Research Article

State-Feedback H_∞ Control for LPV System Using T-S Fuzzy Linearization Approach

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This paper discusses the linear parameter varying (LPV) gain scheduling control problem based on the Takagi-Sugeno (T-S) fuzzy linearization approach. Firstly, the affine nonlinear parameter varying (ANPV) description of a class of nonlinear dynamic processes is defined; that is, at any scheduling parameter, the corresponding system is affine nonlinear as usual. For such a class of ANPV systems, a kind of developed T-S fuzzy modeling procedure is proposed to deal with the nonlinearity, instead of the traditional Jacobian linearization approach. More concretely, the evaluation system for the approximation ability of the novel developed T-S fuzzy modeling procedure is established. Consequently, the LPV T-S fuzzy system is obtained which can approximate the ANPV system with required accuracy. Secondly, the notion of piecewise parameter-dependent Lyapunov function is introduced, and then the stabilization problem and the state-feedback H_∞ control problem of the LPV T-S fuzzy system are studied. The sufficient conditions are given in linear matrix inequalities (LMIs) form. Finally, a numerical example is provided to demonstrate the availability of the above approaches. The simulation results show the high approximation accuracy of the LPV T-S fuzzy system to the ANPV system and the effectiveness of the LPV T-S fuzzy gain scheduling control.

1. Introduction

It is well known that the gain scheduling control is an efficient solution for the control of nonlinear dynamic processes [1, 2]. In particular, due to the advantage to carry forward the stability and dynamic performance analysis, the LPV gain scheduling control has been popularly studied [3–12]. Currently, there are mainly two ways to realize the LPV gain scheduling control: the linear fractional transformation (LFT) gain scheduling technique based on a scaled version of the small gain theorem [5–8] and the quadratic gain scheduling technique based on Lyapunov theory [9–12]. However, both of them adopt the Jacobian linearization approach to deal with the nonlinearity around each steady operating point which means that the number of scheduling parameters is equal to the number of variables relevant to the nonlinearity. As a result, too many scheduling parameters are brought in and too much computation burden is caused for the control design; that is, while the gridding process of scheduling parameters and the parameterization of decision

variables in LMIs are executed, the number of LMIs would have a rapid increase. Besides, due to the local linearization around each steady operating point, the control performance is restricted within the local region which would become a weakness when the controlled variables vary widely. Also, it is hard to guarantee the system stability and control performance during the scheduling process. Hence, there requires a further study on the appropriate linearization approach for the ANPV system which can both efficiently reduce the number of scheduling parameters and enlarge the linearization range to extend the effective region of control performance.

On the other hand, the T-S fuzzy model was firstly proposed by Takagi and Sugeno in 1985 [13] and developed by Sugeno and Kang [14], which was data based. To linearize the nonlinearity, the research on the model-based T-S fuzzy modeling procedure was proposed by Kawamoto et al. [15] via the nonlinear sector method which was an exact modeling procedure with constant consequent parts and nonlinear membership functions. Then, Kluska [16]

exploited the local approximation method to establish T-S fuzzy model with homogeneous linear consequent parts. In order to evaluate the approximation ability, Abonyi and Babuska [17] discussed the ways to analyze the relation between the local and global approximation performances of the T-S fuzzy model. Subsequently, Teixeira and Zak [18] gave the T-S fuzzy modeling procedure with homogeneous linear consequent parts by solving a convex optimization problem. And Tanaka and Wang [19] also realized the similar results through uniform partition to the input space of premise variables and proved the universal approximation ability. However, as far as we know, the T-S fuzzy linearization to the ANPV system has been rarely studied in most of the literature.

If the T-S fuzzy linearization is applied to the ANPV system, the choosing of the scheduling parameters and the premise variables can be separated. The nonlinearity of scheduling parameters can remain and that of premise variables should be linearized within their varying regions. As a result, the number of scheduling parameters can be reduced. Meanwhile, because the T-S fuzzy linearization range can be extended to any optional one, the nonlinearity with widely varying state variables can be dealt with for the nonlinear dynamic processes. Due to the above points, it is valuable to study the T-S fuzzy modeling procedure for the ANPV system. Furthermore, in order to efficiently reduce the number of T-S fuzzy rules, an ununiform partition method is utilized, and the evaluation system is also established to adjust the approximation performance. Then, the ANPV system can be linearized on demand and the LPV T-S fuzzy system with required accuracy can be obtained. However, a challenging control problem for such a class of systems is correspondingly formulated. For the T-S fuzzy control, the piecewise Lyapunov functions have been brought in to improve the solvability compared with that based on the common Lyapunov function [20–22]. In addition, considering the approximation error to the ANPV system, the T-S fuzzy H_∞ control has been also studied [23–26]. Hence, the T-S fuzzy control needs to be generalized to the LPV system while improving the solvability by introducing the notion of piecewise Lyapunov function. In addition, for the sake of improving the control performance, the approximation error of LPV T-S fuzzy system to the ANPV system should be considered.

In the paper, the T-S fuzzy modeling procedure with homogeneous linear consequent parts is discussed for the ANPV system while an ununiform partition method is utilized to reduce the number of T-S fuzzy rules. To adjust the approximation accuracy, the evaluation system for the approximation performance of the LPV T-S fuzzy system is established. Then, in order to improve the solvability of the LPV T-S fuzzy gain scheduling control, the notion of piecewise parameter-dependent Lyapunov function is introduced and the LPV T-S fuzzy gain scheduling control design based on the piecewise parameter-dependent Lyapunov functions is studied. Meanwhile, while taking the approximation error to the ANPV system in consideration, the sufficient conditions of the stabilization problem and the state-feedback H_∞ control problem of the LPV T-S fuzzy system are given in

LMIs form. More concretely, the main contributions of the paper are listed as follows.

- (i) The T-S fuzzy approximation with required accuracy aiming at the ANPV system is studied, and then the nonlinearity of the ANPV system can be dealt with and the LPV T-S fuzzy system with required accuracy can be obtained.
- (ii) The stabilization problem of the LPV T-S fuzzy system is studied by bringing in the piecewise parameter-dependent Lyapunov functions while the sufficient conditions are given in LMIs form.
- (iii) Considering the approximation error to the ANPV system, the state-feedback H_∞ control problem of the LPV T-S fuzzy system is studied based on the piecewise parameter-dependent Lyapunov functions. The sufficient conditions are given in both Riccati inequalities form and LMIs form.

The rest of the paper is organized as follows. In Section 2, the developed T-S fuzzy modeling procedure utilizing ununiform partition method is proposed aiming at the ANPV system. And the evaluation system for the approximation performance of the LPV T-S fuzzy system with homogeneous consequent parts is established. Then, the LPV T-S fuzzy system with required accuracy can be obtained by applying the above ways to the ANPV system. In Section 3, the notion of piecewise parameter-dependent Lyapunov function is introduced and the stabilization problem of the LPV T-S fuzzy system is studied. In Section 4, the state-feedback H_∞ control problem of the LPV T-S fuzzy system is studied and the sufficient conditions are given in both Riccati inequalities form and LMIs form. In Section 5, a numerical example is provided to demonstrate the availability of the above approaches. Section 6 concludes the paper.

2. Establishment of LPV T-S Fuzzy System

In this section, the ANPV description of a class of nonlinear dynamic processes is defined aiming at which kind of novel developed T-S fuzzy modeling procedure is proposed and the corresponding evaluation system for the approximation performance is established. Both can be combined to deal with the nonlinearity of the ANPV system to get the LPV T-S fuzzy system with required accuracy.

For a class of nonlinear dynamic processes, if we can achieve their nonlinear descriptions in mathematical models, the ANPV system can be obtained by choosing the proper variables as scheduling parameters. The general form of it can be defined as

$$\begin{bmatrix} \dot{x} \\ z \end{bmatrix} = \begin{bmatrix} f(x, \theta) + d_0(x, \theta) \omega + g_0(x, \theta) u \\ \varphi(x, \theta) + g_1(x, \theta) u \end{bmatrix}, \quad (1)$$

where θ is a vector of the scheduling parameters which may be system parameters, external inputs, or other parameters; x is a column vector of the state variables with n dimensions; and $f(x, \theta), \dots, g_1(x, \theta)$ are the nonlinear functions of x and θ . Then, referring to the way in [18], the T-S fuzzy modeling

procedure aiming at (1) can be developed to linearize the nonlinearity of the ANPV system.

Considering (1), around the zero state, the Jacobian linearization approach can be easily executed to get homogeneous linear model in the local region. However, for the nonzero states, it would be unavailable. Assume the nonzero operating state x_0^i which can be steady or transient one and corresponds to the i th fuzzy rule, $i = 1, 2, \dots, L$. It should be noted that x_0^i may include the part or all of the state variables and it is gotten from the partition to the input space of premise variables. Firstly, we establish the homogeneous linear model in the vicinity of x_0^i . Because the local linear model should approximate the local dynamics in the vicinity of x_0^i , we can get

$$\begin{aligned} & \begin{bmatrix} f(x, \theta) + d_0(x, \theta)\omega + g_0(x, \theta)u \\ \varphi(x, \theta) + g_1(x, \theta)u \end{bmatrix} \\ & \approx \begin{bmatrix} A_i(\theta) & B_1^i(\theta) & B_2^i(\theta) \\ C_1^i(\theta) & 0 & D_{12}^i(\theta) \end{bmatrix} \begin{bmatrix} x \\ \omega \\ u \end{bmatrix}, \\ & \begin{bmatrix} f(x_0^i, \theta) + d_0(x_0^i, \theta)\omega + g_0(x_0^i, \theta)u \\ \varphi(x_0^i, \theta) + g_1(x_0^i, \theta)u \end{bmatrix} \\ & = \begin{bmatrix} A_i(\theta) & B_1^i(\theta) & B_2^i(\theta) \\ C_1^i(\theta) & 0 & D_{12}^i(\theta) \end{bmatrix} \begin{bmatrix} x_0^i \\ \omega \\ u \end{bmatrix}, \end{aligned} \quad (2)$$

where ω and u are arbitrarily changed variables and $A_i(\theta), \dots, D_{12}^i(\theta)$ are parameter-dependent matrices.

Take the first equation $\dot{x} = f(x, \theta) + d_0(x, \theta)\omega + g_0(x, \theta)u$ for example. Due to that, $f(x, \theta)$, $d_0(x, \theta)$, and $g_0(x, \theta)$ are nonlinear functions of operating state x at any θ , and the matrices $A_i(\theta)$, $B_1^i(\theta)$, and $B_2^i(\theta)$ only depend on x . Moreover, because ω and u are arbitrarily changed, we can uniquely determine that

$$\begin{aligned} d_0(x_0^i, \theta) &= B_1^i(\theta), \\ g_0(x_0^i, \theta) &= B_2^i(\theta), \\ f(x_0^i, \theta) &= A_i(\theta)x_0^i, \\ f(x, \theta) &\approx A_i(\theta)x. \end{aligned} \quad (3)$$

Define $a_l^i(\theta)^T$, a row vector with n dimensions, as the l th row of the matrix $A_i(\theta)$. Then, condition (3) can be equivalently represented as

$$f_l(x, \theta) \approx a_l^i(\theta)^T x, \quad l = 1, 2, \dots, n, \quad (4)$$

where $f_l(x, \theta)$ is the l th row of $f(x, \theta)$. Additionally, assume $f_l(0, \theta) = 0$ and $f_l(x, \theta) \in C^1$, which means that f_l and $\partial f_l / \partial x$ are continuous functions on the compact set $\Theta \subset R^n$. Meanwhile, $f(x_0^i, \theta) = A_i(\theta)x_0^i$ can be equivalently represented as

$$f_l(x_0^i, \theta) = a_l^i(\theta)^T x_0^i, \quad l = 1, 2, \dots, n. \quad (5)$$

Expanding $f_l(x, \theta)$ of (4) around the operating state x_0^i and neglecting second- and higher-order terms, we can get

$$f_l(x_0^i, \theta) + \nabla^T f_l(x_0^i, \theta)(x - x_0^i) \approx a_l^i(\theta)^T x, \quad (6)$$

where $\nabla f_l(x_0^i, \theta)$ is the gradient, a column vector, of $f_l(x, \theta)$ evaluated at x_0^i . Substituting (5) into (6), we can get

$$\nabla^T f_l(x_0^i, \theta)(x - x_0^i) \approx a_l^i(\theta)^T (x - x_0^i), \quad (7)$$

where x is arbitrarily close to x_0^i . From (7), it can be found that the coefficient vector $a_l^i(\theta)$ needs to be estimated. And in the vicinity of x_0^i , an optimization problem can be constructed by defining an optimal index to evaluate the estimation. Here, define that the notation C^m represents the set of functions which is m -order continuous and differentiable on the domain of input variables.

Lemma 1. Consider the following constrained optimization problem:

$$\begin{aligned} & \text{minimize } f(x) \\ & \text{subject to } h(x) = 0, \end{aligned} \quad (8)$$

where $f(x)$ is $f: R^n \rightarrow R$ and $f \in C^1$; it is a convex function on the feasible set $\Omega = \{x \in R^n: h(x) = 0\}$, where $h: R^n \rightarrow R^m$ and $h \in C^1$. Assume that Ω is convex and there exist $x^* \in \Omega$ and $\lambda \in R^m$ such that

$$\nabla_{\Omega} f(x^*) + \lambda \nabla_{\Omega} h(x^*) = 0, \quad (9)$$

where λ is the Lagrange multiplier. Then, x^* is the optimal solution of f over Ω and λ is the sufficient condition for the convex optimization problem [27]. Particularly, when $h(x)$ is affine linear, Ω is convex.

Considering the estimation of $a_l^i(\theta)$ around $x_0^i \neq 0$ in (7), the optimal index for $f_l(x, \theta)$ corresponding to the i th fuzzy rule can be defined as

$$I_{f_l}^i(a_l^i(\theta)) = \frac{1}{2} \|\nabla f_l(x_0^i, \theta) - a_l^i(\theta)\|_2^2. \quad (10)$$

Then, the optimal problem can be constructed as

$$\begin{aligned} & \text{minimize}_{\Omega_{f_l}^i} I_{f_l}^i(a_l^i(\theta)) \\ & \text{subject to } h_{f_l}^i(a_l^i(\theta)) = 0, \end{aligned} \quad (11)$$

where (5) should be fulfilled as an equality constraint and $h_{f_l}^i(a_l^i(\theta)) = f_l(x_0^i, \theta) - a_l^i(\theta)^T x_0^i$.

Transform the objective function in (10) into the form

$$\begin{aligned} I_{f_l}^i(a_l^i(\theta)) &= \frac{1}{2} \|\nabla f_l(x_0^i, \theta) - a_l^i(\theta)\|_2^2 \\ &= \frac{1}{2} (\nabla f_l(x_0^i, \theta) - a_l^i(\theta))^T (\nabla f_l(x_0^i, \theta) - a_l^i(\theta)), \end{aligned} \quad (12)$$

which is a quadratic function of the unknown coefficient vector $a_l^i(\theta)$ on R^n . Obviously, it means that $I_{f_l}^i(a_l^i(\theta))$ is a convex function. Considering the equality constraint $h_{f_l}^i(a_l^i(\theta))$, it is the linear function of $a_l^i(\theta)$, so the feasible set $\Omega_{f_l}^i = \{a_l^i(\theta) \in R^n : h(a_l^i(\theta)) = 0\}$ is convex. Therefore, the optimal problem is a convex optimization problem and can be solved according to Lemma 1.

Computing the derivative of $I_{f_l}^i(a_l^i(\theta))$ and $h_{f_l}^i(a_l^i(\theta))$ about $a_l^i(\theta)$, we can get

$$\begin{aligned} \nabla_{\Omega_{f_l}^i} I_{f_l}^i(a_l^i(\theta)) &= a_l^i(\theta) - \nabla f_l(x_0^i, \theta), \\ \nabla_{\Omega_{f_l}^i} h_{f_l}^i(a_l^i(\theta)) &= x_0^i. \end{aligned} \quad (13)$$

Then, substituting (13) into (9), we can get

$$a_l^i(\theta) - \nabla f_l(x_0^i, \theta) + \lambda_{f_l}^i(\theta) x_0^i = 0. \quad (14)$$

Considering the equality constraint (5), the Lagrange multiplier $\lambda_{f_l}^i(\theta)$ can be represented as

$$\begin{aligned} \lambda_{f_l}^i(\theta) &= \frac{\nabla^T f_l(x_0^i, \theta) x_0^i - a_l^i(\theta)^T x_0^i}{\|x_0^i\|_2^2} \\ &= \frac{\nabla^T f_l(x_0^i, \theta) x_0^i - f_l(x_0^i, \theta)}{\|x_0^i\|_2^2}. \end{aligned} \quad (15)$$

As a result, we can get the column vector

$$\begin{aligned} a_l^i(\theta) &= \nabla f_l(x_0^i, \theta) \\ &+ \frac{f_l(x_0^i, \theta) - \nabla^T f_l(x_0^i, \theta) x_0^i}{\|x_0^i\|_2^2} x_0^i, \quad x_0^i \neq 0. \end{aligned} \quad (16)$$

Then, the i th T-S fuzzy rule corresponding to the operating state x_0^i can be chosen as

$$\begin{aligned} \text{Rule } i: \\ \text{IF } x_1 \text{ is } A_1^i(x_1), \dots, \text{ and } x_n \text{ is } A_n^i(x_n), \\ \text{THEN } \hat{f}_l^i(x, \theta) = a_l^i(\theta)^T x, \end{aligned} \quad (17)$$

where $\hat{f}_l^i(x, \theta)$ represents the i th local estimation of the l th row of $f(x, \theta)$, $a_l^i(\theta)$ is the corresponding coefficient vector of the i th fuzzy rule, a column vector, and $A_j^i(x_j)$ represents the fuzzy set and membership function of the j th premise variable, x_j , in the i th fuzzy rule and $j = 1, 2, \dots, n$.

The activated possibility of the i th fuzzy rule under a group of premise variables x is computed by

$$\omega_i(x) = \prod_{j=1}^n A_j^i(x_j), \quad i = 1, 2, \dots, L, \quad j = 1, 2, \dots, n. \quad (18)$$

FIGURE 1: Projection of $\Theta_{k_1 k_2 \dots k_n}$ on $x_1 x_2$ plane.

The weight coefficients are computed and the center-of-gravity method is used for defuzzification:

$$\alpha_i(x) = \frac{\omega_i(x)}{\sum_{i=1}^L \omega_i(x)}, \quad i = 1, 2, \dots, L, \quad (19)$$

which fulfills that

$$\begin{aligned} \sum_{i=1}^L \alpha_i(x) &= 1, \\ \alpha_i(x) &\geq 0. \end{aligned} \quad (20)$$

Finally, the overall T-S fuzzy model can be obtained as

$$\hat{f}_l(x, \theta) = \sum_{i=1}^L \alpha_i(x) \hat{f}_l^i(x, \theta) = \sum_{i=1}^L \alpha_i(x) a_l^i(\theta)^T x. \quad (21)$$

Usually, based on (21), the evaluation of the overall approximation performance about the nonlinear function $f_l(x, \theta)$ is defined as

$$\|e_{f_l}(x, \theta)\|_2 = \|f_l(x, \theta) - \hat{f}_l(x, \theta)\|_2. \quad (22)$$

Subsequently, a kind of un-uniform partition method is utilized for the above T-S fuzzy modeling procedure which can efficiently reduce the number of T-S fuzzy rules. Meanwhile, define the subset $\Theta_0 = \{x \mid |x_j| < \chi_0\}$ on Θ , where χ_0 is a predefined positive scalar and $a_l^0 = \partial f_l(x, \theta) / \partial x|_{x=0}$ can be gotten at the zero state $x = 0$. Considering the nonzero states, they can be represented as $x_0^{k_1 k_2 \dots k_n} = (x_{01}^{k_1}, x_{02}^{k_2}, \dots, x_{0n}^{k_n})^T$, where $\chi_{0j}^{k_j} = x_{0j}^{k_j+1} - x_{0j}^{k_j}$ is a positive scalar and $k_j \in N^+$ is the serial number of partition for x_j . Define the subregion $\Theta_{k_1 k_2 \dots k_n}$ on Θ (as shown in Figure 1):

$$\Theta_{k_1 k_2 \dots k_n} = \{x \mid x \in \Theta, x_{0j}^{k_j} \leq x_j \leq x_{0j}^{k_j+1}, \chi_{0j}^{k_j} = x_{0j}^{k_j+1} - x_{0j}^{k_j}\}. \quad (23)$$

Then, the coefficient vector in (16) can be represented as

$$\begin{aligned} a_l^{k_1 k_2 \dots k_n}(\theta) &= \nabla f_l(x_0^{k_1 k_2 \dots k_n}, \theta) \\ &\quad - \frac{f_l(x_0^{k_1 k_2 \dots k_n}, \theta) - \nabla^T f_l(x_0^{k_1 k_2 \dots k_n}, \theta) x_0^{k_1 k_2 \dots k_n}}{\|x_0^{k_1 k_2 \dots k_n}\|_2^2} \\ &\quad \times x_0^{k_1 k_2 \dots k_n}, \quad x_0^{k_1 k_2 \dots k_n} \neq 0. \end{aligned} \quad (24)$$

Consequently, the T-S fuzzy rules can be chosen as follows:

Rule 0:

IF x_1 is about 0, ..., and x_n is about 0,

THEN $\hat{f}_l^0(x, \theta) = a_l^0(\theta)^T x$, (25)

Rule $k_1 k_2 \dots k_n$:

IF x_1 is about $x_{01}^{k_1}$, ..., and x_n is about $x_{0n}^{k_n}$,

THEN $\hat{f}_l^{k_1 k_2 \dots k_n}(x, \theta) = a_l^{k_1 k_2 \dots k_n}(\theta)^T x$.

For Rule 0, the activated possibility $\alpha_0(x)$ is 1 inside Θ_0 and 0 outside Θ_0 . And the activated possibility of the Rule $k_1 k_2 \dots k_n$ under a group of premise variables x is computed by

$$\begin{aligned} \omega_{k_1 k_2 \dots k_n}(x) &= \prod_{j=1}^n A_j^{k_1 k_2 \dots k_n}(x_j), \\ \alpha_{k_1 k_2 \dots k_n} &= \frac{\omega_{k_1 k_2 \dots k_n}(x)}{\sum_{k_1 k_2 \dots k_n} \omega_{k_1 k_2 \dots k_n}(x)}, \end{aligned} \quad (26)$$

where the membership function for (26) is given as

$$\begin{aligned} A_j^{k_1 k_2 \dots k_n}(x_j) &= \begin{cases} 1 - \frac{(x_j - x_{0j}^{k_j})}{\lambda_{0j}^{k_j}}, & 0 \leq x_j - x_{0j}^{k_j} \leq \lambda_{0j}^{k_j}, \\ 1 + \frac{(x_j - x_{0j}^{k_j})}{\lambda_{0j}^{k_j-1}}, & 0 \leq x_{0j}^{k_j} - x_j \leq \lambda_{0j}^{k_j-1}, \\ 0, & \text{elsewhere.} \end{cases} \end{aligned} \quad (27)$$

Then, $\hat{f}_l(x, \theta)$ can be written as

$$\hat{f}_l(x, \theta) = \alpha_0(x) a_l^0(\theta)^T x + \sum_{k_1 k_2 \dots k_n} \alpha_{k_1 k_2 \dots k_n} a_l^{k_1 k_2 \dots k_n}(\theta)^T x. \quad (28)$$

Here, the notion of the nonlinear measure along x_j in the partition region $\lambda_{0j}^{k_j}$ can be defined as

$$\begin{aligned} M_{0j}^{k_j}(\lambda_{0j}^{k_j}, \theta) &= \left\| \nabla f_l(x_0^{k_1 \dots k_j+1 \dots k_n}, \theta) \right. \\ &\quad \left. - \nabla f_l(x_0^{k_1 \dots k_j \dots k_n}, \theta) \right\|_2 > 0. \end{aligned} \quad (29)$$

Assuming the boundary of $M_{0j}^{k_j}$ is κ_{0j} , the value of κ_{0j} can be set and the proper length of $\lambda_{0j}^{k_j}$ can be chosen to guarantee $M_{0j}^{k_j} \leq \kappa_{0j}$. Thus, the length of the partition regions, $\lambda_{0j}^{k_j}$ ($k_j \in N^+$), of x_j can be different with the same κ_{0j} .

So far, all of the methods show us how to establish the T-S fuzzy system. However, the approximation performance of the T-S fuzzy system is still unknown. Subsequently, the approximation performance of the T-S fuzzy system will be evaluated. Combining with the above T-S fuzzy modeling procedure, the T-S fuzzy system with required approximation accuracy can be obtained at last.

Then, using (24), the approximation performance of the T-S fuzzy system can be evaluated by

$$\begin{aligned} \|e_{f_l}(x, \theta)\|_2 &= \|f_l(x, \theta) - \hat{f}_l(x, \theta)\|_2 \\ &= \left\| f_l(x, \theta) - \sum_{k_1 k_2 \dots k_n} \alpha_{k_1 k_2 \dots k_n}(x) a_l^{k_1 k_2 \dots k_n}(\theta)^T x \right\|_2 \\ &= \left\| f_l(x, \theta) - \sum_{k_1 k_2 \dots k_n} \alpha_{k_1 k_2 \dots k_n}(x) a_l^{k_1 k_2 \dots k_n}(\theta)^T x_0^{k_1 k_2 \dots k_n} \right. \\ &\quad \left. - \sum_{k_1 k_2 \dots k_n} \alpha_{k_1 k_2 \dots k_n}(x) a_l^{k_1 k_2 \dots k_n}(\theta)^T (x - x_0^{k_1 k_2 \dots k_n}) \right\|_2 \\ &= \left\| f_l(x, \theta) - \sum_{k_1 k_2 \dots k_n} \alpha_{k_1 k_2 \dots k_n}(x) \hat{f}_l(x_0^{k_1 k_2 \dots k_n}, \theta) \right. \\ &\quad \left. - \sum_{k_1 k_2 \dots k_n} \alpha_{k_1 k_2 \dots k_n} a_l^{k_1 k_2 \dots k_n}(\theta)^T (x - x_0^{k_1 k_2 \dots k_n}) \right\|_2 \\ &\leq \sum_{k_1 k_2 \dots k_n} \alpha_{k_1 k_2 \dots k_n}(x) \|f_l(x, \theta) - \hat{f}_l(x_0^{k_1 k_2 \dots k_n}, \theta)\|_2 \\ &\quad + \sum_{k_1 k_2 \dots k_n} \alpha_{k_1 k_2 \dots k_n}(x) \|a_l^{k_1 k_2 \dots k_n}(\theta)^T (x - x_0^{k_1 k_2 \dots k_n})\|_2 \\ &\leq \max_{k_1 k_2 \dots k_n} \|f_l(x, \theta) - \hat{f}_l(x_0^{k_1 k_2 \dots k_n}, \theta)\|_2 \\ &\quad + \max_{k_1 k_2 \dots k_n} \|a_l^{k_1 k_2 \dots k_n}(\theta)^T (x - x_0^{k_1 k_2 \dots k_n})\|_2, \end{aligned} \quad (30)$$

where $x \in \Theta_{k_1 k_2 \dots k_n}$ and

$$\begin{aligned}
 & a_l^{k_1 k_2 \dots k_n}(\theta)^T (x - x_0^{k_1 k_2 \dots k_n}) \\
 &= \nabla^T f_l(x_0^{k_1 k_2 \dots k_n}, \theta) \\
 & \times \left[(x - x_0^{k_1 k_2 \dots k_n}) \right. \\
 & \quad \left. - \frac{\langle (x - x_0^{k_1 k_2 \dots k_n}), x_0^{k_1 k_2 \dots k_n} \rangle}{\|x_0^{k_1 k_2 \dots k_n}\|_2^2} x_0^{k_1 k_2 \dots k_n} \right] \\
 & \quad + \frac{\langle (x - x_0^{k_1 k_2 \dots k_n}), x_0^{k_1 k_2 \dots k_n} \rangle}{\|x_0^{k_1 k_2 \dots k_n}\|_2^2} f_l(x_0^{k_1 k_2 \dots k_n}, \theta). \tag{31}
 \end{aligned}$$

Because $x \in \Theta_{k_1 k_2 \dots k_n}$, the maximum distance between x and any vertex point of $\Theta_{k_1 k_2 \dots k_n}$ is less than $\max_{k_1 k_2 \dots k_n} \sqrt{n} \chi_{0j}^{k_j}$, which means $\|x - x_0^{k_1 k_2 \dots k_n}\|_2 \leq \max_{k_1 k_2 \dots k_n} \sqrt{n} \chi_{0j}^{k_j}$; besides, in the boundary of $M_{0j}^{k_j}$, the max value of $\|f_l(x) - \hat{f}_l(x_0^{k_1 k_2 \dots k_n})\|_2$ can be adjusted to fulfill the required accuracy.

Thus, the approximation error, $e_{f_l}(x, \theta)$, can be befittingly minimized by properly setting $\chi_{0j}^{k_j}$, which can be shown as

$$\begin{aligned}
 \|e_{f_l}(x, \theta)\|_2 &\leq \max_{k_1 k_2 \dots k_n} \|f_l(x, \theta) - \hat{f}_l(x_0^{k_1 k_2 \dots k_n}, \theta)\|_2 \\
 &+ \max_{k_1 k_2 \dots k_n} \|a_l^{k_1 k_2 \dots k_n}(\theta)\|_2 \sqrt{n} \chi_{0j}^{k_j}. \tag{32}
 \end{aligned}$$

As a result, the evaluation system for the approximation performance of the LPV T-S fuzzy system is established. It provides a useful way to attain the required accuracy of the T-S fuzzy approximation. Then, the nonlinearity in (1) can be dealt with completely. Note that at each known operating state x_0^i , one fuzzy rule is established, which approximates the local dynamics around x_0^i . And the i th T-S fuzzy rule can be represented as

Rule i :

IF x_1 is $A_1^i(x_1), \dots$, and x_n is $A_n^i(x_n)$,

$$\text{THEN } \begin{bmatrix} \dot{x} \\ z \end{bmatrix} = \begin{bmatrix} A_i(\theta) & B_1^i(\theta) & B_2^i(\theta) \\ C_1^i(\theta) & 0 & D_{12}^i(\theta) \end{bmatrix} \begin{bmatrix} x \\ \omega \\ u \end{bmatrix}, \tag{33}$$

where $i = 1, 2, \dots, L$. More concretely, it is known that $i = k_1 k_2 \dots k_n$ and $L = \max\{k_1 \cdot k_2 \dots k_n\}$.

Finally, the overall LPV T-S fuzzy model can be represented as

$$\begin{aligned}
 \begin{bmatrix} \dot{x} \\ z \end{bmatrix} &= \begin{bmatrix} A(\alpha, \theta) & B_1(\alpha, \theta) & B_2(\alpha, \theta) \\ C_1(\alpha, \theta) & 0 & D_{12}(\alpha, \theta) \end{bmatrix} \begin{bmatrix} x \\ \omega \\ u \end{bmatrix} \\
 &= \begin{bmatrix} \sum_{i=1}^L \alpha_i(x) A_i(\theta) & \sum_{i=1}^L \alpha_i(x) B_1^i(\theta) & \sum_{i=1}^L \alpha_i(x) B_2^i(\theta) \\ \sum_{i=1}^L \alpha_i(x) C_1^i(\theta) & 0 & \sum_{i=1}^L \alpha_i(x) D_{12}^i(\theta) \end{bmatrix} \begin{bmatrix} x \\ \omega \\ u \end{bmatrix} \\
 &= \sum_{i=1}^L \alpha_i(x) \begin{bmatrix} A_i(\theta) & B_1^i(\theta) & B_2^i(\theta) \\ C_1^i(\theta) & 0 & D_{12}^i(\theta) \end{bmatrix} \begin{bmatrix} x \\ \omega \\ u \end{bmatrix}. \tag{34}
 \end{aligned}$$

Remark 2. It is noted that the T-S fuzzy modeling procedure utilizes an un-uniform partition method to reduce the number of T-S fuzzy rules while guaranteeing the approximation accuracy. And the evaluation system for the approximation performance of the LPV T-S fuzzy system can be used to balance the number of T-S fuzzy rules and the approximation performance of the LPV T-S fuzzy system.

3. Piecewise Parameter-Dependent Quadratic Stabilization of LPV T-S Fuzzy System

The LPV T-S fuzzy system (34) represents a class of complex continuous-time systems in a novel form which has both fuzzy inference and locally analytic linear models. In this section, the notion of piecewise parameter-dependent Lyapunov function is introduced for the stabilization problem of the LPV T-S fuzzy system. Firstly, the r th subspace in the state space can be defined as

$$\overline{X}_r = X_r \cup \widetilde{X}_r, \tag{35}$$

where

$$\begin{aligned}
 X_r &= \{x \mid \alpha_i(x) = \alpha_k(x), i, k = 1, 2, \dots, L, i \neq k\}, \\
 \widetilde{X}_r &= \{x \mid \alpha_i(x) > \alpha_k(x), i, k = 1, 2, \dots, L, i \neq k\}. \tag{36}
 \end{aligned}$$

Note that two subspaces are generated around the i th T-S fuzzy rule for the single state x_j . The schematic about the state x_j can be shown in Figure 2. And then, the relation between the r th subspace and the i th T-S fuzzy rule is $i = k_1 k_2 \dots k_n$ and $r = (2k_1 - 2/2k_1 - 1)(2k_2 - 2/2k_2 - 1) \dots (2k_n - 2/2k_n - 1)$ ($i = 1, 2, \dots, L$; word $r = 1, 2, \dots, \overline{L}$), where $L = \max\{k_1 \cdot k_2 \dots k_n\}$ and $\overline{L} = \max\{2^n \cdot (k_1 - 1) \cdot (k_2 - 1) \dots (k_n - 1)\}$. Then, the overall model of the LPV T-S system in the r th subspace can be represented as

$$\begin{aligned}
 \begin{bmatrix} \dot{x} \\ z \end{bmatrix} &= \begin{bmatrix} \overline{A}_r(\alpha, \theta) & \overline{B}_1^r(\alpha, \theta) & \overline{B}_2^r(\alpha, \theta) \\ \overline{C}_1^r(\alpha, \theta) & 0 & \overline{D}_{12}^r(\alpha, \theta) \end{bmatrix} \begin{bmatrix} x \\ \omega \\ u \end{bmatrix} \\
 &= \begin{bmatrix} A_i(\theta) + \widetilde{A}_r(\alpha, \theta) & B_1^i(\theta) + \widetilde{B}_1^r(\alpha, \theta) & B_2^i(\theta) + \widetilde{B}_2^r(\alpha, \theta) \\ C_1^i(\theta) + \widetilde{C}_1^r(\alpha, \theta) & 0 & D_{12}^i(\theta) + \widetilde{D}_{12}^r(\alpha, \theta) \end{bmatrix} \begin{bmatrix} x \\ \omega \\ u \end{bmatrix} \tag{37}
 \end{aligned}$$

FIGURE 2: Piecewise space partition of x_j .

for $x \in \bar{X}_r$, where

$$\begin{aligned}
 \bar{A}_r(\alpha, \theta) &= \sum_{k \in \phi_r} \alpha_k(x) \bar{A}_{ik}(\theta), \\
 \bar{B}_1^r(\alpha, \theta) &= \sum_{k \in \phi_r} \alpha_k(x) \bar{B}_1^{ik}(\theta), \\
 \bar{B}_2^r(\alpha, \theta) &= \sum_{k \in \phi_r} \alpha_k(x) \bar{B}_2^{ik}(\theta), \\
 \bar{C}_1^r(\alpha, \theta) &= \sum_{k \in \phi_r} \alpha_k(x) \bar{C}_1^{ik}(\theta), \\
 \bar{D}_{12}^r(\alpha, \theta) &= \sum_{k \in \phi_r} \alpha_k(x) \bar{D}_{12}^{ik}(\theta), \\
 \bar{A}_{ik}(\theta) &= A_k(\theta) - A_i(\theta), \\
 \bar{B}_1^{ik} &= B_1^k(\theta) - B_1^i(\theta), \quad \bar{B}_2^{ik} = B_2^k(\theta) - B_2^i(\theta), \\
 \bar{C}_1^{ik} &= C_1^k(\theta) - C_1^i(\theta), \quad \bar{D}_{12}^{ik} = D_{12}^k(\theta) - D_{12}^i(\theta), \\
 \phi_r &= \{k \mid \alpha_k(x) \neq 0, \alpha_i(x) \geq \alpha_k(x)\},
 \end{aligned} \tag{38}$$

and ϕ_r is a set including the indexes of the membership functions which are nonzero and less than $\alpha_i(x)$ around the i th rule in the r th subspace. For the overall model, $\bar{A}_r(\alpha, \theta)$, $\bar{B}_*^r(\alpha, \theta)$, $\bar{C}_*^r(\alpha, \theta)$, and $\bar{D}_*^r(\alpha, \theta)$ represent the interpolation terms produced by interactions between the i th rule and the other rules in the r th subspace.

In order to find the piecewise parameter-dependent Lyapunov function which is continuous across the r th subspace boundary at a fixed θ , the following constant matrix F_r is established from the structure information of the r th subspace, which fulfills

$$F_r x = F_p x, \quad x \in \bar{X}_r \cap \bar{X}_p, \quad r, p = 1, 2, \dots, \bar{L}. \tag{39}$$

Then, the piecewise parameter-dependent Lyapunov function candidates that are continuous across the r th subspace boundary can be parameterized as

$$V(\theta) = x^T P_r(\theta) x, \quad x \in \bar{X}_r, \tag{40}$$

with

$$P_r(\theta) = F_r^T T(\theta) F_r, \tag{41}$$

where $T(\theta)$ is the symmetric matrix and characterizes $P_r(\theta)$ with F_r together.

In order to carry forward the control design, the following upper bounds for the interpolation terms in (37) can be defined as

$$\begin{aligned}
 [\bar{A}_r(\alpha, \theta)] [\bar{A}_r(\alpha, \theta)]^T &\leq E_{rA}(\alpha, \theta) E_{rA}^T(\alpha, \theta), \\
 [\bar{B}_1^r(\alpha, \theta)] [\bar{B}_1^r(\alpha, \theta)]^T &\leq E_{rB_1}(\alpha, \theta) E_{rB_1}^T(\alpha, \theta), \\
 [\bar{B}_2^r(\alpha, \theta)] [\bar{B}_2^r(\alpha, \theta)]^T &\leq E_{rB_2}(\alpha, \theta) E_{rB_2}^T(\alpha, \theta), \\
 [\bar{C}_1^r(\alpha, \theta)] [\bar{C}_1^r(\alpha, \theta)]^T &\leq E_{rC_1}(\alpha, \theta) E_{rC_1}^T(\alpha, \theta), \\
 [\bar{D}_{12}^r(\alpha, \theta)] [\bar{D}_{12}^r(\alpha, \theta)]^T &\leq E_{rD_{12}}(\alpha, \theta) E_{rD_{12}}^T(\alpha, \theta).
 \end{aligned} \tag{42}$$

Since all the information of the interpolation terms in (42) is a priori knowledge, there are many ways to acquire these upper bounds. For example, one simple way is

$$\begin{aligned}
 E_{rA}(\alpha, \theta) E_{rA}^T(\alpha, \theta) \\
 = \left(\sum_{k \in \phi_r} \alpha_k(\bar{x}) \bar{A}_{ik}(\theta) \right) \left(\sum_{k \in \phi_r} \alpha_k(\bar{x}) \bar{A}_{ik}(\theta) \right)^T,
 \end{aligned} \tag{43}$$

where $\bar{x}$ is the state such that $\alpha_k(\bar{x}) = 0.5$.

Definition 3. On the compact set $\Psi \subset R^s$, one has finite nonnegative numbers $\{v_q\}_{q=1}^s$. Then the bounded variations set of scheduling parameters can be defined as

$$\Gamma_\Psi^v = \{\theta \in C^1(R, R^s) \mid \theta \in \Psi, \|\dot{\theta}_q\|_2 \leq v_q, \quad q = 1, \dots, s\}, \tag{44}$$

where C^1 represents the class of functions which are piecewise continuous and one-order differential.

Definition 4. On the compact $\Psi \in R^s$, one has the LPV system

$$\dot{x} = A(\theta) x, \tag{45}$$

where $\theta \in \Gamma_\Psi^v$. If there exists a continuously differentiable symmetric function $P(\theta)$ such that $P(\theta) > 0$ and

$$A^T(\theta) P(\theta) + P(\theta) A(\theta) + \dot{P}(\theta) < 0, \tag{46}$$

for all $\theta \in \Gamma_\Psi^v$, then the continuous function $A(\theta)$ is parametrically dependent quadratically stable (or PDQ stable, for short).

Theorem 5. Consider the LPV T-S fuzzy system (34) with $\omega = u = 0$. If there exists a symmetric matrix $T(\theta)$ satisfying

$$P_r(\theta) > 0, \tag{47}$$

$$\begin{bmatrix} P_r(\theta) A_i^T(\theta) + A_i(\theta) P_r(\theta) - \dot{P}_r(\theta) & P_r(\theta) \\ + \varepsilon_{\bar{A}_r P_r}(\theta) E_{rA}(\alpha, \theta) E_{rA}^T(\alpha, \theta) & \\ P_r(\theta) & -\varepsilon_{\bar{A}_r P_r}(\theta) I \end{bmatrix} < 0 \tag{48}$$

with

$$P_r(\theta) = (F_r^T F_r)^{-1} F_r^T T(\theta) F_r (F_r^T F_r)^{-1} \quad (49)$$

for $r = 1, 2, \dots, \bar{L}$, then the fuzzy system is globally asymptotically stable.

Proof. Define the following Lyapunov function $V(\theta)$:

$$V(\theta) = x^T P_r^{-1}(\theta) x, \quad x \in \bar{X}_r. \quad (50)$$

From (41), we can get

$$P_r^{-1}(\theta) = F_r^T T^{-1}(\theta) F_r. \quad (51)$$

From (50) and (51), there exist constants $\sigma > 0$ and $\beta > 0$ such that

$$\sigma \|x\|_2^2 \leq V(\theta) \leq \beta \|x\|_2^2. \quad (52)$$

Thus, using the conditions (47) and (49), $V(\theta)$ is positive and continuous across the subspace boundary. If it can guarantee that the system (37) is asymptotically stable in each subspace, the global asymptotical stability for the system (34) can be attained. Next, we will demonstrate that $V(\theta)$ can guarantee the asymptotical stability in each subspace.

If there exist a constant $\varepsilon_{XY} > 0$ and matrices X and Y with appropriate dimensions, the following matrix inequality can be gotten:

$$X^T Y + Y^T X \leq \frac{1}{\varepsilon_{XY}} X^T X + \varepsilon Y^T Y. \quad (53)$$

From (53), define the parameter-dependent positive scalar $\varepsilon_{bc}(\theta)$, where the subscript bc means the relation between b and c just like ε_{XY} means the relation between X and Y . In many cases, $\varepsilon_{ab}(\theta)$ can be set as a constant value. Note that b and c are just used for example and do not have any special significance. However, subscripts of all $\varepsilon_*(\theta)$ are marked as follows to clearly show the meaning of $\varepsilon_*(\theta)$.

Then, from (53) we can get

$$\begin{aligned} & P_r(\theta) \left(A_i(\theta) + \tilde{A}_r(\alpha, \theta) \right)^T \\ & + \left(A_i(\theta) + \tilde{A}_r(\alpha, \theta) \right) P_r(\theta) - \dot{P}_r(\theta) \\ & = P_r(\theta) A_i^T(\theta) + P_r(\theta) \tilde{A}_r^T(\alpha, \theta) + A_i(\theta) P_r(\theta) \\ & + \tilde{A}_r(\alpha, \theta) P_r(\theta) - \dot{P}_r(\theta) \\ & \leq P_r(\theta) A_i^T(\theta) + A_i(\theta) P_r(\theta) + \frac{1}{\varepsilon_{\tilde{A}_r, P_r}(\theta)} P_r(\theta) P_r(\theta) \\ & + \varepsilon_{\tilde{A}_r, P_r}(\theta) \tilde{A}_r^T(\alpha, \theta) \tilde{A}_r(\alpha, \theta) - \dot{P}_r(\theta) \\ & \leq P_r(\theta) A_i^T(\theta) + A_i(\theta) P_r(\theta) + \frac{1}{\varepsilon_{\tilde{A}_r, P_r}(\theta)} P_r(\theta) P_r(\theta) \\ & + \varepsilon_{\tilde{A}_r, P_r}(\theta) E_{rA}^T(\alpha, \theta) E_{rA}(\alpha, \theta) - \dot{P}_r(\theta), \end{aligned} \quad (54)$$

for the system (37). Besides, via the Schur complement lemma, it follows from (48) that

$$\begin{aligned} & P_r(\theta) A_i^T(\theta) + A_i(\theta) P_r(\theta) + \frac{1}{\varepsilon_{\tilde{A}_r, P_r}(\theta)} P_r(\theta) P_r(\theta) \\ & + \varepsilon_{\tilde{A}_r, P_r}(\theta) E_{rA}(\alpha, \theta) E_{rA}^T(\alpha, \theta) - \dot{P}_r(\theta) < 0. \end{aligned} \quad (55)$$

Then, from (54) and (55), we have

$$\begin{aligned} & P_r(\theta) \left(A_i(\theta) + \tilde{A}_r(\alpha, \theta) \right)^T \\ & + \left(A_i(\theta) + \tilde{A}_r(\alpha, \theta) \right) P_r(\theta) - \dot{P}_r(\theta) < 0, \end{aligned} \quad (56)$$

which suggests that

$$\begin{aligned} & \left(A_i(\theta) + \tilde{A}_r(\alpha, \theta) \right)^T P_r^{-1}(\theta) \\ & + P_r^{-1}(\theta) \left(A_i(\theta) + \tilde{A}_r(\alpha, \theta) \right) + \dot{P}_r^{-1}(\theta) < 0, \end{aligned} \quad (57)$$

where $\dot{P}_r^{-1}(\theta) = -F_r^T T^{-1}(\theta) \dot{T}(\theta) T^{-1}(\theta) F_r = -P_r^{-1}(\theta) \dot{P}_r(\theta) P_r^{-1}(\theta)$.

Moreover, there exists a constant $\delta > 0$ such that

$$\begin{aligned} & \left(A_i(\theta) + \tilde{A}_r(\alpha, \theta) \right)^T P_r^{-1}(\theta) \\ & + P_r^{-1}(\theta) \left(A_i(\theta) + \tilde{A}_r(\alpha, \theta) \right) + \dot{P}_r^{-1}(\theta) + \delta I < 0. \end{aligned} \quad (58)$$

As a result, we have

$$\begin{aligned} \dot{V}(\theta) & = x^T \left[\left(A_i(\theta) + \tilde{A}_r(\alpha, \theta) \right)^T P_r^{-1}(\theta) + P_r^{-1}(\theta) \right. \\ & \quad \times \left. \left(A_i(\theta) + \tilde{A}_r(\alpha, \theta) \right) + \dot{P}_r^{-1}(\theta) \right] x \\ & \leq x^T (-\delta I) x = -\delta \|x\|_2^2, \end{aligned} \quad (59)$$

which completes the proof of this theorem. $\square$

Remark 6. It is noted that the sufficient conditions in LMIs form are easy to be solved. At a fixed θ , conditions (47) and (49) guarantee that the Lyapunov function is positive and piecewise continuous across each subspace. And the Lyapunov functions solved from condition (48) guarantee that the system (37) is asymptotical in each subspace. All the conditions guarantee that the system (34) is globally asymptotically stable. Besides, in many cases, for the T-S fuzzy control based on the common Lyapunov function, it was hard to find the common Lyapunov function or such a Lyapunov function did not exist at all. Thus, the previous approach can improve the solvability of the LPV T-S fuzzy gain scheduling control.

4. State-Feedback H_∞ Control Design of LPV T-S Fuzzy System

In this section, the state-feedback H_∞ control of the LPV T-S fuzzy system is studied. Considering the approximation error, a controller synthesis approach with H_∞ performance

is presented. In the r th subspace, the state feedback controller can be represented as

$$u = K_r(\theta) x, \quad x \in \bar{X}_r. \quad (60)$$

Substituting (60) into (34), the closed-loop system can be obtained as

$$\dot{x} = A_C(\alpha, \theta) x + B_C(\alpha, \theta) \omega, \quad z = C_C(\alpha, \theta) x, \quad (61)$$

where

$$\begin{aligned} A_C(\alpha, \theta) &= A(\alpha, \theta) + B_2(\alpha, \theta) K(\theta), \quad B_C(\alpha, \theta) = B_1(\alpha, \theta), \\ C_C(\alpha, \theta) &= C_1(\alpha, \theta) + D_{12}(\alpha, \theta) K(\theta). \end{aligned} \quad (62)$$

In the r th subspace, the system (61) can be represented as

$$\begin{aligned} \dot{x} &= A_{Cr}(\alpha, \theta) x + B_{Cr}(\alpha, \theta) \omega, \\ z &= C_{Cr}(\alpha, \theta) x, \quad x \in \bar{X}_r, \end{aligned} \quad (63)$$

where

$$\begin{aligned} A_{Cr}(\alpha, \theta) &= A_i(\theta) + \bar{A}_r(\alpha, \theta) \\ &\quad + (B_2^i(\theta) + \bar{B}_2^r(\alpha, \theta)) K_r(\theta), \\ B_{Cr}(\alpha, \theta) &= B_1^i(\theta) + \bar{B}_1^r(\alpha, \theta), \\ C_{Cr}(\alpha, \theta) &= C_1^i(\theta) + \bar{C}_1^r(\alpha, \theta) \\ &\quad + (D_{12}^i(\theta) + \bar{D}_{12}^r(\alpha, \theta)) K_r(\theta). \end{aligned} \quad (64)$$

When substituting (37) into (1) and considering the approximating error, the closed-loop nonlinear system in the r th subspace can be represented as

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ z \end{bmatrix} &= \begin{bmatrix} f(x, \theta) + d_0(x, \theta) \omega + g_0(x, \theta) u \\ \varphi(x, \theta) + g_1(x, \theta) u \end{bmatrix} \\ &= \begin{bmatrix} \bar{A}_r(\alpha, \theta) & \bar{B}_1^r(\alpha, \theta) & \bar{B}_2^r(\alpha, \theta) \\ \bar{C}_1^r(\alpha, \theta) & 0 & \bar{D}_{12}^r(\alpha, \theta) \end{bmatrix} \begin{bmatrix} x \\ \omega \\ u \end{bmatrix} \\ &\quad + \begin{bmatrix} (f(x, \theta) - \bar{A}_r(\alpha, \theta) x) + (d_0(x, \theta) - \bar{B}_1^r(\alpha, \theta) \omega) \\ (g_0(x, \theta) - \bar{B}_2^r(\alpha, \theta) u) \\ (\varphi(x, \theta) - \bar{C}_1^r(\alpha, \theta) x) + (g_1(x, \theta) - \bar{D}_{12}^r(\alpha, \theta) u) \end{bmatrix} \\ &= \begin{bmatrix} \bar{A}_r(\alpha, \theta) + \bar{B}_2^r(\alpha, \theta) K_r(\theta) & \bar{B}_1^r(\alpha, \theta) \\ \bar{C}_1^r(\alpha, \theta) + \bar{D}_{12}^r(\alpha, \theta) K_r(\theta) & 0 \end{bmatrix} \begin{bmatrix} x \\ \omega \end{bmatrix} \\ &\quad + \begin{bmatrix} (f(x, \theta) - \bar{A}_r(\alpha, \theta) x) + (g_0(x, \theta) - \bar{B}_2^r(\alpha, \theta) K_r(\theta) x) \\ (d_0(x, \theta) - \bar{B}_1^r(\alpha, \theta) \omega) \\ (\varphi(x, \theta) - \bar{C}_1^r(\alpha, \theta) x) + (g_1(x, \theta) - \bar{D}_{12}^r(\alpha, \theta) K_r(\theta) x) \end{bmatrix} \\ &= \begin{bmatrix} A_{Cr}(\alpha, \theta) & B_{Cr}(\alpha, \theta) \\ C_{Cr}(\alpha, \theta) & 0 \end{bmatrix} \begin{bmatrix} x \\ \omega \end{bmatrix} + \begin{bmatrix} \Delta f_r(\theta) + \Delta g_0^r(\theta) + \Delta d_0^r(\theta) \\ \Delta \varphi_r(\theta) + \Delta g_1^r(\theta) \end{bmatrix}, \end{aligned} \quad (65)$$

where

$$\begin{aligned} \Delta f_r(\theta) &= f(x, \theta) - \bar{A}_r(x, \theta) x, \\ \Delta g_0^r(\theta) &= (g_0(x, \theta) - \bar{B}_2^r(\alpha, \theta)) K_r(\theta) x, \\ \Delta d_0^r(\theta) &= (d_0(x, \theta) - \bar{B}_1^r(\alpha, \theta) \omega), \\ \Delta \varphi_r(\theta) &= \varphi(x, \theta) - \bar{C}_1^r(\alpha, \theta) x, \\ \Delta g_1^r(\theta) &= (g_1(x, \theta) - \bar{D}_{12}^r(\alpha, \theta)) K_r(\theta) x. \end{aligned} \quad (66)$$

According to (37) and (65), the approximation errors, $\Delta f_r(\theta)$, $\Delta g_0^r(\theta)$, $\Delta d_0^r(\theta)$, $\Delta \varphi_r(\theta)$, and $\Delta g_1^r(\theta)$, of the LPV T-S fuzzy system can be represented as follows:

$$\begin{aligned} \|\Delta f_r(\theta)\|_2 &= \|f(x, \theta) - \bar{A}_r(x, \theta) x\|_2 \\ &= \left\| f(x, \theta) - \left(A_i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \bar{A}_{ik}(\theta) \right) x \right\|_2 \\ &\leq \left\| \left(\Delta A_i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \Delta \bar{A}_{ik}(\theta) \right) x \right\|_2 \\ &= \|\Delta \bar{A}_r(\theta) x\|_2, \end{aligned} \quad (67)$$

where

$$\begin{aligned} \Delta \bar{A}_r(\theta) &= \Delta A_i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \bar{A}_{ik}(\theta) \\ &= \Delta A_i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) (\Delta A_k(\theta) - \Delta A_i(\theta)) \\ &= \alpha_i(x) \Delta A_i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \Delta A_k(\theta) \\ &= \left(\alpha_i(x) \delta_{A_i^i}(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \delta_{A_i^k}(\theta) \right) \hat{A}_r(\theta) \\ &= \delta_{\bar{A}_r}(\theta) \hat{A}_r(\theta), \\ \|\Delta g_0^r(\theta)\|_2 &= \|(g_0(x, \theta) - \bar{B}_2^r(\alpha, \theta)) K_r(\theta) x\|_2 \\ &= \left\| \left(g_0(x, \theta) - \left(B_2^i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \bar{B}_2^{ik}(\theta) \right) \right) \right. \\ &\quad \left. \times K_r(\theta) x \right\|_2 \\ &\leq \left\| \left(\Delta B_2^i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \Delta \bar{B}_2^{ik}(\theta) \right) K_r(\theta) x \right\|_2 \\ &= \|\Delta \bar{B}_2^r(\theta) K_r(\theta) x\|_2, \end{aligned} \quad (68)$$

where

$$\begin{aligned}
 \Delta \bar{B}_2^r(\theta) &= \Delta B_2^i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \Delta \bar{B}_2^{ik}(\theta) \\
 &= \Delta B_2^i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) (\Delta B_2^k(\theta) - \Delta B_2^i(\theta)) \\
 &= \alpha_i(x) \Delta B_2^i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \Delta B_2^k(\theta) \\
 &= \left(\alpha_i(x) \delta_{B_2^{i,r}}(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \delta_{B_2^{k,r}}(\theta) \right) \hat{B}_2^r(\theta) \\
 &= \delta_{\bar{B}_2^r} \hat{B}_2^r(\theta), \\
 \|\Delta d_0^r(\theta)\|_2 &= \|(d_0(x, \theta) - \bar{B}_1^r(\alpha, \theta)) \omega\|_2 \\
 &= \left\| \left(d_0(x, \theta) - \left(B_1^i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \bar{B}_1^{ik}(\theta) \right) \right) \omega \right\|_2 \\
 &\leq \left\| \left(\Delta B_1^i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \Delta \bar{B}_1^{ik}(\theta) \right) \omega \right\|_2 \\
 &= \|\Delta \bar{B}_1^r(\theta) \omega\|_2,
 \end{aligned} \tag{69}$$

where

$$\begin{aligned}
 \Delta \bar{B}_1^r(\theta) &= \Delta B_1^i(\theta) + \sum_{k \in \phi_r} \alpha_k(\theta) \Delta \bar{B}_1^{ik}(\theta) \\
 &= \Delta B_1^i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) (\Delta B_1^k(\theta) - \Delta B_1^i(\theta)) \\
 &= \alpha_i(x) \Delta B_1^i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \Delta B_1^k(\theta) \\
 &= \left(\alpha_i(x) \delta_{B_1^{i,r}}(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \delta_{B_1^{k,r}}(\theta) \right) \hat{B}_1^r(\theta) \\
 &= \delta_{\bar{B}_1^r} \hat{B}_1^r(\theta), \\
 \Delta \|\varphi_r(\theta)\|_2 &= \|\varphi(x, \theta) - \bar{C}_1^r(\alpha, \theta) x\|_2 \\
 &= \left\| \varphi(x, \theta) - \left(C_1^i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \bar{C}_1^{ik}(\theta) \right) x \right\|_2 \\
 &\leq \left\| \left(\Delta C_1^i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \Delta \bar{C}_1^{ik}(\theta) \right) x \right\|_2 \\
 &= \|\Delta \bar{C}_1^r(\theta) x\|_2,
 \end{aligned} \tag{70}$$

where

$$\begin{aligned}
 \Delta \bar{C}_1^r(\theta) &= \Delta C_1^i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \Delta \bar{C}_1^{ik}(\theta) \\
 &= \Delta C_1^i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) (\Delta C_1^k(\theta) - \Delta C_1^i(\theta)) \\
 &= \alpha_i(x) \Delta C_1^i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \Delta C_1^k(\theta) \\
 &= \left(\alpha_i(x) \delta_{C_1^{i,r}}(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \delta_{C_1^{k,r}}(\theta) \right) \\
 &\quad \times \hat{C}_1^r(\theta) \\
 &= \delta_{\bar{C}_1^r}(\theta) \hat{C}_1^r(\theta), \\
 \|\Delta g_1^r(\theta)\|_2 &= \|(g_1(x, \theta) - \bar{D}_{12}^r(\alpha, \theta)) K_r(\theta) x\|_2 \\
 &= \left\| \left(g_1(x, \theta) - \left(D_{12}^i(\theta) \right. \right. \right. \\
 &\quad \left. \left. \left. + \sum_{k \in \phi_r} \alpha_k(x) \bar{D}_{12}^{ik}(\theta) \right) \right) \right. \\
 &\quad \left. \times K_r(\theta) x \right\|_2 \\
 &\leq \left\| \left(\Delta D_{12}^i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \Delta \bar{D}_{12}^{ik}(\theta) \right) \right. \\
 &\quad \left. \times K_r(\theta) x \right\|_2 \\
 &= \|\Delta \bar{D}_{12}^r(\theta) K_r(\theta) x\|_2,
 \end{aligned} \tag{71}$$

where

$$\begin{aligned}
 \Delta \bar{D}_{12}^r(\theta) &= \Delta D_{12}^i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \Delta \bar{D}_{12}^{ik}(\theta) \\
 &= \Delta D_{12}^i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) (\Delta D_{12}^k(\theta) - \Delta D_{12}^i(\theta)) \\
 &= \alpha_i(x) \Delta D_{12}^i(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \Delta D_{12}^k(\theta) \\
 &= \left(\alpha_i(x) \delta_{D_{12}^{i,r}}(\theta) + \sum_{k \in \phi_r} \alpha_k(x) \delta_{D_{12}^{k,r}}(\theta) \right) \hat{D}_{12}^r(\theta) \\
 &= \delta_{\bar{D}_{12}^r}(\theta) \hat{D}_{12}^r(\theta).
 \end{aligned} \tag{72}$$

Note that not all of the nonlinearities can be given such bounds of approximation error in a linear form. However, for

the nonlinearities satisfying the Lipschitz condition or even the one-order derivable condition, we can usually give the bounds of the approximation error in such a linear form. In addition, it should be noted that $\alpha_i(x) + \sum_{k \in \phi_r} \alpha_k(x) = 1$ and $\|\delta_{A_r^*}(\theta)\|_2 \leq 1, \dots, \|\delta_{\bar{D}_{12}^{*r}}(\theta)\|_2 \leq 1$ for $i = 1, 2, \dots, L$ and $r = 1, 2, \dots, \bar{L}$. Then, we can get

$$\begin{bmatrix} \|\Delta f_r(\theta)\|_2 & \|\Delta d_0^r(\theta)\|_2 & \|\Delta g_0^r(\theta)\|_2 \\ \|\Delta \varphi_r(\theta)\|_2 & 0 & \|\Delta g_1^r(\theta)\|_2 \end{bmatrix} \leq \begin{bmatrix} \|\Delta \bar{A}_r(\theta)x\|_2 & \|\Delta \bar{B}_1^r(\theta)\omega\|_2 & \|\Delta \bar{B}_2^r(\theta)K_r(\theta)x\|_2 \\ \|\Delta \bar{C}_1^r(\theta)x\|_2 & 0 & \|\Delta \bar{D}_{12}^r(\theta)K_r(\theta)x\|_2 \end{bmatrix}, \quad (73)$$

for all the x , and describe the bounding matrices as

$$\begin{bmatrix} \Delta \bar{A}_r(\theta) & \Delta \bar{B}_1^r(\theta) & \Delta \bar{B}_2^r(\theta) \\ \Delta \bar{C}_1^r(\theta) & 0 & \Delta \bar{D}_{12}^r(\theta) \end{bmatrix} = \begin{bmatrix} \delta_{\bar{A}_r}(\theta) \hat{A}_r(\theta) & \delta_{\bar{B}_1^r}(\theta) \hat{B}_1^r(\theta) & \delta_{\bar{B}_2^r}(\theta) \hat{B}_2^r(\theta) \\ \delta_{\bar{C}_1^r}(\theta) \hat{C}_1^r(\theta) & 0 & \delta_{\bar{D}_{12}^r}(\theta) \hat{D}_{12}^r(\theta) \end{bmatrix}, \quad (74)$$

where $\|\delta_{\bar{A}_r}(\theta)\|_2 \leq 1, \dots$ and $\|\delta_{\bar{D}_{12}^r}(\theta)\|_2 \leq 1$ for $i = 1, 2, \dots, L$ and $r = 1, 2, \dots, \bar{L}$.

Theorem 7. For the closed-loop system (61), while considering the approximation error and a positive scalar $\gamma > 0$, if there exist symmetric matrixes $T(\theta)$ and $P_r(\theta) = (F_r^T F_r)^{-1} F_r^T T(\theta) F_r (F_r^T F_r)^{-1}$ satisfying

$$P_r(\theta) > 0, \quad (75)$$

$$\begin{aligned} & P_r(\theta) A_{C_r}^T(\alpha, \theta) + A_{C_r}(\alpha, \theta) P_r(\theta) - \dot{P}_r(\theta) \\ & + \frac{1}{\varepsilon_{\Delta f_r P_r}(\theta)} P_r(\theta) \hat{A}_r^T(\theta) \hat{A}_r(\theta) P_r(\theta) \\ & + \frac{1}{\varepsilon_{\Delta g_0^r P_r}(\theta)} P_r(\theta) (\hat{B}_2^r(\theta) K_r(\theta))^T (\hat{B}_2^r(\theta) K_r(\theta)) P_r(\theta) \\ & + (\varepsilon_{\Delta f_r P_r}(\theta) + \varepsilon_{\Delta g_0^r P_r}(\theta) + \varepsilon_{\Delta d_0^r P_r}(\theta)) \\ & + (1 + \varepsilon_{\Delta \varphi_r C_{C_r}}(\theta) + \varepsilon_{\Delta g_1^r C_{C_r}}(\theta)) \\ & \times P_r(\theta) C_{C_r}^T(\alpha, \theta) C_{C_r}(\alpha, \theta) P_r(\theta) \\ & + \left(1 + \frac{1}{\varepsilon_{\Delta \varphi_r C_{C_r}}(\theta)} + \frac{1}{\varepsilon_{\Delta \varphi_r \Delta g_1^r}(\theta)} \right) \end{aligned}$$

$$\begin{aligned} & \times P_r(\theta) \hat{C}_1^r(\alpha, \theta)^T \hat{C}_1^r(\alpha, \theta) P_r(\theta) \\ & + \left(1 + \frac{1}{\varepsilon_{\Delta g_1^r C_{C_r}}(\theta)} + \frac{1}{\varepsilon_{\Delta \varphi_r \Delta g_1^r}(\theta)} \right) P_r(\theta) \\ & \times \hat{D}_{12}^r(\alpha, \theta)^T \hat{D}_{12}^r(\alpha, \theta) P_r(\theta) \\ & + \left(\gamma^2 I - \frac{1}{\varepsilon_{\Delta d_0^r P_r}(\theta)} \hat{B}_1^r(\theta)^T \hat{B}_1^r(\theta) \right)^{-1} \\ & \times B_{C_r}(\alpha, \theta) B_{C_r}^T(\alpha, \theta) < 0, \end{aligned} \quad (76)$$

$$\gamma^2 I - \frac{1}{\varepsilon_{\Delta d_0^r P_r}(\theta)} \hat{B}_1^r(\theta)^T \hat{B}_1^r(\theta) > 0, \quad (77)$$

the closed-loop system (61) while considering the approximation error has the H_∞ performance with disturbance attenuation γ . And it is globally asymptotically stable when $\omega = 0$.

Proof. From (63), the sufficient condition for (46) can be gotten:

$$A_{C_r}^T(\alpha, \theta) P_r^{-1}(\theta) + P_r^{-1}(\theta) A_{C_r}(\alpha, \theta) + \dot{P}_r^{-1} < 0. \quad (78)$$

According to Theorem 5, if the condition (78) is fulfilled, the system (63) is asymptotically stable. For the system (65) with the approximation error, we can get

$$\begin{aligned} & A_{C_r}^T(\alpha, \theta) P_r^{-1}(\theta) + P_r^{-1}(\theta) A_{C_r}(\alpha, \theta) \\ & + \dot{P}_r^{-1}(\theta) + \frac{1}{\varepsilon_{\Delta f_r P_r}(\theta)} \hat{A}_r^T(\theta) \hat{A}_r(\theta) \\ & + \frac{1}{\varepsilon_{\Delta g_0^r P_r}(\theta)} (\hat{B}_2^r(\theta) K_r(\theta))^T \hat{B}_2^r(\theta) K_r(\theta) \\ & + (\varepsilon_{\Delta f_r P_r}(\theta) + \varepsilon_{\Delta g_0^r P_r}(\theta) + \varepsilon_{\Delta d_0^r P_r}(\theta)) P_r^{-1}(\theta) P_r^{-1}(\theta) \\ & + (1 + \varepsilon_{\Delta \varphi_r C_{C_r}}(\theta) + \varepsilon_{\Delta g_1^r C_{C_r}}(\theta)) C_{C_r}^T(\alpha, \theta) C_{C_r}(\alpha, \theta) \\ & + \left(1 + \frac{1}{\varepsilon_{\Delta \varphi_r C_{C_r}}(\theta)} + \frac{1}{\varepsilon_{\Delta \varphi_r \Delta g_1^r}(\theta)} \right) \hat{C}_1^r(\alpha, \theta)^T \hat{C}_1^r(\alpha, \theta) \\ & + \left(1 + \frac{1}{\varepsilon_{\Delta g_1^r C_{C_r}}(\theta)} + \frac{1}{\varepsilon_{\Delta \varphi_r \Delta g_1^r}(\theta)} \right) \hat{D}_{12}^r(\alpha, \theta)^T \hat{D}_{12}^r(\alpha, \theta) \\ & + \left(\gamma^2 I - \frac{1}{\varepsilon_{\Delta d_0^r P_r}(\theta)} \hat{B}_1^r(\theta)^T \hat{B}_1^r(\theta) \right)^{-1} P_r^{-1}(\theta) B_{C_r}(\alpha, \theta) \\ & \times B_{C_r}^T(\alpha, \theta) P_r^{-1}(\theta) < 0 \end{aligned} \quad (79)$$

from (76), which suggests that the condition (78) is fulfilled. Here, the condition (77) needs to be fulfilled. Then, the

system (65) is asymptotically stable. Next, we will show the disturbance attenuation performance.

Considering the Lyapunov function $V(\theta) = x^T P_r^{-1}(\theta) x$ which is piecewise continuous across the r th subspace boundary, we differentiate $V(\theta)$ and then integrate it from zero to infinity in the r th subspace. Then, we can get

$$\begin{aligned}
& \int_0^\infty \dot{V}(\theta) dt \\
&= \int_0^\infty \left(\dot{x}^T P_r^{-1}(\theta) x + x^T P_r^{-1}(\theta) \dot{x} + x^T \dot{P}_r^{-1}(\theta) x \right) dt \\
&= \int_0^\infty \left[x^T \left(A_{C_r}^T(\alpha, \theta) P_r^{-1}(\theta) + P_r^{-1}(\theta) A_{C_r}(\alpha, \theta) \right. \right. \\
&\quad \left. \left. + \dot{P}_r^{-1}(\theta) \right) x + \Delta f_r^T(\theta) P_r^{-1}(\theta) x \right. \\
&\quad \left. + \Delta g_0^r(\theta)^T P_r^{-1}(\theta) x + \Delta d_0^r(\theta)^T P_r^{-1}(\theta) x \right. \\
&\quad \left. + x^T P_r^{-1}(\theta) \Delta f_r(\theta) + x^T P_r^{-1}(\theta) \Delta g_0^r(\theta) \right. \\
&\quad \left. + x^T P_r^{-1}(\theta) \Delta d_0^r(\theta) + \omega^T B_{C_r}^T(\alpha, \theta) P_r^{-1}(\theta) x \right. \\
&\quad \left. + x^T P_r^{-1}(\theta) B_{C_r}(\alpha, \theta) \omega \right] dt \\
&\leq \int_0^\infty \left[x^T \left(A_{C_r}^T(\alpha, \theta) P_r^{-1}(\theta) + P_r^{-1}(\theta) A_{C_r}(\alpha, \theta) \right. \right. \\
&\quad \left. \left. + \dot{P}_r^{-1}(\theta) \right) x \right. \\
&\quad \left. + \frac{1}{\varepsilon_{\Delta f_r P_r}(\theta)} \Delta f_r^T(\theta) \Delta f_r(\theta) \right. \\
&\quad \left. + \varepsilon_{\Delta f_r P_r}(\theta) x^T P_r^{-1}(\theta) P_r^{-1}(\theta) x \right. \\
&\quad \left. + \frac{1}{\varepsilon_{\Delta g_0^r P_r}(\theta)} \Delta g_0^r(\theta)^T \Delta g_0^r(\theta) \right. \\
&\quad \left. + \varepsilon_{\Delta g_0^r P_r}(\theta) x^T P_r^{-1}(\theta) P_r^{-1}(\theta) x \right. \\
&\quad \left. + \frac{1}{\varepsilon_{\Delta d_0^r P_r}(\theta)} \Delta d_0^r(\theta)^T \Delta d_0^r(\theta) \right. \\
&\quad \left. + \varepsilon_{\Delta d_0^r P_r}(\theta) x^T P_r^{-1}(\theta) P_r^{-1}(\theta) x \right. \\
&\quad \left. + \omega^T B_{C_r}^T(\alpha, \theta) P_r^{-1}(\theta) x + x^T P_r^{-1}(\theta) B_{C_r}(\alpha, \theta) \omega \right] dt \\
&\leq \int_0^\infty \left[x^T \left(A_{C_r}^T(\alpha, \theta) P_r^{-1}(\theta) + P_r^{-1}(\theta) A_{C_r}(\alpha, \theta) \right. \right. \\
&\quad \left. \left. + \dot{P}_r^{-1}(\theta) \right) x \right. \\
&\quad \left. + \frac{1}{\varepsilon_{\Delta f_r P_r}(\theta)} \left(\widehat{A}_r(\theta) x \right)^T \widehat{A}_r(\theta) x + \frac{1}{\varepsilon_{\Delta g_0^r P_r}(\theta)} \right. \\
&\quad \left. \times \left(\widehat{B}_2^r(\theta) K_r(\theta) x \right)^T \widehat{B}_2^r(\theta) K_r(\theta) x \right. \\
&\quad \left. + \frac{1}{\varepsilon_{\Delta d_0^r P_r}(\theta)} \left(\widehat{B}_1^r(\theta) \omega \right)^T \widehat{B}_1^r(\theta) \omega \right.
\end{aligned}$$

$$\begin{aligned}
& \left. + \left(\varepsilon_{\Delta f_r P_r}(\theta) + \varepsilon_{\Delta g_0^r P_r}(\theta) \right) x^T P_r^{-1}(\theta) P_r^{-1}(\theta) x \right. \\
& \left. + \omega^T B_{C_r}^T(\alpha, \theta) P_r^{-1}(\theta) x + x^T P_r^{-1}(\theta) B_{C_r}(\alpha, \theta) \omega \right] dt \\
&\leq \int_0^\infty \left[-x^T \left(\left(1 + \varepsilon_{\Delta \varphi_r C_{Cr}}(\theta) + \varepsilon_{\Delta g_1^r C_{Cr}}(\theta) \right) \right. \right. \\
&\quad \left. \left. \times C_{C_r}^T(\alpha, \theta) C_{C_r}(\alpha, \theta) \right. \right. \\
&\quad \left. \left. + \left(1 + \frac{1}{\varepsilon_{\Delta \varphi_r C_{Cr}}(\theta)} \right. \right. \right. \\
&\quad \left. \left. \left. + \frac{1}{\varepsilon_{\Delta \varphi_r \Delta g_1^r}(\theta)} \right) \widehat{C}_1^r(\alpha, \theta)^T \widehat{C}_1^r(\alpha, \theta) \right. \right. \\
&\quad \left. \left. + \left(1 + \frac{1}{\varepsilon_{\Delta g_1^r C_{Cr}}(\theta)} \right. \right. \right. \\
&\quad \left. \left. \left. + \frac{1}{\varepsilon_{\Delta \varphi_r \Delta g_1^r}(\theta)} \right) \widehat{D}_{12}^r(\alpha, \theta)^T \widehat{D}_{12}^r(\alpha, \theta) \right. \right. \\
&\quad \left. \left. + \left(\gamma^2 I - \frac{1}{\varepsilon_{\Delta d_0^r P_r}(\theta)} \left(\widehat{B}_1^r(\theta) \right)^T \widehat{B}_1^r(\theta) \right)^{-1} P_r^{-1}(\theta) \right. \right. \\
&\quad \left. \left. \times B_{C_r}(\alpha, \theta) B_{C_r}^T(\alpha, \theta) P_r^{-1}(\theta) \right) x \right. \\
&\quad \left. + \frac{1}{\varepsilon_{\Delta d_0^r P_r}(\theta)} \left(\widehat{B}_1^r(\theta) \omega \right)^T \widehat{B}_1^r(\theta) \omega \right. \\
&\quad \left. + \omega^T B_{C_r}^T(\alpha, \theta) P_r^{-1}(\theta) x \right. \\
&\quad \left. + x^T P_r^{-1}(\theta) B_{C_r}(\alpha, \theta) \omega + \gamma^2 \omega^T \omega - \gamma^2 \omega^T \omega \right] dt \\
&\leq \int_0^\infty \left[-z^T z + \gamma^2 \omega^T \omega \right. \\
&\quad \left. - \left\| \left(\gamma^2 I - \frac{1}{\varepsilon_{\Delta d_0^r P_r}(\theta)} \widehat{B}_1^r(\theta)^T \widehat{B}_1^r(\theta) \right)^{1/2} \omega \right. \right. \\
&\quad \left. \left. - \left(\gamma^2 I - \frac{1}{\varepsilon_{\Delta d_0^r P_r}(\theta)} \widehat{B}_1^r(\theta)^T \widehat{B}_1^r(\theta) \right)^{-1/2} \right. \right. \\
&\quad \left. \left. \times B_{C_r}^T(\alpha, \theta) P_r^{-1}(\theta) x \right\|_2^2 \right] dt \\
&\leq \int_0^\infty \left[-z^T z + \gamma^2 \omega^T \omega \right] dt.
\end{aligned}$$

That is

$$V(x(\infty), \theta) - V(x(0), \theta) \leq \int_0^\infty (-z^T z + \gamma^2 \omega^T \omega) dt, \quad (81)$$

which suggests that, with $x(0) = 0$,

$$\|z\|_2 \leq \gamma \|\omega\|_2. \quad (82)$$

Overall, combining with the condition (75) and $P_r(\theta) = (F_r^T F_r)^{-1} F_r^T T(\theta) F_r (F_r^T F_r)^{-1}$, the system (61), while considering the approximation error, has the H_∞ performance with disturbance attenuation γ and it is globally asymptotically stable when $\omega = 0$. Then, the proof of this theorem is completed. $\square$

Remark 8. Considering the approximation error of the LPV T-S fuzzy system and bringing in the piecewise parameter-dependent Lyapunov functions to the LPV T-S fuzzy gain scheduling control, the state-feedback H_∞ control design is discussed here. And the sufficient conditions are given in Riccati inequalities form which can guarantee the H_∞ performance boundary γ of the closed-loop system.

As follows, based on Theorem 7, the equivalent conditions can be obtained in LMIs form which are easier to be solved.

Theorem 9. For the closed-loop system (61), while considering the approximation error and a positive scalar $\gamma > 0$, if there exist a symmetric matrix $T(\theta)$, a set of matrices $Q_r(\theta) = K_r(\theta)P_r(\theta)$ and $P_r(\theta) = (F_r^T F_r)^{-1} F_r^T T(\theta) F_r (F_r^T F_r)^{-1}$ satisfying

$$P_r(\theta) > 0, \quad (83)$$

$$\begin{bmatrix} W_r & P_r(\theta) & Q_r^T(\theta) \\ P_r(\theta) & -M_{P_r}^{-1} & 0 \\ Q_r(\theta) & 0 & -M_{Q_r}^{-1} \end{bmatrix} < 0,$$

where

$$\begin{aligned} W_r &= P_r(\theta) A_i^T(\theta) + A_i(\theta) P_r(\theta) - \dot{P}_r(\theta) \\ &+ Q_r^T(\theta) B_2^i(\theta)^T + B_2^i(\theta) Q_r(\theta) \\ &+ \varepsilon_{\tilde{A}_r P_r}(\theta) E_{rA}(\alpha, \theta) E_{rA}^T(\alpha, \theta) \\ &+ \varepsilon_{\tilde{B}_2 P_r}(\theta) E_{rB_2}(\alpha, \theta) E_{rB_2}^T(\alpha, \theta) \\ &+ \left(\gamma^2 I - \frac{1}{\varepsilon_{\Delta d_0^r P_r}(\theta)} \tilde{B}_1^r(\theta)^T \tilde{B}_1^r(\theta) \right)^{-1} \\ &\times \left(1 + \frac{1}{\varepsilon_{B_1^i \tilde{B}_1^i}(\theta)} \right) B_1^i(\theta) B_1^i(\theta)^T \end{aligned}$$

$$\begin{aligned} &+ \left(\gamma^2 I - \frac{1}{\varepsilon_{\Delta d_0^r P_r}(\theta)} \tilde{B}_1^r(\theta)^T \tilde{B}_1^r(\theta) \right)^{-1} \\ &\times \left(1 + \varepsilon_{B_1^i \tilde{B}_1^i}(\theta) \right) E_{rB_1}(\alpha, \theta) E_{rB_1}(\alpha, \theta)^T, \\ M_{P_r} &= \frac{1}{\varepsilon_{\tilde{A}_r P_r}(\theta)} + \frac{1}{\varepsilon_{\Delta f_r P_r}(\theta)} \tilde{A}_r^T(\theta) \tilde{A}_r(\theta) \\ &+ \left(1 + \frac{1}{\varepsilon_{\Delta \varphi_r C_{C_r}}(\theta)} + \frac{1}{\varepsilon_{\Delta \varphi_r \Delta g_1^r}(\theta)} \right) \\ &\times \tilde{C}_1^T(\alpha, \theta)^T \tilde{C}_1^T(\alpha, \theta) \\ &+ \left(1 + \frac{1}{\varepsilon_{\Delta g_1^r C_{C_r}}(\theta)} + \frac{1}{\varepsilon_{\Delta \varphi_r \Delta g_1^r}(\theta)} \right) \\ &\times \tilde{D}_{12}^T(\alpha, \theta)^T \tilde{D}_{12}^T(\alpha, \theta) \\ &+ \left(1 + \varepsilon_{\Delta \varphi_r C_{C_r}}(\theta) + \varepsilon_{\Delta g_1^r C_{C_r}}(\theta) \right) \\ &\times \left(1 + \frac{1}{\varepsilon_{C_1^i \tilde{C}_1^i}(\theta)} + \frac{1}{\varepsilon_{C_1^i D_{12}^i}(\theta)} + \frac{1}{\varepsilon_{C_1^i \tilde{D}_{12}^i}(\theta)} \right) \\ &\times C_1^i(\theta)^T C_1^i(\theta) \\ &+ \left(1 + \varepsilon_{\Delta \varphi_r C_{C_r}}(\theta) + \varepsilon_{\Delta g_1^r C_{C_r}}(\theta) \right) \\ &\times \left(1 + \varepsilon_{C_1^i \tilde{C}_1^i}(\theta) + \varepsilon_{D_{12}^i \tilde{C}_1^i}(\theta) + \frac{1}{\varepsilon_{\tilde{C}_1^i \tilde{D}_{12}^i}(\theta)} \right) E_{rC_1}^T \\ &\times E_{rC_1}^T(\alpha, \theta) E_{rC_1}(\alpha, \theta), \\ M_{Q_r} &= \frac{1}{\varepsilon_{\tilde{B}_2^i P_r}(\theta)} + \frac{1}{\varepsilon_{\Delta g_0^r P_r}(\theta)} \tilde{B}_2^r(\theta)^T \tilde{B}_2^r(\theta) \\ &+ \left(1 + \varepsilon_{\Delta \varphi_r C_{C_r}}(\theta) + \varepsilon_{\Delta g_1^r C_{C_r}}(\theta) \right) \\ &\times \left(1 + \varepsilon_{C_1^i D_{12}^i}(\theta) + \frac{1}{\varepsilon_{D_{12}^i \tilde{C}_1^i}(\theta)} + \frac{1}{\varepsilon_{D_{12}^i \tilde{D}_{12}^i}(\theta)} \right) \\ &\times D_{12}^i(\theta)^T D_{12}^i(\theta) \\ &+ \left(1 + \varepsilon_{\Delta \varphi_r C_{C_r}}(\theta) + \varepsilon_{\Delta g_1^r C_{C_r}}(\theta) \right) \\ &\times \left(1 + \varepsilon_{C_1^i \tilde{D}_{12}^i}(\theta) + \varepsilon_{\tilde{C}_1^i \tilde{D}_{12}^i}(\theta) + \varepsilon_{D_{12}^i \tilde{D}_{12}^i}(\theta) \right) \\ &\times E_{rD_{12}}^T(\alpha, \theta) E_{rD_{12}}(\alpha, \theta), \end{aligned} \quad (84)$$

and condition (77), the closed-loop system (61), while considering the approximation error, has the H_∞ performance with disturbance attenuation γ and it is globally asymptotically stable when $\omega = 0$. Moreover, the controller gain in the r th subspace is given by

$$K_r(\theta) = Q_r(\theta) P_r^{-1}(\theta). \quad (85)$$

Proof. According to condition (76) of Theorem 7, we know that the closed-loop system (61), while considering the approximation error, has the H_∞ performance boundary γ .

Then, substituting (37) into (76), we can get

$$\begin{aligned}
& P_r(\theta) A_{Cr}^T(\alpha, \theta) + A_{Cr}(\alpha, \theta) P_r(\theta) \\
&= P_r(\theta) \left(A_i(\theta) + \tilde{A}_r(\alpha, \theta) \right. \\
&\quad \left. + (B_2^i(\theta) + \tilde{B}_2^r(\alpha, \theta)) K_r(\theta) \right)^T \\
&\quad + \left(A_i(\theta) + \tilde{A}_r(\alpha, \theta) + (B_2^i(\theta) + \tilde{B}_2^r(\alpha, \theta)) \right. \\
&\quad \left. \times K_r(\theta) \right) P_r(\theta) \\
&= P_r(\theta) A_i^T(\theta) + A_i(\theta) P_r(\theta) + Q_r^T(\theta) B_2^i(\theta)^T \\
&\quad + B_2^i(\theta) Q_r(\theta) + P_r(\theta) \tilde{A}_r^T(\alpha, \theta) \\
&\quad + \tilde{A}_r(\alpha, \theta) P_r(\theta) + Q_r^T(\theta) \tilde{B}_2^r(\alpha, \theta)^T \\
&\quad + \tilde{B}_2^r(\alpha, \theta) Q_r(\theta) \\
&\leq P_r(\theta) A_i^T(\theta) + A_i(\theta) P_r(\theta) \\
&\quad + Q_r^T(\theta) B_2^i(\theta)^T + B_2^i(\theta) Q_r(\theta) \\
&\quad + \varepsilon_{\tilde{A}_r P_r}(\theta) E_{rA}(\alpha, \theta) E_{rA}^T(\alpha, \theta) \\
&\quad + \varepsilon_{\tilde{B}_2^r P_r}(\theta) E_{rB_2}(\alpha, \theta) E_{rB_2}^T(\alpha, \theta) \\
&\quad + \frac{1}{\varepsilon_{\tilde{A}_r P_r}(\theta)} P_r(\theta) P_r(\theta) + \frac{1}{\varepsilon_{\tilde{B}_2^r P_r}(\theta)} Q_r^T(\theta) Q_r(\theta).
\end{aligned} \tag{86}$$

Besides, we also have

$$\begin{aligned}
& P_r(\theta) C_{Ci}^T(\alpha, \theta) C_{Ci}(\alpha, \theta) P_r(\theta) \\
&= P_r(\theta) \left[C_1^i(\theta)^T C_1^i(\theta) + C_1^i(\theta)^T \tilde{C}_1^r(\alpha, \theta) \right. \\
&\quad \left. + \tilde{C}_1^r(\alpha, \theta)^T C_1^i(\theta) \right. \\
&\quad \left. + \tilde{C}_1^r(\alpha, \theta)^T \tilde{C}_1^r(\alpha, \theta) \right] P_r(\theta) \\
&\quad + P_r(\theta) C_1^i(\theta)^T D_{12}^i(\theta) Q_r(\theta) \\
&\quad + P_r(\theta) C_1^i(\theta)^T \tilde{D}_{12}^r(\alpha, \theta) Q_r(\theta) \\
&\quad + P_r(\theta) \tilde{C}_1^r(\alpha, \theta)^T D_{12}^i(\theta) Q_r(\theta) \\
&\quad + P_r(\theta) \tilde{C}_1^r(\alpha, \theta)^T \tilde{D}_{12}^r(\alpha, \theta) Q_r(\theta) \\
&\quad + Q_r^T(\theta) D_{12}^i(\theta)^T C_1^i(\theta) P_r(\theta) \\
&\quad + Q_r^T(\theta) D_{12}^i(\theta)^T \tilde{C}_1^r(\alpha, \theta) P_r(\theta) \\
&\quad + Q_r^T(\theta) \tilde{D}_{12}^r(\alpha, \theta)^T C_1^i(\theta) P_r(\theta)
\end{aligned}$$

$$\begin{aligned}
& + Q_r^T(\theta) \tilde{D}_{12}^r(\alpha, \theta)^T \tilde{C}_1^r(\alpha, \theta) P_r(\theta) \\
& + Q_r^T(\theta) \left[D_{12}^i(\theta)^T D_{12}^i(\theta) \right. \\
&\quad \left. + D_{12}^i(\theta)^T \tilde{D}_{12}^r(\alpha, \theta) \right. \\
&\quad \left. + \tilde{D}_{12}^r(\alpha, \theta)^T D_{12}^i(\theta) \right. \\
&\quad \left. + \tilde{D}_{12}^r(\alpha, \theta)^T \tilde{D}_{12}^r(\alpha, \theta) \right] Q_r(\theta) \\
&\leq P_r(\theta) \left[\left(1 + \frac{1}{\varepsilon_{C_1^i \tilde{C}_1^r}(\theta)} + \frac{1}{\varepsilon_{C_1^i D_{12}^i}(\theta)} \right. \right. \\
&\quad \left. \left. + \frac{1}{\varepsilon_{C_1^i \tilde{D}_{12}^r}(\theta)} \right) C_1^i(\theta)^T C_1^i(\theta) \right. \\
&\quad \left. + \left(1 + \varepsilon_{C_1^i \tilde{C}_1^r}(\theta) + \varepsilon_{D_{12}^i \tilde{C}_1^r}(\theta) \right. \right. \\
&\quad \left. \left. + \frac{1}{\varepsilon_{\tilde{C}_1^r \tilde{D}_{12}^r}(\theta)} \right) E_{rC_1}^T \right. \\
&\quad \left. \times (\alpha, \theta) E_{rC_1}(\alpha, \theta) \right] P_r(\theta) \\
&\quad + Q_r^T(\theta) \left[\left(1 + \varepsilon_{C_1^i D_{12}^i}(\theta) + \frac{1}{\varepsilon_{D_{12}^i \tilde{C}_1^r}(\theta)} \right. \right. \\
&\quad \left. \left. + \frac{1}{\varepsilon_{D_{12}^i \tilde{D}_{12}^r}(\theta)} \right) D_{12}^i(\theta)^T D_{12}^i(\theta) \right. \\
&\quad \left. + \left(1 + \varepsilon_{C_1^i \tilde{D}_{12}^r}(\theta) + \varepsilon_{\tilde{C}_1^r \tilde{D}_{12}^r}(\theta) \right. \right. \\
&\quad \left. \left. + \varepsilon_{D_{12}^i \tilde{D}_{12}^r}(\theta) \right) \right. \\
&\quad \left. \times E_{rD_{12}}^T(\alpha, \theta) E_{rD_{12}}(\alpha, \theta) \right] Q_r(\theta),
\end{aligned} \tag{87}$$

$$\begin{aligned}
& B_{Cr}(\alpha, \theta) B_{Cr}^T(\alpha, \theta) \\
&= B_1^i(\theta) B_1^i(\theta)^T + B_1^i(\theta) \tilde{B}_1^r(\alpha, \theta)^T \\
&\quad + \tilde{B}_1^r(\alpha, \theta) B_1^i(\theta)^T + \tilde{B}_1^r(\alpha, \theta) \tilde{B}_1^r(\alpha, \theta)^T \\
&\leq \left(1 + \frac{1}{\varepsilon_{B_1^i \tilde{B}_1^r}(\theta)} \right) B_1^i(\theta) B_1^i(\theta)^T \\
&\quad + \left(1 + \varepsilon_{B_1^i \tilde{B}_1^r}(\theta) \right) E_{rB_1}(\alpha, \theta) E_{rB_1}^T(\alpha, \theta).
\end{aligned} \tag{88}$$

Substituting (86), (87), and (88) into (76), the complete representation of (76) can be gotten. Then, by defining W_r , M_{P_r} , and M_{Q_r} , and using the Schur complement lemma, we can get the sufficient conditions in LMIs form as (83) and (80). The proof of this theorem is completed. $\square$

Remark 10. Based on Theorem 7, the conveniently solvable conditions are obtained in LMIs form in Theorem 9. For some parameter-dependent variables, such as $\varepsilon_*(\theta)$, they can be set as constant values in many cases by which the computation can be further simplified. However, subscripts of all $\varepsilon_*(\theta)$ are marked clearly in Theorem 5, Theorem 7, and Theorem 9 in order to show the meaning of them.

Remark 11. From Theorem 9, the sufficient conditions in LMIs form have been given. Though the solving of the LPV T-S fuzzy gain scheduling control design is different from the solving of the existing LPV gain scheduling control design, most of the procedures are similar. According to [9, 28, 29], the solution of the existing LPV gain scheduling control design has been given. Then, the main procedures of the LPV gain scheduling control design can be shown as follows. Firstly, according to the property of θ , the gridding process may be executed for the nonlinear form of θ or not. In particular, for the affine linear form of θ , the multicongruity property can be used to guarantee the continuous properties of continuously scheduled θ instead of the gridding process and the number of the LMIs can be reduced which can simplify the computation. Secondly, the parametrization of decision variables in LMIs needs to be executed. Through the two steps, infinite LMIs with infinite decision variables can be gotten. By solving the LMIs, the piecewise parameter-dependent functions can be obtained and then the controller can be determined.

5. Numerical Example

In order to show the availability of the approaches, a numerical example is provided in this subsection. Consider the ANPV system as

$$\begin{aligned}\dot{x} &= f(x, \theta) + B_1 \omega + B_2 u, \\ z &= \varphi(x, \theta),\end{aligned}\quad (89)$$

where $f(x, \theta) = \begin{bmatrix} 0.0005\theta x_1^2 + 0.0008x_2 \\ x_1 - 0.5x_2 \end{bmatrix}$, $\varphi(x, \theta) = 0.5\theta - x_1$, $B_1 = [0.0001 \ 0]^T$, $B_2 = [0 \ 1]^T$, and $\omega = \sin(t)$ is the sinusoidal interference signal.

Here, define that $\theta = vt + \tau$ is a linear function about time t and set the variation rate $\|v\|_2 \leq 1$. Assume that θ changes in the range $[-2, 2]$ and the variation range of x_1 is $[-1, 1]$. From (89), it can be found that the T-S fuzzy linearization is only used to deal with the nonlinearity about x_1 . Using the T-S fuzzy modeling procedure given before, the input space of x_1 can be partitioned into $[-1, 0]$ and $[0, 1]$. Then, three T-S fuzzy rules can be obtained as follows:

Rule 1: If x_1 is about -1 ,

$$\text{then } \begin{cases} \dot{x} = \begin{bmatrix} -0.0005\theta & 0.0008 \\ 1 & -0.5 \end{bmatrix} x + \begin{bmatrix} 0.0001 \\ 0 \end{bmatrix} \omega + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ z = [-0.5\theta - 1 \ 0] x, \end{cases}\quad (90)$$

Rule 2: If x_1 is about 0,

$$\text{then } \begin{cases} \dot{x} = \begin{bmatrix} 0 & 0.0008 \\ 1 & -0.5 \end{bmatrix} x + \begin{bmatrix} 0.0001 \\ 0 \end{bmatrix} \omega + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ z = [-1 \ 0] x, \end{cases}\quad (91)$$

Rule 3: If x_1 is about 1,

$$\text{then } \begin{cases} \dot{x} = \begin{bmatrix} 0.0005\theta & 0.0008 \\ 1 & -0.5 \end{bmatrix} x + \begin{bmatrix} 0.0001 \\ 0 \end{bmatrix} \omega + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ z = [0.5\theta - 1 \ 0] x. \end{cases}\quad (92)$$

In most cases, the approximation of the LPV T-S fuzzy system to the ANPV system usually has high accuracy, so the approximation error can often be ignored. Here, three T-S fuzzy rules are used at -1 , 0 , and 1 of x_1 to approximate the nonlinear term x_1^2 ; the maximum absolute error is $1.1102e - 16$. Thus, the approximation error can be ignored.

Meanwhile, using the definition in (35)-(36) and the triangular membership functions, four piecewise subspaces are obtained which are named $\bar{X}_1$, $\bar{X}_2$, $\bar{X}_3$, and $\bar{X}_4$ (as shown in Figure 3). Then, the T-S fuzzy model in each subspace can be represented as

$$\bar{X}_1 : \begin{cases} \dot{x} = \left(\begin{bmatrix} -0.0005\theta & 0.0008 \\ 1 & -0.5 \end{bmatrix} + \alpha_2(x_1) \begin{bmatrix} 0.0005\theta & 0 \\ 0 & 0 \end{bmatrix} \right) x \\ \quad + \begin{bmatrix} 0.0001 \\ 0 \end{bmatrix} \omega + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ z = ([-0.5\theta - 1 \ 0] + \alpha_2(x_1) [0.5\theta \ 0]) x, \quad x_1 \in [-1, -0.5], \end{cases}\quad (93)$$

$$\bar{X}_2 : \begin{cases} \dot{x} = \left(\begin{bmatrix} 0 & 0.0008 \\ 1 & -0.5 \end{bmatrix} + \alpha_1(x_1) \begin{bmatrix} -0.0005\theta & 0 \\ 0 & 0 \end{bmatrix} \right) x \\ \quad + \begin{bmatrix} 0.0001 \\ 0 \end{bmatrix} \omega + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ z = ([-1 \ 0] + \alpha_1(x_1) [-0.5\theta \ 0]) x, \quad x_1 \in [-0.5, 0], \end{cases}\quad (94)$$

$$\bar{X}_3 : \begin{cases} \dot{x} = \left(\begin{bmatrix} 0 & 0.0008 \\ 1 & -0.5 \end{bmatrix} + \alpha_3(x_1) \begin{bmatrix} 0.0005\theta & 0 \\ 0 & 0 \end{bmatrix} \right) x \\ \quad + \begin{bmatrix} 0.0001 \\ 0 \end{bmatrix} \omega + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ z = ([-1 \ 0] + \alpha_3(x_1) [0.5\theta \ 0]) x, \quad x_1 \in [0, 0.5], \end{cases}\quad (95)$$

$$\bar{X}_4 : \begin{cases} \dot{x} = \left(\begin{bmatrix} 0.0005\theta & 0.0008 \\ 1 & -0.5 \end{bmatrix} + \alpha_2(x_1) \begin{bmatrix} -0.0005\theta & 0 \\ 0 & 0 \end{bmatrix} \right) x \\ \quad + \begin{bmatrix} 0.0001 \\ 0 \end{bmatrix} \omega + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ z = ([0.5\theta - 1 \ 0] + \alpha_2(x_1) [-0.5\theta \ 0]) x, \quad x_1 \in [0.5, 1]. \end{cases}\quad (96)$$

According to the method in [22], it can be set that $\alpha_k(\bar{x}, \theta) = 0.5$ and the matrices of $E_{rA}(\theta)$ and $E_{rC_1}(\theta)$ for the i th ($i = 1, 2, 3$) fuzzy rule in the r th ($r = 1, 2, 3, 4, k \in \phi_r$) subspace can be obtained. Using the detailed method in [20], the constant matrices F_r can be given.

When the LPV T-S fuzzy system is linear with the scheduling parameter θ , we can parameterize the system

matrices $\bar{A}_r(\theta)$ and $\bar{C}_1^r(\theta)$ and decision variables $P_r(\theta)$ and $Q_r(\theta)$ in the r th subspace as

$$\begin{aligned}\bar{A}_r(\theta) &= \bar{A}_r^0 + \theta \bar{A}_r^1, & \bar{C}_1^r(\theta) &= \bar{C}_{1,0}^r + \theta \bar{C}_{1,1}^r, \\ \bar{P}_r(\theta) &= \bar{P}_r^0 + \theta \bar{P}_r^1, & \bar{Q}_r(\theta) &= \bar{Q}_r^0 + \theta \bar{Q}_r^1,\end{aligned}\quad (97)$$

where $\bar{A}_r^0, \bar{A}_r^1, \bar{C}_{1,0}^r, \bar{C}_{1,1}^r, \bar{P}_r^0, \bar{P}_r^1, \bar{Q}_r^0$, and $\bar{Q}_r^1$ are constant values and $r = 1, 2, 3, 4$.

Besides, the gridding process is avoided when solving the parameter-dependent Lyapunov functions $P_r(\theta)$ in r th subspace $\bar{X}_r$. Utilizing the multiconvexity property in [30], it just needs to validate the LMIs at the vertexes of θ . Here, the two vertexes at $\theta = -2$ and $\theta = 2$ are tested, respectively.

Following the design algorithm with $\gamma = 0.9$, the solution to the LMIs when $\varepsilon_*(\theta) = 10$ can be obtained:

$$\begin{aligned}P_1(\theta) &= \begin{bmatrix} 0.0265 + 0.0777\theta & -0.2 + 0.996\theta \\ -0.2 + 0.996\theta & 4.475 + 1.5562\theta \end{bmatrix}, \\ Q_1(\theta) &= [0.1476 + 0.0840\theta \quad 10.9778 + 0.0191\theta], \\ P_2(\theta) &= \begin{bmatrix} -0.0126 + 0.3006\theta & -0.6849 + 0.0309\theta \\ -0.6849 + 0.0309\theta & 3.6378 + 0.8595\theta \end{bmatrix}, \\ Q_2(\theta) &= [0.0997 + 0.0373\theta \quad 7.2065 - 0.1058\theta], \\ P_3(\theta) &= \begin{bmatrix} -0.0126 + 0.3006\theta & -0.6849 + 0.0309\theta \\ -0.6849 + 0.0309\theta & 3.6378 + 0.8595\theta \end{bmatrix}, \\ Q_3(\theta) &= [0.0997 + 0.0373\theta \quad 7.2065 - 0.1058\theta], \\ P_4(\theta) &= \begin{bmatrix} -0.0176 + 0.0651\theta & -0.1457 - 1.0275\theta \\ -0.1457 - 1.0275\theta & 6.5439 + 1.5280\theta \end{bmatrix}, \\ Q_4(\theta) &= [0.0423 - 0.0114\theta \quad 10.8734 + 0.0179\theta].\end{aligned}\quad (98)$$

Then, the state-feedback controller can be determined as $u = K_r(\theta)x$, where $K_r(\theta) = Q_r(\theta)P_r^{-1}(\theta)$ and $r = 1, 2, 3, 4$.

Because the effective performance region of the controller about x_1 is set as $[-1, 1]$, the scheduling values of θ are chosen to linearly vary from 0.3 to 1.3 with the interval of 0.1 to validate the effectiveness of the controller. The state responses of x_1 and x_2 and the evaluation response of z are shown in Figures 4, 5, and 6, respectively.

Meanwhile, in order to validate the effectiveness of the controller during the scheduling process of θ , θ is scheduled from 0.5 to 1 at 0.5 seconds, which guarantees that the variation rate of θ is 1. The responses of x_1 , x_2 , and z are shown in Figures 7, 8, and 9, respectively.

From Figures 4, 5, and 6, they suggest the effectiveness of the parameter-dependent controller. At different scheduling points of θ , the system can be stabilized by the state-feedback controller with H_∞ performance boundary $\gamma = 0.9$. From Figures 7, 8, and 9, the performance when switching at different θ points is validated. The time-domain response curves illustrate the stability and control performance of the system when continuously scheduling. Obviously, all the above results suggest the availability of sufficient conditions in Theorems 5, 7, and 9 for the control design of the LPV T-S fuzzy system.

FIGURE 3: Piecewise space partition of x_1 .

FIGURE 4: State response of x_1 .

6. Conclusion

In order to improve the existing LPV gain scheduling control, the T-S fuzzy modeling procedure was adopted to deal with the nonlinearity and relevant control design was studied in the paper. For the ANPV description of the nonlinear dynamic processes, a kind of developed T-S fuzzy modeling procedure with homogeneous linear consequent parts was proposed to carry forward the linearization. As a result, greater flexibility was obtained by which the number of scheduling parameters could be reduced and the approximation accuracy could be raised in any optional region. Moreover, the un-uniform partition method was utilized and the evaluation system for the approximation ability of the newly developed T-S fuzzy modeling procedure was established. Through this way, the number of fuzzy rules could be decreased while guaranteeing the approximation performance, and the LPV T-S fuzzy system was obtained with required accuracy. For the LPV T-S fuzzy gain scheduling control, the notion of piecewise parameter-dependent Lyapunov function was introduced to improve the solvability. Subsequently, the stabilization problem and the

FIGURE 5: State response of x_2 . FIGURE 8: State response of x_2 in scheduling. FIGURE 6: Evaluation response of z . FIGURE 9: Evaluation response of z in scheduling. FIGURE 7: State response of x_1 in scheduling.

state-feedback H_∞ control problem of the LPV T-S fuzzy system were discussed. The sufficient conditions were given in LMIs form and the approximation error was considered for the H_∞ control. Note that the novel LPV T-S fuzzy system can represent a great class of nonlinear dynamic processes. Based on the LPV T-S fuzzy system framework, more problems could be further studied following such a line, such as the output feedback control design and the corresponding applications.

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References

- [1] J. S. Shamma and M. Athans, "Analysis of gain scheduled control for nonlinear plants," *IEEE Transactions on Automatic Control*, vol. 35, no. 8, pp. 898–907, 1990.
- [2] W. J. Rugh, "Analytical framework for gain scheduling," *IEEE Control Systems Magazine*, vol. 11, no. 1, pp. 79–84, 1991.
- [3] J. S. Shamma and M. Athans, "Guaranteed properties of gain scheduled control for linear parameter-varying plants," *Automatica*, vol. 27, no. 3, pp. 559–564, 1991.
- [4] J. S. Shamma and M. Athans, "Gain scheduling: potential hazards and possible remedies," *IEEE Control Systems Magazine*, vol. 12, no. 3, pp. 101–107, 1992.
- [5] P. Apkarian and P. Gahinet, "A convex characterization of gain-scheduled H_∞ controllers," *IEEE Transactions on Automatic Control*, vol. 40, no. 5, pp. 853–864, 1995.
- [6] A. Packard, "Gain scheduling via linear fractional transformations," *Systems & Control Letters*, vol. 22, no. 2, pp. 79–92, 1994.
- [7] C. W. Scherer, "LPV control and full block multipliers," *Automatica*, vol. 37, no. 3, pp. 361–375, 2001.
- [8] G. Scorletti and L. El Ghaoui, "Improved LMI conditions for gain scheduling and related control problems," *International Journal of Robust and Nonlinear Control*, vol. 8, no. 10, pp. 845–877, 1998.
- [9] P. Apkarian and R. J. Adams, "Advanced gain-scheduling techniques for uncertain systems," *IEEE Transactions on Control Systems Technology*, vol. 6, no. 1, pp. 21–32, 1998.
- [10] P. Apkarian, P. Gahinet, and G. Becker, "Self-scheduled H_∞ control of linear parameter-varying systems: a design example," *Automatica*, vol. 31, no. 9, pp. 1251–1261, 1995.
- [11] G. Becker and A. Packard, "Robust performance of linear parametrically varying systems using parametrically-dependent linear feedback," *Systems & Control Letters*, vol. 23, no. 3, pp. 205–215, 1994.
- [12] F. Wu, "A generalized LPV system analysis and control synthesis framework," *International Journal of Control*, vol. 74, no. 7, pp. 745–759, 2001.
- [13] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Transactions on Systems, Man and Cybernetics*, vol. 15, no. 1, pp. 116–132, 1985.
- [14] M. Sugeno and G. T. Kang, "Structure identification of fuzzy model," *Fuzzy Sets and Systems*, vol. 28, no. 1, pp. 15–33, 1988.
- [15] S. Kawamoto, K. Tada, A. Ishigame, and T. Taniguchi, "An approach to stability analysis of second order fuzzy systems," in *Proceedings of the IEEE International Conference on Fuzzy Systems*, pp. 1427–1434, March 1992.
- [16] J. Kluska, *Analytical Methods in Fuzzy Modeling and Control*, Springer, Berlin, Germany, 2009.
- [17] J. Abonyi and R. Babuska, "Local and global identification and interpretation of parameters in Takagi-Sugeno fuzzy models," in *Proceedings of the 9th IEEE International Conference on Fuzzy Systems*, vol. 2, pp. 835–840, May 2000.
- [18] M. C. M. Teixeira and S. H. Zak, "Stabilizing controller design for uncertain nonlinear systems using fuzzy models," *IEEE Transactions on Fuzzy Systems*, vol. 7, no. 2, pp. 133–142, 1999.
- [19] K. Tanaka and H. O. Wang, *Fuzzy Control Systems Design and Analysis: A Linear Matrix Inequality Approach*, John Wiley & Sons, New York, NY, USA, 2004.
- [20] M. Johansson, A. Rantzer, and K. E. Årzén, "Piecewise quadratic stability of fuzzy systems," *IEEE Transactions on Fuzzy Systems*, vol. 7, no. 6, pp. 713–722, 1999.
- [21] G. Feng, "Controller synthesis of fuzzy dynamic systems based on piecewise Lyapunov functions," *IEEE Transactions on Fuzzy Systems*, vol. 11, no. 5, pp. 605–612, 2003.
- [22] G. Feng, " H_∞ controller design of fuzzy dynamic systems based on piecewise Lyapunov functions," *IEEE Transactions on Systems, Man, and Cybernetics B*, vol. 34, no. 1, pp. 283–292, 2004.
- [23] B. S. Chen, C. S. Tseng, and H. J. Uang, "Robustness design of nonlinear dynamic systems via fuzzy linear control," *IEEE Transactions on Fuzzy Systems*, vol. 7, no. 5, pp. 571–585, 1999.
- [24] N. S. D. Arrifano and V. A. Oliveira, "Robust H_∞ fuzzy control approach for a class of Markovian jump nonlinear systems," *IEEE Transactions on Fuzzy Systems*, vol. 14, no. 6, pp. 738–754, 2006.
- [25] Z. Lin, Y. Lin, and W. Zhang, " H_∞ stabilisation of nonlinear stochastic active fault-tolerant control systems: fuzzy-interpolation approach," *IET Control Theory & Applications*, vol. 4, no. 10, pp. 2003–2017, 2010.
- [26] Z. Lin, J. Liu, W. Zhang, and Y. Niu, "Stabilization of interconnected nonlinear stochastic Markovian jump systems via dissipativity approach," *Automatica*, vol. 47, no. 12, pp. 2796–2800, 2011.
- [27] E. K. P. Chong and S. H. Zak, *An Introduction to Optimization*, John Wiley & Sons, New Jersey, NJ, USA, 2013.
- [28] F. Wu, *Control of linear parameter varying systems [Ph.D. thesis]*, University of California at Berkeley, Berkeley, Calif, USA, 1995.
- [29] F. Wu, X. H. Yang, A. Packard, and G. Becker, "Induced L_2 -norm control for LPV systems with bounded parameter variation rates," *International Journal of Robust and Nonlinear Control*, vol. 6, no. 9–10, pp. 983–998, 1996.
- [30] P. Gahinet, P. Apkarian, and M. Chilali, "Affine parameter-dependent Lyapunov functions and real parametric uncertainty," *IEEE Transactions on Automatic Control*, vol. 41, no. 3, pp. 436–442, 1996.