Research Article

A Fully Discrete Symmetric Finite Volume Element Approximation of Nonlocal Reactive Flows in Porous Media

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We study symmetric finite volume element approximations for two-dimensional parabolic integrodifferential equations, arising in modeling of nonlocal reactive flows in porous media. It is proved that symmetric finite volume element approximations are convergent with optimal order in L^2 -norm. Numerical example is presented to illustrate the accuracy of our method.

1. Introduction

In this paper, we consider symmetric finite volume element discretizations of the following initial value problem for the operator equation for $u = u(t)$:

$$u_t + \mathcal{A}u + \int_0^t \mathcal{B}(t, s)u(s)ds = f(t), \quad u(0) = u_0, \quad (1)$$

where $\mathcal{A}$ is a strongly elliptic differential operator and $\mathcal{B}$ is a second-order elliptic differential operator in space. The operators $\mathcal{A}$ and $\mathcal{B}$ incorporate Dirichlet and Neumann boundary conditions. The problem (1) is an abstract form of an initial boundary value problem for a parabolic integrodifferential equation.

This model is very important in the transport of reactive and passive contaminants in aquifers, an area of active interdisciplinary research of mathematicians, engineers, and life scientists. From a mathematical point of view, the evolution of either a passive or reactive chemical within a velocity field exhibiting strong variation on many scales defies representation using classical Fickian theory. The evolution of a chemical in such a velocity field when modeled by Fickian-type theories leads to a dispersion tensor whose magnitude depends upon the time scales of observation. In order to avoid such difficulty, a new class of nonlocal models of transport have been derived. In this case, the constitutive relations involve either integrals or higher order derivatives, which take multiscales into consideration. We refer to [1, 2] for the derivation

of the mathematical models and for the precise hypotheses and analysis.

Mathematical formulations of this kind also arise naturally in various engineering models, such as nonlocal reactive transport in underground water flows in porous media [3], heat conduction, radioactive nuclear decay in fluid flows [4], non-Newtonian fluid flows, or viscoelastic deformations of materials with memory (in particular polymers) [5], semiconductor modeling [6], and biotechnology. One very important characteristic of all these models is that they all express a conservation of a certain quantity (mass, momentum, heat, etc.) in any moment for any subdomain. This in many applications is the most desirable feature of the approximation method when it comes to the numerical solution of the corresponding initial boundary value problem.

This type of equations has been extensively treated by finite element, finite difference, and collocation methods in the last years [7–12].

The finite element method conserves the flux approximately; therefore, in the asymptotic limit (i.e., when the grid step-size tends to zero) it produces adequate results. However, this could be a disadvantage when relatively coarse grids are used. Perhaps the most important property of the finite volume method is that it exactly conserves the approximate flux (heat, mass, etc.) over each computational cell. This important property combined with adequate accuracy and ease of implementation has contributed to the recent renewed interest in the method. The theoretical framework and

the basic tools for the analysis of the finite volume element methods have been developed in the last decade (see, e.g., [13–19]). Reference [20] has given the finite volume element approximations of the problem (1). But in a general case the coefficient matrix of the linear system obtained from the finite volume element method is not symmetric. In this paper, we study symmetric modified finite volume element approximations for two-dimensional parabolic integrodifferential equations, arising in modeling of nonlocal reactive flows in porous media. It is proved that symmetric modified finite volume element approximations are convergent with optimal order in L^2 -norm.

Throughout this paper we use C (without or with subscript or superscript) to denote a generic constant independent of the discretization parameter.

2. Fully Discrete Symmetric Finite Volume Element Scheme

Consider the following initial boundary value problem: find $u = u(x, t)$ such that

$$\begin{aligned} u_t - \nabla \cdot (A \nabla u) - \int_0^t \nabla \cdot (B \nabla u(s)) ds &= f, \quad (x, t) \in \Omega \times (0, T], \\ u(x, t) &= 0, \quad (x, t) \in \partial\Omega \times (0, T], \\ u(x, 0) &= u_0(x), \quad x \in \Omega, \end{aligned} \quad (2)$$

where Ω is a bounded polygonal domain in R^2 with boundary $\partial\Omega$. $A = \{a_{ij}(x)\}$ is a 2×2 real-valued symmetric and uniformly positive definite matrix, $B = \{b_{ij}(x, t, s)\}$ is a 2×2 matrix. $f = f(x, t)$ and $u_0(x)$ are known functions, which are assumed to be smooth so that problem (2) has a unique solution in a certain Sobolev space.

The problem (2) can be written in the form (1) by introducing the operators $\mathcal{A} : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ and $\mathcal{B} : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$:

$$\begin{aligned} (\mathcal{A}u, v) &= \int_{\Omega} A(x) \nabla u \cdot \nabla v dx, \\ (\mathcal{B}u, v) &= \int_{\Omega} B(x, t, s) \nabla u \cdot \nabla v dx, \end{aligned} \quad (3)$$

for all $v \in H_0^1(\Omega)$ and for any $t, s \in (0, T)$.

We assume that Ω is a convex polygonal domain. The domain Ω is split into triangular finite elements K . The elements K are considered to be a closed set, and the triangulation is denoted by $\mathcal{T}_h$. Then $\bar{\Omega} = \bigcup_{K \in \mathcal{T}_h} K$ and N_h denotes all nodes or vertices:

$$N_h = \{p : p \text{ is a vertex of element, } K \in \mathcal{T}_h \text{ and } p \in \bar{\Omega}\}. \quad (4)$$

In order to accommodate Dirichlet boundary conditions, we will also need the set of vertices that are internal to Ω , denoted by N_h^0 , that is, $N_h^0 = N_h \cap \Omega$. For a given vertex x_i , we define by $\Pi(i)$ the index set of all neighbors of x_i in N_h .

For a given triangulation $\mathcal{T}_h$, we construct a dual mesh $\mathcal{T}_h^*$ based upon $\mathcal{T}_h$ whose elements are called control volumes. In the finite volume methods, there are various ways to introduce the control volumes. Almost all approaches can be described in the following general scheme: in each triangle $K \in \mathcal{T}_h$ a point q is selected; similarly, on each of the three edges $\overline{x_i x_j}$ of K a point x_{ij} is selected; then q is connected with the points x_{ij} by straight lines (see Figure 1).

Thus, around each vertex $x_j \in N_h^0$, we associate a control volume $V_j \in \mathcal{T}_h^*$, which consists of the union of the sub-elements $K \in \mathcal{T}_h$, which have x_j as a vertex. Also, let γ_{ij} denote the interface of two control volumes V_i and V_j : $\gamma_{ij} = V_i \cap V_j$, $j \in \Pi(i)$. We call the partition regular or quasi-uniform, if there exists a positive constant $C > 0$ such that $C^{-1}h^2 \leq \text{meas}(V_i) \leq Ch^2$, for all $V_i \in \mathcal{T}_h^*$. Here, h is the maximal diameter of all elements $K \in \mathcal{T}_h$. In the first (and most popular) control volume partition, the point q is chosen to be the mediacenter (the center of gravity or centroid) of the finite element K and the points x_{ij} are chosen to be the midpoints of the edges of K . This type of control volume can be introduced for any finite element partition $\mathcal{T}_h$ and leads to relatively simple calculations. Besides, if the finite element partition $\mathcal{T}_h$ is locally regular, that is, there is a constant C such that $Ch_K^2 \leq \text{meas}(K) \leq h_K^2$, $h_K = \text{diam}(K)$, for all elements $K \in \mathcal{T}_h$, then the finite volume partition $\mathcal{T}_h^*$ is also locally regular.

In this paper, we will also use the construction of the control volumes in which the point q is the circumcenter of the element K . Then obviously, γ_{ij} are the perpendicular bisectors of the three edges of K . This construction requires that all finite elements are triangles of acute type, which we will assume whenever such a triangulation is used.

We define the linear finite element space S_h :

$$S_h = \{v \in C(\Omega) : v|_K \text{ is linear for all } K \in \mathcal{T}_h \text{ and } v|_{\partial\Omega} = 0\}, \quad (5)$$

and its dual volume element space S_h^* :

$$\begin{aligned} S_h^* &= \{v \in L^2(\Omega) : v|_V \\ &\text{is constant for all } V \in \mathcal{T}_h^* \text{ and } v|_{\partial\Omega} = 0\}. \end{aligned} \quad (6)$$

Obviously, $S_h = \text{span}\{\phi_i(x) : x_i \in N_h^0\}$ and $S_h^* = \text{span}\{\chi_i(x) : x_i \in N_h^0\}$, where ϕ_i are the standard nodal linear basis functions associated with the node x_i and χ_i are the characteristic functions of the volume V_i . Let $I_h : C(\Omega) \rightarrow S_h$ be the piecewise linear interpolation operator and let $I_h^* : C(\Omega) \rightarrow S_h^*$ be the piecewise constant interpolation operator. That is,

$$\begin{aligned} I_h u &= \sum_{x_i \in N_h^0} u(x_i) \phi_i(x), \\ I_h^* u &= \sum_{x_i \in N_h^0} u(x_i) \chi_i(x). \end{aligned} \quad (7)$$

FIGURE 1: A control volume.

The semidiscrete finite volume element approximation u_h of (2) is a solution to the problem: find $u_h(t) \in S_h$, for $t > 0$ such that

$$\begin{aligned} (u_{h,t}, v_h) + A(u_h, v_h) + \int_0^t B(t, s; u_h(s), v_h) ds \\ = (f, v_h), \quad v_h \in S_h^*, \\ u_h(0) = u_{0,h} \in S_h. \end{aligned} \quad (8)$$

Here, the bilinear forms $A(u, v)$ and $B(t, s; u, v)$ are defined by

$$\begin{aligned} A(u, v) &= - \sum_{x_i \in N_h} v_i \int_{\partial V_i} A(x) \nabla u \cdot \mathbf{n} dx, \\ (u, v) &\in S_h \times S_h^*, \\ B(t, s; u, v) &= - \sum_{x_i \in N_h} v_i \int_{\partial V_i} B(x, t, s) \nabla u \cdot \mathbf{n} dx, \\ (u, v) &\in S_h \times S_h^*, \end{aligned} \quad (9)$$

where $\mathbf{n}$ denotes the outer-normal direction to the domain under consideration.

It is more convenient to rewrite (8) in the following form, which we use in the exposition:

$$\begin{aligned} (u_{h,t}, I_h^* v_h) + A(u_h, I_h^* v_h) + \int_0^t B(t, s; u_h(s), I_h^* v_h) ds \\ = (f, I_h^* v_h), \quad v_h \in S_h. \end{aligned} \quad (10)$$

Next, we define the fully discrete time stepping schemes. Let $\Delta t > 0$ be a time-step size and $t_n = n\Delta t$, $u^n = u(t_n)$, and $\partial_t u^n = (u^n - u^{n-1})/\Delta t$. $u_h(0) \in S_h$ is an approximation of u_0 , such that $\|u_h(0) - u_0\| \leq Ch^2 \|u_0\|_2$.

The backward Euler scheme is defined to be the solution of $u_h^n \in S_h$ such that

$$\begin{aligned} \left(\frac{u_h^n - u_h^{n-1}}{\Delta t}, I_h^* v_h \right) + A(u_h^n, I_h^* v_h) \\ + \sum_{k=0}^{n-1} \omega_{n,k} B(t_n, t_k; u_h^k, I_h^* v_h) = (f^n, I_h^* v_h), \end{aligned} \quad (11)$$

where $\omega_{n,j}$ are the weights, and the quadrature error is given by for any smooth functions g and M and its error

$$q_n(g) = \int_0^{t_n} M(t_n, s) g(s) ds - \sum_{k=0}^{n-1} \omega_{n,k} M_{n,k} g(t_k) \quad (12)$$

satisfies

$$|q_n(g)| \leq C\Delta t \int_0^{t_n} (|g| + |g'|) dt. \quad (13)$$

Then we present fully discrete symmetric finite volume element scheme as the following: find $u_h^n \in S_h$, such that

$$\begin{aligned} (\partial_t u_h^n, I_h^* v_h) + \bar{A}(u_h^n, I_h^* v_h) + \sum_{k=0}^{n-1} \omega_{n,k} B(t_n, t_k; u_h^k, I_h^* v_h) \\ = (f^n, I_h^* v_h) + \bar{A}(u_h^{n-1}, I_h^* v_h) - A(u_h^{n-1}, I_h^* v_h), \end{aligned} \quad (14)$$

where $\bar{A}(u, v)$ is defined by

$$\begin{aligned} \bar{A}(u, v) &= - \sum_{x_i \in N_h} v_i \int_{\partial V_i} \bar{A}(x) \nabla u \cdot \mathbf{n} dx, \\ (u, v) &\in S_h \times S_h^*, \end{aligned} \quad (15)$$

$$\bar{A}|_K = \bar{A}_K, \quad \bar{A}_K = \frac{1}{\text{meas}(K)} \int_K A(x) dx. \quad (16)$$

We note that $\bar{A}(u_h^{n-1}, I_h^* v_h) - A(u_h^{n-1}, I_h^* v_h)$ is a modified term.

3. Some Auxiliary Results

Lemma 1 (see [17]). *If the matrix $A = (a_{ij})_{2 \times 2}$ is constant over each element $K \in \mathcal{T}_h$, then*

$$- \sum_{x_i \in N_h} v_i \int_{\partial V_i} A \nabla u \cdot \mathbf{n} ds = \int_{\Omega} A \nabla u \cdot \nabla v dx, \quad \forall u, v \in S_h. \quad (17)$$

Proposition 2. *Consider the following:*

$$- \sum_{x_i \in N_h} v_i \int_{\partial V_i} \bar{A} \nabla u_h^n \cdot \mathbf{n} ds = \int_{\Omega} A \nabla u_h^n \cdot \nabla v dx, \quad \forall u_h, v \in S_h. \quad (18)$$

Proof. By the definition of (16), $\bar{A}$ is piecewise constant on $\mathcal{T}_h$. Apply Lemma 1, and note that $\nabla u_h, \nabla v$ are constants on K , we can get (18). $\square$

It can be deduced from Proposition 2 that the coefficient matrix of (14) is symmetric.

Remark 3. For each $K \in \mathcal{T}_h$, if we select a point $x_K \in K$ and replace (16) by $\bar{A}_K = A(x_K)$, for all $K \in \mathcal{T}_h$, $\bar{A}|_K = \bar{A}_K$ from the analysis given in the following we can get the same error estimates as in Theorem 12.

We define a space $V^h = \{v \in L^\infty : v|_K = \text{constant, for all } K \in \mathcal{T}_h\}$, then introduce $P_h^0: C(\bar{\Omega}) \rightarrow V^h$ by $P_h^0 v = v(q_K)$ or $P_h^0 v = (1/|K|) \int_K v dx$, for all $K \in \mathcal{T}_h$.

Lemma 4 (see [21]). Define $A_h = (P_h^0 a_{ij})_{2 \times 2}$, $a_h(u_h, v) = \int_\Omega A_h \nabla u_h \cdot \nabla v dx$. Then there exist two positive constants C_1 and β , such that

$$\begin{aligned} |a_h(w, v)| &\leq C_1 \|w\|_1 \|v\|_1, \quad \forall w, v \in S_h, \\ a_h(v, v) &\geq \beta \|v\|_1^2, \quad \forall v \in S_h. \end{aligned} \quad (19)$$

Next we define some discrete norms on S_h :

$$\begin{aligned} \|u_h\|_{0,h}^2 &= (u_h, u_h)_{0,h} = (I_h^* u_h, I_h^* u_h), \\ |u_h|_{1,h}^2 &= \sum_{x_i \in N_h} \sum_{x_j \in \Pi(i)} \text{meas}(V_i) \left(\frac{u_i - u_j}{d_{ij}} \right)^2, \end{aligned} \quad (20)$$

$$\|u_h\|_{1,h}^2 = \|u_h\|_{0,h}^2 + |u_h|_{1,h}^2, \quad \|u_h\|^2 = (u_h, I_h^* u_h),$$

where $d_{ij} = d(x_i, x_j)$, the distance between x_i and x_j . Obviously, these norms are well defined for $u_h \in S_h^*$ as well and $\|u_h\|_{0,h} = \|u_h\|$.

Lemma 5 (see [20]). There exist two positive constants C_2, C_3 , independent of h , such that

$$\begin{aligned} C_2 \|v\|_{0,h} &\leq \|v_h\| \leq C_3 \|v\|_{0,h}, \quad \forall v_h \in S_h, \\ C_2 \|v_h\| &\leq \|v_h\| \leq C_3 \|v_h\|, \quad \forall v_h \in S_h, \\ C_2 \|v_h\|_{1,h} &\leq \|v_h\|_1 \leq C_3 \|v_h\|_{1,h}, \quad \forall v_h \in S_h. \end{aligned} \quad (21)$$

Lemma 6 (see [20]). Assume that the jumps (if any) of coefficient matrices $A(x)$ and $B(x, t, s)$ are aligned with the finite element partition $\mathcal{T}_h$, and over each element $K \in \mathcal{T}_h$ their entries are $W^{1,\infty}(K)$ -functions. Then,

(a) there are positive constants h_0 and c_0, c_1 , independent of h and u , such that for all $0 < h \leq h_0$,

$$\begin{aligned} |A(u_h, I_h^* v_h)| &\leq c_1 |u_h|_{1,h} |v_h|_{1,h}, \quad \forall u_h, v_h \in S_h, \\ A(v_h, I_h^* v_h) &\geq c_0 \|v_h\|_{1,h}^2, \quad \forall v_h \in S_h; \end{aligned} \quad (22)$$

(b) for $T > 0$ is fixed there is a constant $C = C(T) > 0$ independent of h and u , such that for $0 < t \leq T$,

$$|B(t, s; u_h, I_h^* v_h)| \leq C |u_h|_{1,h} |v_h|_{1,h}, \quad \forall u_h, v_h \in S_h. \quad (23)$$

4. L^2 -Norm Error Estimate

Now we introduce linear functions $l_{ij}(u)$, which are used in the error analysis of the finite volume element method:

$$l_{ij}(u) = - \int_{\gamma_{ij}} A \nabla (I_h u - u) \cdot \mathbf{n} dx, \quad \gamma_{ij} = V_i \cap V_j. \quad (24)$$

The following estimate is a simple consequence of the Bramble-Hilbert lemma.

Lemma 7 (see [20]). If $u \in H^2(\Omega)$, then there is a positive constant $C > 0$, independent of h , such that for $e_{ij} = \cup\{K \mid K \cap \gamma_{ij} \neq \emptyset, K \in \mathcal{T}_h\}$,

$$|l_{ij}(u)| \leq Ch \|A\|_{0,\infty} |u|_{2,e_{ij}}. \quad (25)$$

Lemma 8 (see [20]). (a) If $u \in H^2(\Omega)$, then there exists a positive constant $C > 0$, independent of h and u , such that

$$|A(u - I_h u, I_h^* v_h)| \leq Ch \|u\|_2 |v_h|_{1,h}, \quad v_h \in S_h. \quad (26)$$

(b) If $u(\cdot) \in L^\infty(H^2)$, then, for $T > 0$ fixed there is a constant $C = C(T) > 0$, independent of h and u , such that for $0 < t \leq T$

$$|B(t, s; u - I_h u, I_h^* v_h)| \leq Ch \|u\|_2 |v_h|_{1,h}, \quad v_h \in S_h. \quad (27)$$

For any fixed $t > 0$, one can define the Ritz projection function $u(x, t)$ and the operator $R_h : H_0^1 \cap H^2 \rightarrow S_h$, such that

$$A(u - R_h u, I_h^* v_h) = 0, \quad \forall v_h \in S_h. \quad (28)$$

Remark 9. If the partition $\mathcal{T}_h^*$ is regular (quasiuniform) and u is H^2 -regular, then

$$\|u - R_h u\|_1 \leq Ch \|u\|_2. \quad (29)$$

However, these estimates for the Ritz projection lead to suboptimal error estimates for the finite volume element solution of the integrodifferential equation. In order to obtain optimal order estimates, we need a projection that also takes into account the integral term. This type of projection was called by Cannon and Lin [7] the Ritz-Volterra projection and has been used in the analysis of the finite element method for integrodifferential equations.

We define the Ritz-Volterra projection $V_h u$ of a function $u(x, t)$ defined on the cylinder $\Omega \times [0, T]$. The Ritz-Volterra projection $V_h : L^\infty(H_0^1 \cap H^2) \rightarrow L^\infty(S_h)$ is defined for $t \geq 0$ by

$$\begin{aligned} A(u - V_h u, I_h^* v_h) &+ \int_0^t B(t, s; u(s) - V_h u(s), I_h^* v_h) ds = 0, \quad \forall v_h \in S_h. \end{aligned} \quad (30)$$

Lemma 10 (see [20]). Assume that $u \in L^\infty(H_0^1 \cap H^2)$ for $T > 0$ fixed, there is a constant $C = C(T) > 0$ independent of h and u , such that for all $0 < t \leq T$,

$$\|(u - V_h u)(t)\|_1 \leq Ch \left(\|u(t)\|_2 + \int_0^t \|u(s)\|_2 ds \right). \quad (31)$$

Now we consider an L^2 -estimate for the Ritz-Volterra projection. This estimate is optimal with respect to the order of convergence, but requires $W^{3,p}$ -regularity of the solution. Therefore, it is suboptimal with respect to the regularity of the solution and can be useful for p close to 1. Namely, we have the following result.

Lemma 11 (see [20]). *Assume that, for some $p > 1$, $u \in L^\infty(W^{3,p}(\Omega))$. Then for $T > 0$ fixed there exists a positive constant $C = C(T) > 0$ independent of h and u , such that, for all $0 < t \leq T$,*

$$\|(u - V_h u)(t)\| \leq Ch^2 \left(\|u(t)\|_{3,p} + \int_0^t \|u(s)\|_{3,p} ds \right). \quad (32)$$

Then we prove the following L^2 -norm error estimate.

Theorem 12. *Let u and u_h^n be the solution of (2) and (14), respectively. Then we have a constant $C > 0$ independent of h , Δt , and u , such that*

$$\begin{aligned} \|u^n - u_h^n\| &\leq Ch^2 \left(\|u_0\|_{3,p} + \int_0^{t_n} \|u_t\|_{3,p} ds \right) \\ &\quad + C\Delta t \int_0^{t_n} (\|u_{tt}\| + \|u_t\|_1) ds. \end{aligned} \quad (33)$$

Proof. Let $\theta^n = u_h^n - V_h u^n$, $\rho^n = u^n - V_h u^n$, from (2), (14), and (30), we have

$$\begin{aligned} &(\partial_t \theta^n, I_h^* v_h) + \bar{A}(\theta^n, I_h^* v_h) + \sum_{k=0}^{n-1} w_{n,k} B(t_n, t_k; \theta^k, I_h^* v_h) \\ &= (u_t^n - \partial_t u^n, I_h^* v_h) + (\partial_t \rho^n, I_h^* v_h) \\ &\quad + [\bar{A}(\theta^{n-1}, I_h^* v_h) - A(\theta^{n-1}, I_h^* v_h)] \\ &\quad - \Delta t [\bar{A}(\partial_t V_h u^n, I_h^* v_h) - A(\partial_t V_h u^n, I_h^* v_h)] \\ &\quad + \left[\int_0^{t_n} B(t_n, s; V_h u^n(s), I_h^* v_h) ds \right. \\ &\quad \left. - \sum_{k=0}^{n-1} w_{n,k} B(t_n, t_k; V_h u^k, I_h^* v_h) \right]. \end{aligned} \quad (34)$$

Let $\tau^n = u_t^n - \partial_t u^n + \partial_t \rho^n$,

$$\begin{aligned} L^{\bar{A}-A}(\theta^{n-1}, I_h^* v_h) &= [\bar{A}(\theta^{n-1}, I_h^* v_h) - A(\theta^{n-1}, I_h^* v_h)], \\ L^{\bar{A}-A}(\partial_t V_h u^n, I_h^* v_h) &= [\bar{A}(\partial_t V_h u^n, I_h^* v_h) \\ &\quad - A(\partial_t V_h u^n, I_h^* v_h)], \\ q^n(V_h u, I_h^* v_h) &= \left[\int_0^{t_n} B(t_n, s; V_h u^n(s), I_h^* v_h) ds \right. \\ &\quad \left. - \sum_{k=0}^{n-1} w_{n,k} B(t_n, t_k; V_h u^k, I_h^* v_h) \right]. \end{aligned} \quad (35)$$

Setting $v_h = \theta^n$ in (34), we get

$$\begin{aligned} &(\partial_t \theta^n, I_h^* \theta^n) + \bar{A}(\theta^n, I_h^* \theta^n) \\ &= (\tau^n, I_h^* \theta^n) + L^{\bar{A}-A}(\theta^{n-1}, I_h^* \theta^n) \\ &\quad - \Delta t L^{\bar{A}-A}(\partial_t V_h u^n, I_h^* \theta^n) + q^n(V_h u, I_h^* \theta^n) \\ &\quad - \sum_{k=0}^{n-1} w_{n,k} B(t_n, t_k; \theta^k, I_h^* \theta^n). \end{aligned} \quad (36)$$

We estimate the terms on the right-hand side:

$$\begin{aligned} |L^{\bar{A}-A}(\theta^{n-1}, I_h^* \theta^n)| &\leq Ch \|A\|_{1,\infty} \int_\Omega |\nabla \theta^{n-1}| |\nabla \theta^n| dx \\ &\leq Ch \|A\|_{1,\infty} \|\nabla \theta^{n-1}\| \|\nabla \theta^n\| \\ &\leq C \|\theta^{n-1}\|^2 + \varepsilon |\theta^n|_1^2, \\ |\Delta t L^{\bar{A}-A}(\partial_t V_h u^n, I_h^* \theta^n)| &\leq Ch \Delta t \|\nabla \partial_t V_h u^n\| \|\nabla \theta^n\| \\ &\leq C \Delta t \|\nabla \partial_t V_h u^n\| \|\theta^n\| \\ &\leq C \Delta t \int_{t_{n-1}}^{t_n} \|u_t\|_1^2 ds + C \|\theta^n\|^2, \\ |q^n(V_h u, I_h^* \theta^n)| &\leq \varepsilon |\theta^n|_1^2 + C \left(\Delta t \int_0^{t_n} |D_t V_h u(s)|_1 ds \right)^2 \\ &\leq \varepsilon |\theta^n|_1^2 + C \left(\Delta t \int_0^{t_n} \|u_t(s)\|_1 ds \right)^2, \\ \left| \sum_{k=0}^{n-1} w_{n,k} B(t_n, t_k; \theta^k, I_h^* \theta^n) \right| &\leq \varepsilon |\theta^n|_1^2 + C \sum_{k=0}^{n-1} \Delta t |\theta^k|_1^2, \\ |(\tau^n, I_h^* \theta^n)| &\leq C \|\theta^n\|^2 + C \Delta t \int_{t_{n-1}}^{t_n} \|u_{tt}(s)\|^2 ds \\ &\quad + Ch^4 \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \|u_t(s)\|_{3,p}^2 ds. \end{aligned} \quad (37)$$

Let $\varepsilon = C_0/6$, and use numerical quadrature error estimates to get

$$\begin{aligned} &\frac{\|\theta^n\|^2 - \|\theta^{n-1}\|^2}{2\Delta t} + C_0 |\theta^n|_1^2 \\ &\leq \frac{C_0}{2} |\theta^n|_1^2 + C \sum_{k=0}^{n-1} \Delta t |\theta^k|_1^2 + C \left(\Delta t \int_0^{t_n} \|u_t(s)\|_1 ds \right)^2 \\ &\quad + C \|\theta^{n-1}\|^2 + C \|\theta^n\|^2 + C \Delta t \int_{t_{n-1}}^{t_n} \|u_t\|_1^2 dt \\ &\quad + C \Delta t \int_{t_{n-1}}^{t_n} \|u_{tt}(s)\|^2 ds + Ch^4 \frac{1}{\Delta t} \int_{t_{n-1}}^{t_n} \|u_t(s)\|_{3,p}^2 ds. \end{aligned} \quad (38)$$

FIGURE 2: Triangulation and circumcenter dual mesh.

TABLE 1: Error behavior for $\Delta t/h^2 = 3.90625e - 4$.

Δt	h	$\ u(x, T) - u_h\ $	γ_h
1/10	1/16	$6.853070480441e - 3$	
1/40	1/32	$1.690155085494e - 3$	4.0
1/160	1/64	$4.225622073870e - 4$	4.0
1/640	1/128	$1.052607231633e - 4$	4.0

TABLE 2: Error behavior for $\Delta t/h^2 = 1.953125e - 4$.

Δt	h	$\ u(x, T) - u_h\ $	γ_h
1/20	1/16	$4.671190948063e - 3$	
1/80	1/32	$1.150735715485e - 3$	4.0
1/320	1/64	$2.880323203978e - 4$	4.0
1/1280	1/128	$7.164754698408e - 5$	4.0

Thus, multiplying each term on both sides of (38) by Δt , summing over n , and employing Gronwall's lemma, we obtain

$$\begin{aligned} \|\theta^n\| &\leq C \|\theta^0\| + Ch^2 \int_0^{t_n} \|u_t\|_{3,p} ds \\ &\quad + C\Delta t \int_0^{t_n} (\|u_{tt}\| + \|u_t\|_1) ds. \end{aligned} \quad (39)$$

And $\|\theta^0\| \leq Ch^2 \|u_0\|_{3,p}$; hence (33) follows from the above analysis and Lemma 11. $\square$

5. Numerical Experiments

In this section, we present a numerical example for solving the problem (2) by using the symmetric modified finite volume element scheme presented in Section 2. Let $\Omega = (0, 1) \times (0, 1)$, $T = 1$, T_h be Delaunay triangulation generated by EasyMesh [22] over Ω with mesh size h as shown in Figure 2 and time step be Δt .

We consider the case of $a_{11} = x_1 + x_2 + 3$, $a_{12} = x_1 + x_2 + 4$, $a_{21} = a_{22} = -(x_1 + x_2)$, $b_{11} = 1$, $b_{12} = 1/2$, $b_{21} = 0$, the exact solution $u(x, t) = (t^2 + \sin(\pi t)) \sin(\pi x_1) \sin(\pi x_2)$, and

the initial function $u(x, 0) = 0$. Denote the numerical solution of $u(x, T)$ by u_h and set $\gamma_h = \|u(x, T) - u_{2h}\| / \|u(x, T) - u_h\|$ with fixed ratio $\Delta t/h^2$. The numerical results are presented in Tables 1 and 2. It is observed that the results support our theory.

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