delta}\int_\eta^1 k(\eta,s)\,ds+\int_\eta^1\kappa(s)\,ds\right]\right.\\
&\quad\left.+\frac{1}{\Delta}\int_\eta^1 k(\eta,s)\,ds\right\}.
\end{aligned}\tag{51}$$

Theorem 7. *If $H_1f^0<1<H_2f_\infty$, then the BVP (3) has at least one positive solution.*

Proof. Since $H_1f^0<1$, there exists $\varepsilon_1>0$ such that

$$H_1\left(f^0+\varepsilon_1\right)\leq1.\tag{52}$$

By the definition of f^0, we may choose $\rho_1>0$ so that

$$f(t,x)\leq\left(f^0+\varepsilon_1\right)x,\quad t\in[0,1],\ x\in[0,\rho_1].\tag{53}$$

Let $\Omega_1=\{u\in C[0,1]:\|u\|<\rho_1\}$. Then, for any $u\in K\cap\partial\Omega_1$,

$$0\leq u(t)\leq\|u\|=\rho_1,\quad t\in[0,1],\tag{54}$$

which together with (52) and (53) implies that

$$\begin{aligned}
(Su)(t) &= \frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta-\rho} \\
&\quad\times\left[\frac{\rho'}{\Delta}\int_0^1 k(\eta,s)f(s,u(\alpha(s)))\,ds\right. \\
&\qquad\left.+\int_0^1 \kappa(s)f(s,u(\alpha(s)))\,ds\right] \\
&\quad+\frac{\gamma+(\beta-\gamma)t}{\Delta}\int_0^1 k(\eta,s)f(s,u(\alpha(s)))\,ds \\
&\quad+\int_0^1 k(t,s)f(s,u(\alpha(s)))\,ds \\
&\le \frac{1+(\beta-\gamma)(1-\eta)}{\Delta-\rho} \\
&\quad\times\left[\frac{\rho'}{\Delta}\int_0^1 k(\eta,s)f(s,u(\alpha(s)))\,ds\right. \\
&\qquad\left.+\int_0^1 \kappa(s)f(s,u(\alpha(s)))\,ds\right] \\
&\quad+\frac{\beta}{\Delta}\int_0^1 k(\eta,s)f(s,u(\alpha(s)))\,ds \\
&\quad+\frac{1}{2}\int_0^1 (1+s)(1-s)^2 f(s,u(\alpha(s)))\,ds \\
&\le \left(f^0+\varepsilon_1\right)\left\{\frac{1+(\beta-\gamma)(1-\eta)}{\Delta-\rho}\right. \\
&\qquad\times\left[\frac{\rho'}{\Delta}\int_0^1 k(\eta,s)u(\alpha(s))\,ds\right. \\
&\qquad\left.+\int_0^1 \kappa(s)u(\alpha(s))\,ds\right] \\
&\qquad+\frac{\beta}{\Delta}\int_0^1 k(\eta,s)u(\alpha(s))\,ds \\
&\qquad\left.+\frac{1}{2}\int_0^1 (1+s)(1-s)^2 u(\alpha(s))\,ds\right\} \\
&\le \left(f^0+\varepsilon_1\right)\|u\|\left\{\frac{1+(\beta-\gamma)(1-\eta)}{\Delta-\rho}\right. \\
&\qquad\times\left[\frac{\rho'}{\Delta}\int_0^1 k(\eta,s)\,ds\right. \\
&\qquad\left.+\int_0^1 \kappa(s)\,ds\right] \\
&\qquad\left.+\frac{\beta}{\Delta}\int_0^1 k(\eta,s)\,ds+\frac{5}{24}\right\} \\
&= H_1\left(f^0+\varepsilon_1\right)\|u\| \le \|u\|, \quad t\in[0,1].
\end{aligned} \tag{55}$$

This shows that

$$\|Su\| \le \|u\|, \quad u\in K\cap\partial\Omega_1. \tag{56}$$

On the other hand, since $1 < H_2 f_\infty$, there exists $\varepsilon_2 > 0$ such that

$$H_2\left(f_\infty-\varepsilon_2\right) \ge 1. \tag{57}$$

By the definition of f_∞, we may choose $\overline{\rho_2} > 0$ so that

$$f(t,x) \ge \left(f_\infty-\varepsilon_2\right)x, \quad t\in[\eta,1],\ x\in[\overline{\rho_2},+\infty). \tag{58}$$

Let $\rho_2 = \max\{2\rho_1, \overline{\rho_2}/\Gamma\}$ and $\Omega_2 = \{u\in C[0,1] : \|u\| < \rho_2\}$. Then, for any $u\in K\cap\partial\Omega_2$,

$$u(t) \ge \Gamma\|u\| = \Gamma\rho_2 \ge \overline{\rho_2}, \quad t\in[\eta,1], \tag{59}$$

which together with (57) and (58) implies that

$$\begin{aligned}
(Su)(\eta) &= \frac{1}{\Delta-\rho}\left[\frac{\rho'}{\Delta}\int_0^1 k(\eta,s)f(s,u(\alpha(s)))\,ds\right. \\
&\qquad\left.+\int_0^1 \kappa(s)f(s,u(\alpha(s)))\,ds\right] \\
&\quad+\frac{1}{\Delta}\int_0^1 k(\eta,s)f(s,u(\alpha(s)))\,ds \\
&\ge \frac{1}{\Delta-\rho}\left[\frac{\rho'}{\Delta}\int_\eta^1 k(\eta,s)f(s,u(\alpha(s)))\,ds\right. \\
&\qquad\left.+\int_\eta^1 \kappa(s)f(s,u(\alpha(s)))\,ds\right] \\
&\quad+\frac{1}{\Delta}\int_\eta^1 k(\eta,s)f(s,u(\alpha(s)))\,ds
\end{aligned}$$

$$\begin{aligned}
&\geq (f_\infty - \varepsilon_2)\left\{\frac{1}{\Delta-\rho}\left[\frac{\rho'}{\Delta}\int_\eta^1 k(\eta,s)\,u(\alpha(s))\,ds\right.\right.\\
&\qquad\left.\left.+\int_\eta^1 \kappa(s)\,u(\alpha(s))\,ds\right]\right.\\
&\qquad\left.+\frac{1}{\Delta}\int_\eta^1 k(\eta,s)\,u(\alpha(s))\,ds\right\}\\
&\geq (f_\infty - \varepsilon_2)\,\Gamma\|u\|\left\{\frac{1}{\Delta-\rho}\left[\frac{\rho'}{\Delta}\int_\eta^1 k(\eta,s)\,ds\right.\right.\\
&\qquad\left.\left.+\int_\eta^1 \kappa(s)\,ds\right]\right.\\
&\qquad\left.+\frac{1}{\Delta}\int_\eta^1 k(\eta,s)\,ds\right\}\\
&= H_2(f_\infty - \varepsilon_2)\,\|u\| \geq \|u\|.
\end{aligned} \tag{60}$$

This indicates that

$$\|Su\| \geq \|u\|, \quad u \in K \cap \partial\Omega_2. \tag{61}$$

Therefore, it follows from (56), (61), and Theorem 1 that the operator S has one fixed point $u \in K \cap (\overline{\Omega_2} \setminus \Omega_1)$, which is a positive solution of the BVP (3). □

Theorem 8. *If $H_1 f^\infty < 1 < H_2 f_0$, then the BVP (3) has at least one positive solution.*

Proof. Since $H_2 f_0 > 1$, there exists $\varepsilon_3 > 0$ such that

$$H_2(f_0 - \varepsilon_3) \geq 1. \tag{62}$$

By the definition of f_0, we may choose $\rho_3 > 0$ so that

$$f(t,x) \geq (f_0 - \varepsilon_3)\,x, \quad t \in [\eta,1],\ x \in [0,\rho_3]. \tag{63}$$

Let $\Omega_3 = \{u \in C[0,1] : \|u\| < \rho_3\}$. Then, for any $u \in K \cap \partial\Omega_3$,

$$\Gamma\|u\| \leq u(t) \leq \|u\| = \rho_3, \quad t \in [\eta,1], \tag{64}$$

which together with (62) and (63) implies that

$$\begin{aligned}
(Su)(\eta) &= \frac{1}{\Delta-\rho}\left[\frac{\rho'}{\Delta}\int_0^1 k(\eta,s)\,f(s,u(\alpha(s)))\,ds\right.\\
&\qquad\left.+\int_0^1 \kappa(s)\,f(s,u(\alpha(s)))\,ds\right]\\
&\quad+\frac{1}{\Delta}\int_0^1 k(\eta,s)\,f(s,u(\alpha(s)))\,ds\\
&\geq \frac{1}{\Delta-\rho}\left[\frac{\rho'}{\Delta}\int_\eta^1 k(\eta,s)\,f(s,u(\alpha(s)))\,ds\right.\\
&\qquad\left.+\int_\eta^1 \kappa(s)\,f(s,u(\alpha(s)))\,ds\right]\\
&\quad+\frac{1}{\Delta}\int_\eta^1 k(\eta,s)\,f(s,u(\alpha(s)))\,ds\\
&\geq (f_0 - \varepsilon_3)\left\{\frac{1}{\Delta-\rho}\left[\frac{\rho'}{\Delta}\int_\eta^1 k(\eta,s)\,u(\alpha(s))\,ds\right.\right.\\
&\qquad\left.\left.+\int_\eta^1 \kappa(s)\,u(\alpha(s))\,ds\right]\right.\\
&\qquad\left.+\frac{1}{\Delta}\int_\eta^1 k(\eta,s)\,u(\alpha(s))\,ds\right\}\\
&\geq (f_0 - \varepsilon_3)\,\Gamma\|u\|\left\{\frac{1}{\Delta-\rho}\left[\frac{\rho'}{\Delta}\int_\eta^1 k(\eta,s)\,ds\right.\right.\\
&\qquad\left.\left.+\int_\eta^1 \kappa(s)\,ds\right]\right.\\
&\qquad\left.+\frac{1}{\Delta}\int_\eta^1 k(\eta,s)\,ds\right\}\\
&= H_2(f_0 - \varepsilon_3)\,\|u\| \geq \|u\|.
\end{aligned} \tag{65}$$

This shows that

$$\|Su\| \geq \|u\|, \quad u \in K \cap \partial\Omega_3. \tag{66}$$

On the other hand, since $H_1 f^\infty < 1$, there exists $\varepsilon_4 > 0$ so that

$$H_1(f^\infty + \varepsilon_4) < 1. \tag{67}$$

By the definition of f^∞, we may choose $\overline{\rho_4} > 0$ such that

$$f(t,x) \leq (f^\infty + \varepsilon_4)\,x, \quad t \in [0,1],\ x \in [\overline{\rho_4},+\infty), \tag{68}$$

which implies that

$$f(t,x) \leq M + (f^\infty + \varepsilon_4)\,x, \quad t \in [0,1],\ x \in [0,+\infty), \tag{69}$$

where

$$M = \max\{f(t,x) : t \in [0,1], x \in [0,\overline{\rho_4}]\}. \tag{70}$$

Now, we choose $\rho_4 = \max\{2\rho_3, MH_1/(1-H_1(f^\infty+\varepsilon_4))\}$, and let $\Omega_4 = \{u \in C[0,1] : \|u\| < \rho_4\}$. Then, for any $u \in K \cap \partial\Omega_4$,

$$0 \le u(t) \le \|u\|, \quad t \in [0,1], \tag{71}$$

which together with (69) implies that

$$\begin{aligned}
(Su)(t) &= \frac{1-(\beta-\gamma)\eta+(\beta-\gamma)t}{\Delta-\rho} \\
&\quad\times\left[\frac{\rho'}{\Delta}\int_0^1 k(\eta,s) f(s,u(\alpha(s)))\,ds \right. \\
&\qquad\left. + \int_0^1 \kappa(s) f(s,u(\alpha(s)))\,ds\right] \\
&\quad+ \frac{\gamma+(\beta-\gamma)t}{\Delta}\int_0^1 k(\eta,s) f(s,u(\alpha(s)))\,ds \\
&\quad+ \int_0^1 k(t,s) f(s,u(\alpha(s)))\,ds \\
&\le \frac{1+(\beta-\gamma)(1-\eta)}{\Delta-\rho} \\
&\quad\times\left[\frac{\rho'}{\Delta}\int_0^1 k(\eta,s) f(s,u(\alpha(s)))\,ds \right. \\
&\qquad\left. + \int_0^1 \kappa(s) f(s,u(\alpha(s)))\,ds\right] \\
&\quad+ \frac{\beta}{\Delta}\int_0^1 k(\eta,s) f(s,u(\alpha(s)))\,ds \\
&\quad+ \frac{1}{2}\int_0^1 (1+s)(1-s)^2 f(s,u(\alpha(s)))\,ds \\
&\le \left[M+(f^\infty+\varepsilon_4)\|u\|\right]\left\{\frac{1+(\beta-\gamma)(1-\eta)}{\Delta-\rho}\right. \\
&\qquad\times\left[\frac{\rho'}{\Delta}\int_0^1 k(\eta,s)\,ds \right. \\
&\qquad\left. + \int_0^1 \kappa(s)\,ds\right] \\
&\qquad\left. + \frac{\beta}{\Delta}\int_0^1 k(\eta,s)\,ds + \frac{5}{24}\right\} \\
&= \left[M+(f^\infty+\varepsilon_4)\|u\|\right]H_1 \le \|u\|, \quad t \in [0,1].
\end{aligned} \tag{72}$$

This indicates that

$$\|Su\| \le \|u\|, \quad u \in K \cap \partial\Omega_4. \tag{73}$$

Therefore, it follows from (66), (73), and Theorem 1 that the operator S has one fixed point $u \in K \cap (\overline{\Omega_4} \setminus \Omega_3)$, which is a positive solution of the BVP (3). □

Example 9. Consider the following BVP:

$$\begin{gathered}
u'''(t) + t\left[\frac{3359u(\sqrt{t})}{913e^{u(\sqrt{t})}} + \frac{5881(u(\sqrt{t}))^2}{17(u(\sqrt{t})+1)}\right] = 0, \\
t \in (0,1), \\
u(0) = \frac{1}{4}u\left(\frac{1}{2}\right) + \int_0^1 u(t)(2t-1)\,dt, \qquad u''(0) = 0, \\
u(1) = \frac{1}{2}u\left(\frac{1}{2}\right) + \int_0^1 u(t)(2t-1)\,dt.
\end{gathered} \tag{74}$$

In view of $d\Lambda(t) = (2t-1)dt$, we have

$$\begin{gathered}
\int_0^1 d\Lambda(t) = 0, \quad \int_0^1 t\,d\Lambda(t) = \frac{1}{6}, \quad \kappa(s) = \frac{1}{12}s^2(1-s)^2, \\
s \in [0,1].
\end{gathered} \tag{75}$$

At the same time, since $\gamma = 1/4$ and $\beta = \eta = 1/2$, a simple calculation shows that

$$\begin{gathered}
\Delta = \frac{5}{8}, \qquad \rho = \rho' = \frac{1}{24}, \qquad \Gamma = \frac{2}{7}, \\
\int_0^1 k(\eta,s)\,ds = \frac{1}{16}, \qquad \int_\eta^1 k(\eta,s)\,ds = \frac{1}{96}, \\
\int_0^1 \kappa(s)\,ds = \frac{1}{360}, \qquad \int_\eta^1 \kappa(s)\,ds = \frac{1}{720}.
\end{gathered} \tag{76}$$

So,

$$H_1 = \frac{913}{3360}, \qquad H_2 = \frac{17}{2940}. \tag{77}$$

If we let $f(t,x) = t[(3359x/913e^x) + (5881x^2/17(x+1))]$, $(t,x) \in [0,1] \times [0,+\infty)$, then it is easy to compute that

$$f^0 = \frac{3359}{913}, \qquad f_\infty = \frac{5881}{34}, \tag{78}$$

which together with (77) implies that

$$H_1 f^0 = \frac{3359}{3360} < 1 < H_2 f_\infty = \frac{5881}{5880}. \tag{79}$$

Therefore, it follows from Theorem 7 that the BVP (74) has at least one positive solution.

Acknowledgment

This paper is supported by the Natural Science Foundation of Gansu Province of China (1208RJZA240).

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