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Research Article

Option Pricing under Risk-Minimization Criterion in an Incomplete Market with the Finite Difference Method

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We study option pricing with risk-minimization criterion in an incomplete market where the dynamics of the risky underlying asset is governed by a jump diffusion equation with stochastic volatility. We obtain the Radon-Nikodym derivative for the minimal martingale measure and a partial integro-differential equation (PIDE) of European option. The finite difference method is employed to compute the European option valuation of PIDE.

1. Introduction

Option pricing problem is one of the predominant concerns in the financial market. Since the advent of the justly celebrated Black-Scholes option pricing formula in [1], there has been an increasing amount of the literature describing the theory and its practice. In the Black-Scholes model, the appreciation rate and the volatility rate are assumed to be constants. However, more and more empirical evidence has revealed that these assumptions are not consistent with reality. Hence, many option valuation models which relax some of the restrictive assumptions in the Black-Scholes framework have been proposed and tested, such as the stochastic volatility models in [2–8], the jump diffusion or the Lévy process models in [9–12]. The model to be studied in this paper takes into account not only stochastic volatility based on Heston's model in [8], but also the jump diffusion case. Therefore, several new problems and difficulties are exposed in our model.

Different from the Black-Scholes framework which uses jump diffusion to describe the price dynamics of underlying asset, the market of our model is incomplete; that is, it is not possible to replicate the payoff of every contingent claim by a portfolio, and there are several equivalent martingale measures. How to choose a consistent pricing measure from the set of equivalent martingale measures becomes an important problem. This means that we need to find some criterions to determine one from the set of equivalent martingale measures in some economically or mathematically motivated fashion. Föllmer and Leukert (2000), Kallsen (1999), Cvitanić et al. (2001), and Bielecki and Jeanblanc (2008) in [13–16] identified a unique equivalent martingale measure by utility maximization. Then, the option valuation under the minimal martingale measure was further developed by several researchers. Schweizer (1991) and Föllmer and Schweizer (1991) in [17, 18] found that under the minimal martingale measure, a unique risk-minimizing (or optimal) strategy hedging of contingent claims in incomplete market exists. In our paper, we name the criterion under the minimal martingale measure as a *risk-minimization criterion*. Thus, in the incomplete market, option pricing is approximately possible with risk-minimization criterion. As presented in this paper, our work is based on the task of Föllmer and Schweizer in [17, 18], and the purpose of this paper is to find the minimal martingale measure and the measure switch of asset prices processes with stochastic volatility and jump diffusion. By employing the minimal martingale measure, we obtain the Radon-Nikodym derivative and a partial integro-differential equation (PIDE) for the European option.

However, it is difficult to get the exact solution of the PIDE in our model. Several numerical methods have been proposed to solve the PIDE approximately. The methods

include numerical integration by Chiarella and Ziogas (2009) in [19], finite elements by Matache et al. (2005) in [20], the method of lines by Meyer (1998) in [21], and the finite difference methods (FDM) including those by Carr and Hirsa (2003) in [22], d'Halluin et al. (2004) in [23], and Briani et al. (2004) in [24]. This paper employs the FDM to compute the valuation of European option, and American option. Then further studies on using other methods to compute the solution of PIDE and derive the pricing formula of another option will be included in our future study.

The rest of the paper is organized as follows. In Section 2, we present the model for the underlying market and Doob-Meyer decomposition of the risky asset. In Section 3, we investigate an explicit representation of the density process of the minimal martingale measure. In Section 4, we derive a PIDE for the European option. The European option pricing with FDM is then studied in Section 5. Numerical results are shown in Section 6, and conclusions are given in Section 7.

2. The Model

In this paper, we consider the financial market with the following two basic assets:

(i) *a Bond* whose price B_t at time t is given by

$$dB_t = r_t B_t dt, \quad B_0 = 1; \tag{1}$$

(ii) *a Stock* whose price S_t at time t is given by

$$\frac{dS_t}{S_{t^-}} = \mu_t dt + \sqrt{V_t} dW_t^S + \int_{R_0} (y-1)\,\overline{N}(du,dy), \quad S_0 > 0,$$
$$dV_t = \kappa(\varphi - V_t)\,dt + \sigma\sqrt{V_t} dW_t^V, \quad V_0 > 0. \tag{2}$$

Here, $t \in [0,T]$, $y \in R_0 \subset \mathbb{R}/\{0\}$, and $T > 0$; on the filtered complete space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, there are W_t^S, W_t^V which both are 1-dimensional Brownian motions with $dW_t^S dW_t^V = \rho dt$ and $\overline{N}(dt,dy) = N(dt,dy) - \nu(dy)dt$ which is the compensated jump measure of $\eta(\cdot)$. $N(dt,dy)$ is the jump measure, and $\nu(dy)$ is the Lévy measure of the Lévy process $\eta(\cdot)$. $\eta(t)$ is given by $\eta(t) = \int_0^t \int_{R_0} (y-1)\overline{N}(du,dy)$, $t \geq 0$. Additionally, $\int_{R_0} (y-1)^2 \nu(dy) < \infty$.

From above settings, the risk is at least two dimensional including the Brownian motion and Lévy process but only one risky asset in the market. It is impossible to exactly replicate the payoff of a given option by a dynamic portfolio strategy of the basic assets which is self-financing. We all know that if the dynamic portfolio strategy exists, then the initial cost of the portfolio must equal the price of the option, and the market is called complete; otherwise, an arbitrage opportunity exists, and the market is called incomplete. Therefore, the market of our model is incomplete since the perfect replication is impossible.

Compared with the Black-Scholes model, our model remedies some serious drawbacks of the Black-Scholes model, such as the constant volatility assumption and market completeness assumption. Bakshi et al. in [10] systematically analyzed the performance of the stochastic volatility model, the jump diffusion model, and the stochastic interest rate model, and concluded that a model with stochastic volatility and the jump diffusion was a better alternative to the Black-Scholes model, because the former not only performed far better but also was practically implementable. An excellent innovation of our model is combining stochastic volatility model and the jump diffusion model. Consequently, our model is better than the Black-Scholes model to portray realistic financial markets in theory and practice.

Now, we want to find a unique optimal strategy hedging of contingent claims under *risk-minimization criterion*. Based on [18], it is equivalent to find the minimal martingale measure from the set of equivalent martingale measures and then obtain approximate prices of contingent claims.

With the Doob-Meyer decomposition, the discounted risky asset price process, $\widehat{S}_t = e^{-\int_0^t r_s ds} S_t$, is a special semimartingale and can be written as

$$\widehat{S}_t = \widehat{S}_0 + M_t + A_t, \tag{3}$$

with

$$M_t = \int_0^t \widehat{S}_{u^-} \sqrt{V_u} dW_u^S + \int_0^t \int_{R_0} \widehat{S}_{u^-} y \overline{N}(du,dy), \tag{4}$$

$$A_t = \int_0^t \widehat{S}_{u^-} (\mu_u - r_u)\,du, \tag{5}$$

where M_t is the martingale part of $\widehat{S}_t$ and A_t is the predictable process of finite variation.

3. Minimal Martingale Measure

We introduce the nations of minimal martingale measure in this section. Föllmer and Schweizer (1991) [18] noticed that the optimal hedging strategy can be computed in terms of the minimal martingale measure. Furthermore, it is uniquely determined. Hence, under the minimal martingale measure, the Radon-Nikodym derivative can be found and computed. Before that, we define the minimal martingale measure.

Definition 1 (see [18]). A local martingale measure $\widetilde{P}$, equivalent to the original measure P, is called minimal if $\widetilde{P} = P$ on $\mathcal{F}_t$ and if any square-integrable P-martingale L which is P orthogonal to M remains a local martingale under $\widetilde{P}$.

Theorem 2 (see [18]). *(i) The minimal martingale measure $\widetilde{P}$ is uniquely determined.*

(ii) $\widetilde{P}$ exists if and only if there exists a predictable process β_t that satisfies

$$Z_t = \frac{d\widetilde{P}}{dP} = 1 + \int_0^t \beta_t dM_t. \tag{6}$$

Using Theorem 2, we obtain the following theorem for computing the Radon-Nikodym derivative.

Theorem 3. *The Radon-Nikodym derivative under the minimal martingale measure $\widetilde{P}$ is*

$$\begin{aligned} Z_t = \exp\Bigg\{ & -\int_0^t \theta_t \sqrt{V_u} dW_u^S - \frac{1}{2}\int_0^t \theta_u^2 V_u du \\ & + \int_0^t \int_{R_0} \ln\left(1-\theta_u(y-1)\right) N(du,dy) \\ & + \int_0^t \int_{R_0} \theta_u (y-1)\, \nu(dy)\, du \Bigg\}. \end{aligned} \tag{7}$$

Proof. The theory of the Girsanov transformation shows that the predictable process of bounded variation can also be computed in terms of Z_t:

$$-dA_t = \frac{1}{Z_{t-}} d\langle M, Z\rangle_t. \tag{8}$$

Throughout this paper, we make use of the notations that $\langle A, B\rangle_t$ defined the quadratic variation process between A and B and denote $\langle A\rangle_t \doteq \langle A, A\rangle_t$. Under $\widetilde{P}$, the predictable process of bounded variation in the Doob-Meyer decomposition of M is given by

$$\frac{1}{Z_{t-}} d\langle M, Z\rangle_t = \frac{1}{Z_{t-}} d\langle M\rangle_t = -dA_t. \tag{9}$$

Using (6) and (9), we have

$$Z_t = 1 - \int_0^t Z_{u-} \frac{dA_u}{d\langle M\rangle_u} dM_u. \tag{10}$$

Denote $dY_u = -(dA_u/d\langle M\rangle_u)dM_u$, then (10) can be written as

$$Z_t = 1 + \int_0^t Z_{u-} dY_u. \tag{11}$$

From (4), we get

$$\begin{aligned} \langle M\rangle_t &= \Bigg\langle \int_0^t \widehat{S}_{u^-} \sqrt{V_u} dW_u^S \\ &\quad + \int_0^t \int_{R_0} \widehat{S}_{u^-} (y-1) \overline{N}(du,dy) \Bigg\rangle \\ &= \int_0^t \widehat{S}_{u^-}^2 \left(\sqrt{V_u}\right)^2 du \\ &\quad + \int_0^t \int_{R_0} \widehat{S}_{u^-}^2 (y-1)^2 \nu(dy)\, du \\ &= \int_0^t \widehat{S}_{u^-}^2 \left(V_u + \int_{R_0} (y-1)^2 \nu(dy)\right) du. \end{aligned} \tag{12}$$

Hence,

$$\begin{aligned} dY_u &= -\frac{dA_u}{d\langle M\rangle_u} dM_u \\ &= -\frac{\widehat{S}_{u^-}(\mu_u - r_u)\, du}{\widehat{S}_{u^-}^2 \left(\nu_u + \int_{R_0} (y-1)^2 \nu(dy)\right) du} \\ &\quad \times \left[\widehat{S}_{u^-}\left(\sqrt{V_u} dW_u^S + \int_{R_0} y\overline{N}(du,dy)\right)\right] \\ &= -\frac{(\mu_u - r_u)\left(\sqrt{V_u} dW_u^S + \int_{R_0} (y-1)\overline{N}(du,dy)\right)}{V_u + \int_{R_0} (y-1)^2 \nu(dy)}. \end{aligned} \tag{13}$$

From (11), we know that Z_t is the Doléans-Dade exponential. Thus, we obtain

$$\begin{aligned} Z_t &= 1 + \int_0^t Z_{u-} dY_u, \quad Z_0 = 1, \\ dY_u &= -\theta_u \left(\sqrt{V_u} dW_u^S + \int_{R_0} (y-1)\overline{N}(du,dy)\right), \end{aligned} \tag{14}$$

where

$$\theta_u = \frac{(\mu_u - r_u)}{V_u + \int_{R_0} (y-1)^2 \nu(dy)}. \tag{15}$$

Solving (14), we obtain Z_t in Theorem 3. □

Remark 4. The Brown motions under the minimal martingale measure $\widetilde{P}$ are

$$\begin{aligned} \widetilde{W}_t^S &= W_t^S + \int_0^t \theta_t \sqrt{V_u} du, \\ \widetilde{W}_t^V &= W_t^V + \rho \int_0^t \theta_t \sqrt{V_u} du, \end{aligned} \tag{16}$$

and the compensatory of $N(du, dy)$ is

$$\begin{aligned} \widetilde{\nu}(dy)\, du &= \left(1 - \theta_u (y-1)\right) \nu(dy)\, du, \\ \widetilde{N}(du,dy) &= N(du,dy) - \widetilde{\nu}(dy)\, du. \end{aligned} \tag{17}$$

Remark 5. Equation (2) under the minimal martingale measure $\widetilde{P}$ is written as

$$\begin{aligned} \frac{dS_t}{S_{t^-}} &= \mu_t dt + \sqrt{V_t} d\widetilde{W}_t^S \\ &\quad + \int_{R_0} (y-1)\widetilde{N}(dt,dy) - \theta_t V_t dt \\ &\quad - \int_{R_0} \theta_t (y-1)^2 \nu(dy)\, dt, \quad S_0 > 0, \\ dV_t &= \kappa(\varphi - V_t)\, dt + \sigma\sqrt{V_t} d\widetilde{W}_t^V - \rho\sigma\theta_t V_t dt. \end{aligned} \tag{18}$$

To guarantee that $\widehat{S}_t$ is a martingale under the minimal martingale measure $\widetilde{P}$, the following corollary is necessary.

Corollary 6. *Under the minimal martingale measure $\widetilde{P}$, $\widehat{S}_t$ is a martingale if and only if*

$$\mu_t - r_t - \theta_t V_t - \int_{R_0} \theta_t (y-1)^2 \nu(dy) = 0. \tag{19}$$

Proof. Substituting $S_t = e^{\int_0^t r_s ds}\widehat{S}_t$ into (18), since $\widehat{S}_t$ is a martingale, the drift term must be identical to zero. Then, we can get (19). □

4. Partial Integro-Differential Equation for European Call Option

Under the minimal martingale measure $\widetilde{P}$, the price of the European call option $C(t, S_t, V_t)$ at time t with strike price K and maturity date T is given by

$$C(t, S_t, V_t) = \mathbb{E}^{\widetilde{P}}\left[e^{-\int_t^T r_u du}(S_T - K)^+ \mid \mathcal{F}_t\right], \tag{20}$$

and $C(T, S_T, V_T) = (S_T - K)^+$.

By the fact that the discounted price of the European call option is a martingale under $\widetilde{P}$, we can obtain the following theorem.

Theorem 7. *The price of the European call option satisfies the following PIDE:*

$$\begin{aligned} 0 = & -r_t C(t, S_t, V_t) + \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S} r_t S_{t^-} \\ & + \frac{\partial C}{\partial V}(\kappa(\varphi - V_t) - \rho\sigma\theta_t V_t) \\ & + \frac{1}{2}\frac{\partial^2 C}{\partial S^2} V_t S_{t^-}^2 + \frac{\partial^2 C}{\partial S \partial V} S_{t^-} V_t \sigma\rho \\ & + \frac{1}{2}\frac{\partial^2 C}{\partial V^2} V_t \sigma^2 + \int_{R_0}\Big(C(t, S_{t^-}y, V_{t^-}) - C(t, S_{t^-}, V_{t^-}) \\ & \qquad - (y-1)\frac{\partial C}{\partial S} S_{t^-}\Big)\widetilde{\nu}(dy), \end{aligned} \tag{21}$$

and $C(T, S_T, V_T) = (S_T - K)^+$.

Proof. The total derivative of the discounted option price is

$$\begin{aligned} & d\left(e^{-\int_0^t r_u du} C(t, S_t, V_t)\right) \\ & = -r_t e^{-\int_0^t r_u du} C(t, S_t, V_t)\, dt \\ & \quad + e^{-\int_0^t r_u du}\frac{\partial C}{\partial t} dt + e^{-\int_0^t r_u du}\frac{\partial C}{\partial S} dS^C \\ & \quad + e^{-\int_0^t r_u du}\frac{\partial C}{\partial V} dV^C + \frac{1}{2} e^{-\int_0^t r_u du}\frac{\partial^2 C}{\partial S^2} dS^C dS^C \\ & \quad + e^{-\int_0^t r_u du}\frac{\partial^2 C}{\partial S \partial V} dS^C dV^C + \frac{1}{2} e^{-\int_0^t r_u du}\frac{\partial^2 C}{\partial V^2} dV^C dV^C \\ & \quad + e^{-\int_0^t r_u du}\int_{R_0}(C(t, S_{t^-}y, V_{t^-}) \\ & \qquad - C(t, S_{t^-}, V_{t^-}))\, N(dt, dy) \\ & = e^{-\int_0^t r_u du}\Big\{-r_t C(t, S_t, V_t)\, dt + \frac{\partial C}{\partial t} dt \\ & \qquad + \frac{\partial C}{\partial S} S_{t^-}\Big(\mu_t dt + \sqrt{V_t} d\widetilde{W}_t^S - \theta_t V_t dt \\ & \qquad\qquad - \int_{R_0}\theta_t (y-1)^2 \nu(dy)\, dt \\ & \qquad\qquad - \int_{R_0}(y-1)\widetilde{\nu}(dy)\, dt\Big) \\ & \qquad + \frac{\partial C}{\partial V}\Big(\kappa(\varphi - V_t)\, dt + \sigma\sqrt{V_t} d\widetilde{W}_t^V \\ & \qquad\qquad - \rho\sigma\theta_t V_t dt\Big) \\ & \qquad + \frac{1}{2}\frac{\partial^2 C}{\partial S^2} V_t S_{t^-}^2 dt + \frac{\partial^2 C}{\partial S \partial V} S_{t^-} V_t \sigma\rho dt \\ & \qquad + \frac{1}{2}\frac{\partial^2 C}{\partial V^2} V_t \sigma^2 dt \\ & \qquad + \int_{R_0}(C(t, S_{t^-}y, V_{t^-}) \\ & \qquad\qquad - C(t, S_{t^-}, V_{t^-}))\, \widetilde{N}(dt, dy) \\ & \qquad + \int_{R_0}(C(t, S_{t^-}y, V_{t^-}) \\ & \qquad\qquad - C(t, S_{t^-}, V_{t^-}))\, \widetilde{\nu}(dy)\, dt\Big\} \\ & = e^{-\int_0^t r_u du}\Big\{-r_t C(t, S_t, V_t) + \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S} r_t S_{t^-} \\ & \qquad + \frac{\partial C}{\partial V}(\kappa(\varphi - V_t) - \rho\sigma\theta_t V_t) \\ & \qquad + \frac{1}{2}\frac{\partial^2 C}{\partial S^2} V_t S_{t^-}^2 + \frac{\partial^2 C}{\partial S \partial V} S_{t^-} V_t \sigma\rho \\ & \qquad + \frac{1}{2}\frac{\partial^2 C}{\partial V^2} V_t \sigma^2 \\ & \qquad + \int_{R_0}\Big(C(t, S_{t^-}y, V_{t^-}) \\ & \qquad\qquad - C(t, S_{t^-}, V_{t^-}) \\ & \qquad\qquad - (y-1)\frac{\partial C}{\partial S} S_{t^-}\Big)\widetilde{\nu}(dy)\Big\}\, dt \\ & + e^{-\int_0^t r_u du}\Big\{\frac{\partial C}{\partial S} S_{t^-}\sqrt{v_t} d\widetilde{W}_t^S + \frac{\partial C}{\partial V}\sigma\sqrt{V_t} d\widetilde{W}_t^V \\ & \qquad + \int_{R_0}(C(t, S_{t^-}y, V_{t^-}) \\ & \qquad\qquad - C(t, S_{t^-}, V_{t^-}))\, \widetilde{N}(dt, dy)\Big\}. \end{aligned} \tag{22}$$

We make the drift term zero, since the discounted price of the European put option is a martingale. Then, we obtain in Theorem 7. □

5. Finite Difference Scheme of European Call Option

5.1. The Problem. In this section, we will derive the finite difference scheme for PIDE (21). Before that, we assume that $r_t = r$, $\mu_t = \mu$, $R_0 = [0, \infty]$ and $\nu(dy)dt = \lambda f(y)dydt$. In this paper, we take Merton's jump diffusion model with the density $f(y)$ which is from the log-normal distribution:

$$f(y) = \frac{1}{y\delta\sqrt{2\pi}} \exp\left(-\frac{(\ln y - \gamma)^2}{2\delta^2}\right). \tag{23}$$

Furthermore, using error function, we denote that

$$\begin{aligned} K_0 &:= \int_{R_0} \nu(dy) = 1, \\ K_1 &:= \int_{R_0} (y-1)\nu(dy) = e^{(\delta^2/2)+\gamma} - 1, \\ K_2 &:= \int_{R_0} (y-1)^2 \nu(dy) \\ &= \frac{1}{\sqrt{1-2\delta^2}} e^{\gamma^2/(1-2\delta^2)} - 2e^{(\delta^2/2)+\gamma} + 1, \\ \theta_u &:= \frac{\mu - r}{V_u + \int_{R_0} (y-1)^2 \nu(dy)} = \frac{\mu - r}{V_u + K_2}, \end{aligned} \tag{24}$$

where r, μ, λ, δ, γ, K_0, K_1, and K_2 are constants, and $\delta^2 < 1/2$.

From (21) and (19), we get the PIDE with original parameters in the original model (2) as follows:

$$\begin{aligned} 0 = &-(r_t + K_0 - \theta_t K_1) C(t, S_t, V_t) \\ &+ \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}(r_t - K_1 + \theta_t K_2) S_{t^-} \\ &+ \frac{\partial C}{\partial V}(\kappa(\varphi - V_t) - \rho\sigma\theta_t V_t) \\ &+ \frac{1}{2}\frac{\partial^2 C}{\partial S^2} V_t S_{t^-}^2 + \frac{\partial^2 C}{\partial S \partial V} S_{t^-} V_t \sigma\rho + \frac{1}{2}\frac{\partial^2 C}{\partial V^2} V_t \sigma^2 \\ &+ \int_{R_0} C(t, S_{t^-} y, V_{t^-}) \nu(dy) \\ &- \theta_t \int_{R_0} C(t, S_{t^-} y, V_{t^-})(y-1)\nu(dy). \end{aligned} \tag{25}$$

To represent conveniently, we denote that

$$\begin{aligned} \mathrm{I}(t, S, V) &:= \int_0^\infty C(t, S_{t^-}, V_{t^-}) f(y)\, dy, \\ \mathrm{II}(t, S, V) &:= \int_0^\infty C(t, S_{t^-}, V_{t^-})(y-1) f(y)\, dy. \end{aligned} \tag{26}$$

Then, (25) can be written as

$$\begin{aligned} 0 = &-(r + K_0 - \theta K_1) C(t, S, V) + \frac{\partial C}{\partial t} \\ &+ \frac{\partial C}{\partial S}(r - K_1 + \theta K_2) S \\ &+ \frac{\partial C}{\partial V}(\kappa(\varphi - V) - \rho\sigma\theta V) \\ &+ \frac{1}{2}\frac{\partial^2 C}{\partial S^2} V S^2 + \frac{\partial^2 C}{\partial S \partial V} S V \sigma\rho \\ &+ \frac{1}{2}\frac{\partial^2 C}{\partial V^2} V\sigma^2 + \lambda \mathrm{I}(t, S, V) - \lambda\theta \mathrm{II}(t, S, V), \\ &C(T, S, V) = (S - K)^+, \\ &C(t, 0, V) = 0, \\ &\lim_{S\to\infty} C(t, S, V) = (S_{\max} - K) e^{-r(T-t)}, \\ &\lim_{V\to\infty} C(t, S, V) = S, \\ 0 = &-\left(r + K_0 - \frac{(\mu - r) K_1}{K_2}\right) C(t, S, 0) \\ &+ \frac{\partial C}{\partial t} + \frac{\partial C}{\partial S}(r - K_1 + (\mu - r)) S \\ &+ \frac{\partial C}{\partial V}\kappa\varphi + \lambda \mathrm{I}(t, S, 0) - \lambda \frac{(\mu - r)}{K_2} \mathrm{II}(t, S, 0). \end{aligned} \tag{27}$$

Now, we denote $\tau := T - t$, $\widetilde{C}(\tau, S, V) := C(T - \tau, S, V) = C(t, S, V)$, $\widetilde{\mathrm{I}}(t, S, V) := \mathrm{I}(T - \tau, S, V) = \mathrm{I}(t, S, V)$, and $\widetilde{\mathrm{II}}(t, S, V) := \mathrm{II}(T - \tau, S, V) = \mathrm{II}(t, S, V)$. Then we have the following problem equivalent to (27).

Problem 1. Consider

$$\begin{aligned} \frac{\partial \widetilde{C}}{\partial \tau} = &-(r + K_0 - \theta K_1) \widetilde{C}(\tau, S, V) \\ &+ \frac{\partial \widetilde{C}}{\partial S}(r - K_1 + \theta K_2) S \\ &+ \frac{\partial \widetilde{C}}{\partial V}(\kappa(\varphi - V) - \rho\sigma\theta V) + \frac{1}{2}\frac{\partial^2 \widetilde{C}}{\partial S^2} V S^2 \\ &+ \frac{\partial^2 \widetilde{C}}{\partial S \partial V} S V \sigma\rho + \frac{1}{2}\frac{\partial^2 \widetilde{C}}{\partial V^2} V\sigma^2 \\ &+ \lambda \widetilde{\mathrm{I}}(\tau, S, V) - \lambda\theta \widetilde{\mathrm{II}}(\tau, S, V), \\ &\widetilde{C}(0, S, V) = (S - K)^+, \\ &\widetilde{C}(\tau, 0, V) = 0, \\ &\lim_{S\to\infty} \widetilde{C}(\tau, S, V) = (S_{\max} - K) e^{-r\tau}, \\ &\lim_{V\to\infty} \widetilde{C}(\tau, S, V) = S, \end{aligned}$$

$$\begin{aligned}0 = & -\left(r + K_0 - \frac{(\mu - r) K_1}{K_2}\right) \widetilde{C}(\tau, S, 0) \\ & + \frac{\partial \widetilde{C}}{\partial \tau} + \frac{\partial \widetilde{C}}{\partial S}(r - K_1 + (\mu - r)) S \\ & + \frac{\partial \widetilde{C}}{\partial V} \kappa\varphi + \lambda \widetilde{\mathrm{I}}(\tau, S, 0) - \lambda \frac{(\mu - r)}{K_2} \widetilde{\mathrm{II}}(\tau, S, 0).\end{aligned} \tag{28}$$

5.2. Discretization. To solve (38), we use a finite difference scheme with the following nation and approximations:

$$\begin{gathered}\tau : 0 \le \cdots n\Delta t \cdots \le (N-1)\Delta t, \quad n = 0, 1, \ldots, N-1, \\ S : 0 \le \cdots i\Delta X \cdots \le (I-1)\Delta X = S_{\max}, \quad i = 0, 1, \ldots, I-1, \\ V : 0 \le \cdots j\Delta V \cdots \le J\Delta V = V_{\max}, \quad j = 0, 1, \ldots, J-1, \\ U_{i,j}^n = \widetilde{C}(n\Delta t, i\Delta X, j\Delta V), \\ \mathrm{I}_{i,j}^n = \widetilde{\mathrm{I}}(n\Delta t, i\Delta X, j\Delta V), \\ \mathrm{II}_{i,j}^n = \widetilde{\mathrm{II}}(n\Delta t, i\Delta X, j\Delta V), \\ \frac{\partial \widetilde{C}}{\partial \tau}(\tau, S, V) \approx \frac{U_{i,j}^{n+1} - U_{i,j}^n}{\Delta t}, \\ \frac{\partial \widetilde{C}}{\partial S}(\tau, S, V) \approx \frac{U_{i+1,j}^n - U_{i-1,j}^n}{2\Delta X}, \\ \frac{\partial^2 \widetilde{C}}{\partial S^2}(\tau, S, V) \approx \frac{U_{i+1,j}^n - 2U_{i,j}^n + U_{i-1,j}^n}{(\Delta X)^2}, \\ \frac{\partial^2 \widetilde{C}}{\partial S \partial V} = \frac{U_{i+1,j+1}^n + U_{i-1,j-1}^n - U_{i-1,j+1}^n - U_{i+1,j-1}^n}{4\Delta X \Delta V}, \\ \frac{\partial \widetilde{C}}{\partial V}(\tau, S, V) \approx \frac{U_{i,j+1}^n - U_{i,j-1}^n}{2\Delta V}, \\ \frac{\partial^2 \widetilde{C}}{\partial V^2}(\tau, S, V) \approx \frac{U_{i,j+1}^n - 2U_{i,j}^n + U_{i,j-1}^n}{(\Delta V)^2}.\end{gathered} \tag{29}$$

Now, we evaluate the integral term. First, by performing a change of variable $z := S \cdot y$, $y = z/S$ and $dy = dz/S$, the $\widetilde{\mathrm{I}}(\tau, S_t, V_t)$ and $\widetilde{\mathrm{II}}(\tau, S_t, V_t)$ can be discretized by using the linear interpolation. Thus,

$$\begin{gathered}z : 0 \le \cdots k\Delta X \cdots \le (I-2)\Delta X, \quad k = 0, 1, \ldots, I-2, \\ \widetilde{\mathrm{I}} = \int_{R_0} \frac{\widetilde{C}(\tau, z, V) f(z/S)}{S dz} \\ \approx \int_0^\infty \frac{\widetilde{C}(n\Delta t, z, j\Delta V) f(z/i\Delta X)}{(i\Delta X) dz}, \\ \widetilde{\mathrm{II}} = \int_{R_0} \frac{\widetilde{C}(\tau, z, V)(z/S - 1) f(z/S)}{S dz} \\ \approx \int_0^\infty \frac{\widetilde{C}(n\Delta t, z, j\Delta V)(z/(i\Delta X) - 1) f(z/i\Delta X)}{(i\Delta X) dz}.\end{gathered} \tag{30}$$

Now, by using linear interpolation, we get approximation the following:

$$\begin{gathered}\mathrm{I}_{i,j}^n \approx A_{i,j}^n = \sum_{k=0}^{I-2} \widetilde{A}_{i,j}^{n,k}, \\ \mathrm{II}_{i,j}^n \approx B_{i,j}^n = \sum_{k=0}^{I-2} \widetilde{B}_{i,j}^{n,k},\end{gathered} \tag{31}$$

where

$$\begin{aligned}\widetilde{A}_{i,j}^{n,k} = & \int_{k\Delta X}^{(k+1)\Delta X} \frac{\widetilde{C}(n\Delta t, z, j\Delta V) f(z/i\Delta X)}{(i\Delta X) dz} \\ = & \int_{k\Delta X}^{(k+1)\Delta X} \frac{(k+1)\Delta X - z}{\Delta X} \\ & \times \frac{\widetilde{C}(n\Delta t, k\Delta X, j\Delta V) f(z/i\Delta X)}{(i\Delta X) dz} \\ & + \int_{k\Delta X}^{(k+1)\Delta X} \frac{z - k\Delta X}{\Delta X} \\ & \times \frac{\widetilde{C}(n\Delta t, (k+1)\Delta X, j\Delta V) f(z/i\Delta X)}{(i\Delta X) dz} \\ = & \frac{1}{2}\left[k\widetilde{C}(n\Delta t, (k+1)\Delta X, j\Delta V)\right. \\ & \left. - (k+1)\widetilde{C}(n\Delta t, k\Delta X, j\Delta V)\right] \\ & \cdot \left[\operatorname{erf}\left(\frac{(\gamma - \ln((k+1)/i))}{\sqrt{2}\delta}\right)\right. \\ & \left. - \operatorname{erf}\left(\frac{(\gamma - \ln(k/i))}{\sqrt{2}\delta}\right)\right] \\ & + \frac{1}{2} i e^{\delta^2/2+\gamma}\left[\widetilde{C}(n\Delta t, k\Delta X, j\Delta V)\right. \\ & \left. - \widetilde{C}(n\Delta t, (k+1)\Delta X, j\Delta V)\right] \\ & \cdot \left[\operatorname{erf}\left(\frac{(\gamma - \ln((k+1)/i) + \delta^2)}{\sqrt{2}\delta}\right)\right. \\ & \left. - \operatorname{erf}\left(\frac{(\gamma - \ln(k/i) + \delta^2)}{\sqrt{2}\delta}\right)\right].\end{aligned} \tag{32}$$

Denote the following:

$$\begin{aligned}
E_{i,j}^{n,k} &= \frac{1}{2}\left[kU_{k+1,j}^{n} - (k+1)\,U_{k,j}^{n}\right],\\
F_{i,j}^{n,k} &= \frac{1}{2}\left[U_{k,j}^{n} - U_{k+1,j}^{n}\right],\\
G_{i}^{k} &= \left[\operatorname{erf}\left(\frac{(\gamma - \ln((k+1)/i))}{\sqrt{2}\delta}\right)\right.\\
&\qquad \left. - \operatorname{erf}\left(\frac{(\gamma - \ln(k/i))}{\sqrt{2}\delta}\right)\right],\\
H_{i}^{k} &= \left[\operatorname{erf}\left(\frac{(\gamma - \ln((k+1)/i) + \delta^2)}{\sqrt{2}\delta}\right)\right.\\
&\qquad \left. - \operatorname{erf}\left(\frac{(\gamma - \ln(k/i) + \delta^2)}{\sqrt{2}\delta}\right)\right].
\end{aligned} \tag{33}$$

Then, we have $A_{i,j}^{n,k} = E_{i,j}^{n,k} * G_i^k + ie^{(\delta^2/2)+\gamma}F_{i,j}^{n,k} * H_i^k$

$$\begin{aligned}
&\widetilde{B}_{i,j}^{n,k}\\
&= \int_{k\Delta X}^{(k+1)\Delta X} \frac{\widetilde{C}(n\Delta t, z, j\Delta V)\,(z/(i\Delta X)-1)\,f\,(z/(i\Delta X)-1)}{(i\Delta X)\,dz}\\
&= \int_{k\Delta X}^{(k+1)\Delta X} \frac{(k+1)\,\Delta X - z}{\Delta X}\\
&\qquad \times \widetilde{C}(n\Delta t, k\Delta X, j\Delta V)\\
&\qquad \times \left(\frac{z}{i\Delta X} - 1\right)\frac{1}{z\delta\sqrt{2\pi}}\\
&\qquad \times \exp\left(-\frac{(\ln(z/i\Delta X) - \gamma)^2}{2\delta^2}\right)dz\\
&\quad + \int_{k\Delta X}^{(k+1)\Delta X} \frac{z - k\Delta X}{\Delta X}\\
&\qquad \times \widetilde{C}(n\Delta t, (k+1)\,\Delta X, j\Delta V)\left(\frac{z}{i\Delta X} - 1\right)\\
&\qquad \times \frac{1}{z\delta\sqrt{2\pi}}\exp\left(-\frac{(\ln(z/i\Delta X) - \gamma)^2}{2\delta^2}\right)dz\\
&= \frac{1}{2}e^{(\delta^2/2)+\gamma}\left[k\widetilde{C}(n\Delta t, (k+1)\,\Delta X, j\Delta V)\right.\\
&\qquad \left. - (k+1)\,\widetilde{C}(n\Delta t, k\Delta X, j\Delta V)\right]\\
&\quad \cdot \left[\operatorname{erf}\left(\frac{(\gamma - \ln((k+1)/i) + \delta^2)}{\sqrt{2}\delta}\right)\right.\\
&\qquad \left. - \operatorname{erf}\left(\frac{(\gamma - \ln(k/i) + \delta^2)}{\sqrt{2}\delta}\right)\right]\\
&\quad + \frac{1}{2i\sqrt{1-2\delta^2}}e^{\gamma^2/(1-2\delta^2)}\\
&\quad \times \left[\widetilde{C}(n\Delta t, k\Delta X, j\Delta V)\right.\\
&\qquad \left. - \widetilde{C}(n\Delta t, (k+1)\,\Delta X, j\Delta V)\right]\\
&\quad \cdot \left[\operatorname{erf}\left(\sqrt{1-2\delta^2}\frac{(\gamma - \ln((k+1)/i))}{\sqrt{2}\delta}\right.\right.\\
&\qquad \left.\left. - \frac{\sqrt{2}\delta\gamma}{\sqrt{1-2\delta^2}}\right)\right.\\
&\qquad \left. - \operatorname{erf}\left(\sqrt{1-2\delta^2}\frac{(\gamma - \ln(k/i))}{\sqrt{2}\delta}\right.\right.\\
&\qquad \left.\left. - \frac{\sqrt{2}\delta\gamma}{\sqrt{1-2\delta^2}}\right)\right] - \widetilde{A}_{i,j}^{n,k}.
\end{aligned} \tag{34}$$

Denote the following:

$$\begin{aligned}
L_i^k &= \left[\operatorname{erf}\left(\sqrt{1-2\delta^2}\frac{(\gamma - \ln((k+1)/i))}{\sqrt{2}\delta}\right.\right.\\
&\qquad \left.\left. - \frac{\sqrt{2}\delta\gamma}{\sqrt{1-2\delta^2}}\right)\right.\\
&\qquad \left. - \operatorname{erf}\left(\sqrt{1-2\delta^2}\frac{(\gamma - \ln(k/i))}{\sqrt{2}\delta}\right.\right.\\
&\qquad \left.\left. - \frac{\sqrt{2}\delta\gamma}{\sqrt{1-2\delta^2}}\right)\right].
\end{aligned} \tag{35}$$

Then, we have $B_{i,j}^{n,k} = e^{(\delta^2/2)+\gamma}E_{i,j}^{n,k} * H_i^k + (1/i\sqrt{1-2\delta^2})\,e^{\gamma^2/(1-2\delta^2)}F_{i,j}^{n,k} * L_i^k - A_{i,j}^{n,k}$, where $\operatorname{erf}(\cdot)$ is the error function defined by $\operatorname{erf}(x) = (2/\sqrt{\pi})\int_0^x e^{-y^2}dy$ with $\operatorname{erf}(\infty) = 1$ and $\operatorname{erf}(-x) = -\operatorname{erf}(x)$.

Then,

$$A_{i,j}^{n} = \sum_{k=0}^{I-2}\widetilde{A}_{i,j}^{n,k}, \qquad B_{i,j}^{n} = \sum_{k=0}^{I-2}\widetilde{B}_{i,j}^{n,k}. \tag{36}$$

Finally, discretization of (27) gives

$$\begin{aligned}
U_{i,j}^{n+1} &= -\left[-1 + \Delta t\left(r + K_0 - \frac{\mu - r}{j\Delta V + K_2}K_1\right)\right.\\
&\qquad \left. + \Delta t j\Delta V i^2 + \frac{\Delta t j\sigma^2}{\Delta V}\right]U_{i,j}^{n}\\
&\quad + \left[\frac{\Delta t i}{2}\left(r - K_1 + \frac{\mu - r}{j\Delta V + K_2}K_2\right) + \frac{\Delta t j\Delta V i^2}{2}\right]U_{i+1,j}^{n}\\
&\quad - \left[\frac{\Delta t i}{2}\left(r - K_1 + \frac{\mu - r}{j\Delta V + K_2}K_2\right) - \frac{\Delta t j\Delta V i^2}{2}\right]U_{i-1,j}^{n}\\
&\quad + \left[\frac{\Delta t}{2\Delta V}\left(\kappa(\varphi - j\Delta V) - \rho\sigma\frac{\mu - r}{j\Delta V + K_2}j\Delta V\right)\right.\\
&\qquad \left. + \frac{\Delta t j\sigma^2}{2\Delta V}\right]U_{i,j+1}^{n}
\end{aligned}$$

$$
\begin{aligned}
&- \left[\frac{\Delta t}{2\Delta V} \left(\kappa (\varphi - j\Delta V) - \rho\sigma \frac{\mu - r}{j\Delta V + K_2} j\Delta V \right) \right. \\
&\quad \left. - \frac{\Delta t j \sigma^2}{2\Delta V} \right] U_{i,j-1}^n \\
&+ \left[\frac{\Delta t i j \sigma \rho}{4} \right] U_{i+1,j+1}^n + \left[\frac{\Delta t i j \sigma \rho}{4} \right] U_{i-1,j-1}^n \\
&- \left[\frac{\Delta t i j \sigma \rho}{4} \right] U_{i-1,j+1}^n - \left[\frac{\Delta t i j \sigma \rho}{4} \right] U_{i+1,j-1}^n \\
&+ \lambda \Delta t A_{i,j}^n - \frac{\lambda \Delta t (\mu - r)}{j\Delta V + K_2} B_{i,j}^n.
\end{aligned} \tag{37}
$$

It equals

$$
\begin{aligned}
U_{i,j}^{n+1} &= c U_{i,j}^n + d U_{i+1,j}^n + e U_{i-1,j}^n + g U_{i,j+1}^n + h U_{i,j-1}^n \\
&\quad + q \left(U_{i+1,j+1}^n + U_{i-1,j-1}^n - U_{i-1,j+1}^n - U_{i+1,j-1}^n \right) \\
&\quad + a A_{i,j}^n + b B_{i,j}^n,
\end{aligned} \tag{38}
$$

where

$$
\begin{aligned}
c(i,j) &= 1 - \Delta t \left(r + K_0 - \frac{\mu - r}{j\Delta V + K_2} K_1 \right) \\
&\quad - \Delta t j \Delta V i^2 - \frac{\Delta t j \sigma^2}{\Delta V}, \\
d(i,j) &= \frac{\Delta t i}{2} \left(r - K_1 + \frac{\mu - r}{j\Delta V + K_2} K_2 \right) + \frac{\Delta t j \Delta V i^2}{2}, \\
e(i,j) &= -\frac{\Delta t i}{2} \left(r - K_1 + \frac{\mu - r}{j\Delta V + K_2} K_2 \right) + \frac{\Delta t j \Delta V i^2}{2}, \\
g(i,j) &= \frac{\Delta t}{2\Delta V} \left(\kappa (\varphi - j\Delta V) - \rho\sigma \frac{\mu - r}{j\Delta V + K_2} j\Delta V \right) + \frac{\Delta t j \sigma^2}{2\Delta V}, \\
h(i,j) &- \frac{\Delta t}{2\Delta V} \left(\kappa (\varphi - j\Delta V) - \rho\sigma \frac{\mu - r}{j\Delta V + K_2} j\Delta V \right) + \frac{\Delta t j \sigma^2}{2\Delta V}, \\
q(i,j) &= \frac{\Delta t i j \sigma \rho}{4}, \qquad a(i,j) = \lambda \Delta t, \\
b(i,j) &= -\frac{\lambda \Delta t (\mu - r)}{j\Delta V + K_2}.
\end{aligned} \tag{39}
$$

6. Numerical Results

In this section, we adopted the basic set of appropriate parameter values of the model shown in Table 1 as our basic set of parameter values to solve the finite difference equation (38).

Figure 1 shows the payoff with the price and the volatility. The solution of (38) is shown in Figure 2. All numerical experiments have been implemented by MATLAB R2011b software on a 2.0-GHz Intel Core PC.

Table 1: Basic set of appropriate parameter values of the model.

μ	0.23
κ	3.46
φ	$(0.0894)^2$
δ	0.0001
ρ	−0.82
T	1
$S_{\max}$	200
$V_{\max}$	1
r	0.039
σ	0.14
λ	0.47
γ	−0.086
K	100
N	2000
I	30
J	30

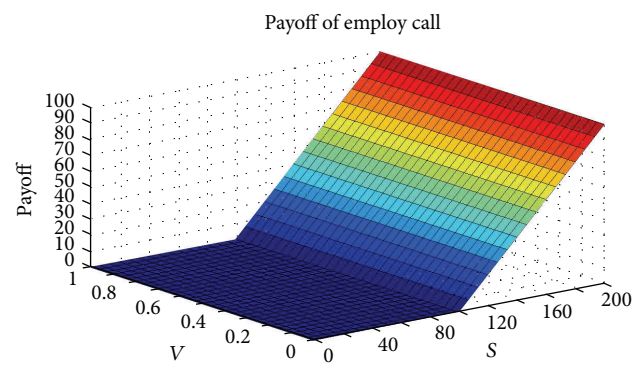

Figure 1: The payoff with the price and the volatility.

Table 2: European option prices.

Price	I
8.7858	10
9.3014	20
9.6929	30
9.7006	40

Now, we suppose that the underlying stock price process follows (2) and there is a European call option with $S_0 = 100$, $\sqrt{V_0} = 0.35$, and, other parameters in Table 1. Table 2 shows the European option prices solved by FDM with different I which stands for the different accuracy.

7. Conclusion

With risk-minimization criterion, we employ the minimal martingale measure to solve the pricing problem in an incomplete market. Then, we obtain the Radon-Nikodym derivative under the minimal martingale measure and a

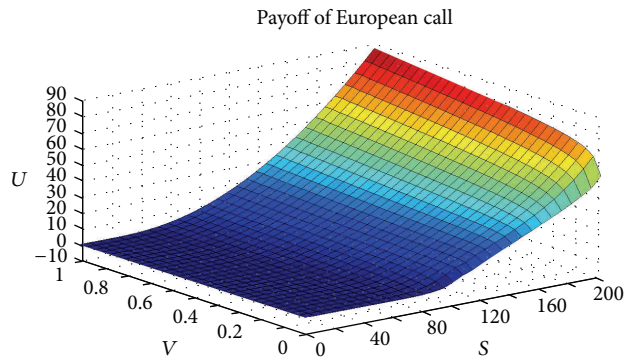

Figure 2: The solution of (38).

partial integro-differential equation (PIDE) for the European option. Since it is difficult to get the exact solution of PIDE in our model, a FDM scheme is proposed to compute the solution approximately. Finally, we complete a European call option price, and it is shown that our method is stable and locally accurate.

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