Research Article

Hopf Bifurcation of an Improved SLBS Model under the Influence of Latent Period

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A model applicable to describe the propagation of computer virus is developed and studied, along with the latent time incorporated. We regard time delay as a bifurcating parameter to study the dynamical behaviors including local asymptotical stability and local Hopf bifurcation. By analyzing the associated characteristic equation, Hopf bifurcation occurs when the time delay passes through a sequence of critical values. A formula for determining the direction of the Hopf bifurcation and the stability of bifurcating periodic solutions is given by using the normal form method and center manifold theorem. Finally, illustrative examples are given to support the theoretical results.

1. Introduction

With the rapid development of computer technologies and network applications, the Internet has become a powerful mechanism for propagating computer virus. Because of this, computers connected to the Internet become much vulnerable to digital threats.

It is quite urgent to understand how computer viruses spread over the Internet and to propose effective measures to cope with this issue. To achieve this goal, and in view of the fact that the spread of virus among computers resembles that of biological virus among a population, it is suitable to establish dynamical models describing the propagation of computer viruses across the Internet by appropriately modifying epidemic models [1].

Based on the fact that infectivity is one of the common features shared by computer viruses and their biological counterparts [2], some classic epidemic models were established for computer virus propagation, such as the SIRS model [3–7], the SEIR model [8, 9], the SEIRS model [10], the SEIQV model [11], and the SEIQRS model [12]. In [13–15] the authors made the following assumptions.

(H1) All computers connected to the Internet are partitioned into three compartments: susceptible computers (S-computers), infected computers that are

latent (*L*-computers), and infected computers that are breaking out (*B*-computers).

(H2) All newly connected computers are virus-free.

(H3) External computers are connected to the Internet at constant rate δ . Meanwhile, internal computers are disconnected from the Internet at rate δ .

(H4) Each virus-free computer is infected by a virulent computer at constant rate β , and the ratio of previously virus-free computers that are infected exactly at time t is $\beta S(L+B)$ [16].

(H5) Breaking-out computers are cured at constant rate γ .

(H6) Latent computers break out at constant rate α .

According to the above assumptions, the authors of [14, 15] proposed the proposed the following SLBS model, which is formulated as

$$\begin{aligned}\dot{S} &= \delta - \beta S(L+B) + \gamma B - \delta S, \\ \dot{L} &= \beta S(L+B) - \alpha L - \delta L, \\ \dot{B} &= \alpha L - \gamma B - \delta B.\end{aligned}\tag{1}$$

It is well known that some computer viruses would delay a period to break out after the computer is infected. However,

the above model fails to consider the concrete time of the delay. Thus, this paper aims to establish a model to incorporate the unconsidered factor, by adding a delay item to the above model. First, we give another assumption as (H7): L -computers turn out to be B -computers with constant time delay τ .

According to the above assumptions (H1)–(H7), the new model with time delay is formulated as

$$\begin{aligned}\dot{S} &= \delta - \beta S(L+B) + \gamma B - \delta S, \\ \dot{L} &= \beta S(L+B) - \alpha L(t-\tau) - \delta L, \\ \dot{B} &= \alpha L(t-\tau) - \gamma B - \delta B.\end{aligned}\quad (2)$$

Here, we let $S(t)$, $L(t)$, and $B(t)$ represent the percentage of S -, L -, and B -computers in all internal computers at time t , respectively. Then we get $S(t) + L(t) + B(t) \equiv 1$ and consider the following equivalent two-dimensional subsystem:

$$\begin{aligned}\dot{L} &= \beta(1-L-B)(L+B) - \alpha L(t-\tau) - \delta L, \\ \dot{B} &= \alpha L(t-\tau) - \gamma B - \delta B.\end{aligned}\quad (3)$$

The initial conditions of (3) are given by $L(\theta) = \phi_1(\theta) > 0$, $B(\theta) = \phi_2(\theta) > 0$, and $\theta \in [-\tau, 0]$, where $(\phi_1(\theta), \phi_2(\theta)) \in C([-\tau, 0], R_+^2)$, the Banach space of the continuous functions mapping the interval $[-\tau, 0]$ into $R_+^2 = \{(x_1, x_2) : x_i \geq 0, i = 1, 2\}$.

The remainder of this paper is organized as follows. In Section 2, the stability of trivial solutions and the existence of Hopf bifurcation are discussed. In Section 3, a formula for determining the direction of Hopf bifurcation and the stability of bifurcating periodic solutions will be given by using the normal form and center manifold theorem introduced by Hassard et al. in [17]. In Section 4, numerical simulations aimed at justifying the theoretical analysis will be reported.

2. Stability of the Equilibria and Existence of Hopf Bifurcation

This section is intended to study model (3) theoretically, by analyzing the stability of its solutions and the existence of Hopf bifurcation. For the convenience of the following description, we define the basic reproduction number of system (3) as

$$R_0 = \frac{\beta(\alpha + \gamma + \delta)}{(\alpha + \delta)(\gamma + \delta)}.\quad (4)$$

We have the following result with respect to the stable state of system (3).

Theorem 1. Consider system (3) with $\tau = 0$. Then the unique virus-free equilibrium $E_0 = (0, 0)$ is globally asymptotically stable if $R_0 < 1$, whereas E_0 becomes unstable and the unique positive equilibrium $E_* = ((\gamma + \delta)(1 - 1/R_0)/(\alpha + \delta + \gamma), \alpha(1 - 1/R_0)/(\alpha + \delta + \gamma))$ is locally asymptotically stable if $R_0 > 1$.

The proof is omitted here (see [14] for details).

The linearized equations of (3) are as follows:

$$\begin{aligned}\dot{L} &= \left[-\beta + \frac{2\beta}{R_0} - \delta\right]L + \left[-\beta + \frac{2\beta}{R_0}\right]B - \alpha L(t-\tau), \\ \dot{B} &= \alpha L(t-\tau) - \gamma B - \delta B.\end{aligned}\quad (5)$$

The determinant of the Jacobian matrix of the system (5) at E_* is given by $|yE - A_1 - B_1 e^{-y\tau}| = 0$, where

$$\begin{aligned}A_1 &= \begin{bmatrix} -\beta + \frac{2\beta}{R_0} - \delta & -\beta + \frac{2\beta}{R_0} \\ 0 & -\gamma - \delta \end{bmatrix}, \\ B_1 &= \begin{bmatrix} -\alpha & 0 \\ \alpha & 0 \end{bmatrix}.\end{aligned}\quad (6)$$

Let $d = -\beta + (2(\alpha + \delta)(\gamma + \delta))/(\alpha + \gamma + \delta)$, and we can obtain the following characteristic equation:

$$y^2 + m_1 y + m_0 + (n_1 y + n_0) e^{-y\tau} = 0,\quad (7)$$

where $m_1 = \gamma + 2\delta - d$, $m_0 = (\gamma + \delta)(\delta - d)$, $n_1 = \alpha$, $n_0 = \alpha(\gamma + \delta - d)$.

Theorem 2. Suppose that $R_0 > 1$, $\gamma + \delta \geq \alpha$, and $(\gamma + \delta)^2(\delta - d)^2 - \alpha^2(\gamma + \delta - d)^2 < 0$ hold; then the positive equilibrium E_* is asymptotically stable for $\tau \in [0, \tau_0)$ and system (3) undergoes a Hopf bifurcation at E_* when $\tau = \tau_0$.

Proof. Suppose that $y = i\omega$ ($\omega > 0$) is a root of (7); then separating the real and imaginary parts of (7), we have

$$\begin{aligned}m_1 \omega &= n_0 \sin \omega \tau - n_1 \omega \cos \omega \tau, \\ \omega^2 - m_0 &= n_0 \cos \omega \tau + n_1 \omega \sin \omega \tau.\end{aligned}\quad (8)$$

Adding up the squares of (8) yields

$$\omega^4 + (m_1^2 - 2m_0 - n_1^2)\omega^2 + m_0^2 - n_0^2 = 0.\quad (9)$$

Since $\gamma + \delta \geq \alpha$, we derive the following equations:

$$\begin{aligned}m_1^2 - 2m_0 - n_1^2 &= (\gamma + 2\delta - d)^2 - 2(\gamma + \delta)(\delta - d) - \alpha^2 \\ &= (\delta - d)^2 + (\gamma + \delta + \alpha)(\gamma + \delta - \alpha) > 0, \\ m_0^2 - n_0^2 &= (\gamma + \delta)^2(\delta - d)^2 - \alpha^2(\gamma + \delta - d)^2 < 0.\end{aligned}\quad (10)$$

Therefore, (9) exists a unique positive solution ω_0 , and the characteristic equation (7) has a pair of pure imaginary roots $\pm i\omega_0$. By (8), we have

$$\begin{aligned}\tau_n &= \frac{1}{\omega_0} \arccos \frac{n_0(\omega_0^2 - m_0) - m_1 n_1 \omega_0^2}{n_0^2 + n_1^2 \omega_0^2} \\ &\quad + \frac{2n\pi}{\omega_0}, \quad n = 0, 1, 2, \dots\end{aligned}\quad (11)$$

By Theorem 1, for $\tau = 0$, the positive equilibrium E_* is locally asymptotically stable, and hence by Butler's Lemma [18], E_* remains stable for $\tau < \tau_0$. Now, we need to show

$$\left. \frac{d(\operatorname{Re} y)}{d\tau} \right|_{\tau=\tau_0} > 0. \quad (12)$$

This will signify that there exists at least one eigenvalue with positive real part for $\tau > \tau_0$. Moreover, the conditions for Hopf bifurcation [19] are then satisfied yielding the required periodic solution. Now, by differentiating (9) with respect to τ , we get

$$\begin{aligned} (2y + m_1 + n_1 e^{-y\tau} - \tau(n_1 y + n_0) e^{-y\tau}) \frac{dy}{d\tau} \\ = y(n_1 y + n_0) e^{-y\tau}. \end{aligned} \quad (13)$$

This gives

$$\begin{aligned} \left[\frac{dy}{d\tau} \right]^{-1} &= \frac{2y + m_1 + n_1 e^{-y\tau} - \tau(n_1 y + n_0) e^{-y\tau}}{y(n_1 y + n_0) e^{-y\tau}} \\ &= \frac{2y + m_1}{y(n_1 y + n_0) e^{-y\tau}} + \frac{n_1}{y(n_1 y + n_0)} - \frac{\tau}{y} \\ &= \frac{y^2 - m_0}{-y^2(y^2 + m_1 y + m_0)} + \frac{-n_0}{y^2(n_1 y + n_0)} - \frac{\tau}{y}. \end{aligned} \quad (14)$$

Thus,

$$\begin{aligned} \operatorname{sign} \left\{ \frac{d(\operatorname{Re} y)}{d\tau} \right\}_{\tau=\tau_k} &= \operatorname{sign} \left\{ \operatorname{Re} \left[\frac{dy}{d\tau} \right]^{-1} \right\}_{y=i\omega_0} \\ &= \operatorname{sign} \left\{ \operatorname{Re} \left[\frac{y^2 - m_0}{-y^2(y^2 + m_1 y + m_0)} \right] \right. \\ &\quad \left. + \operatorname{Re} \left[\frac{-n_0}{y^2(n_1 y + n_0)} \right] \right\}_{y=i\omega_0} \\ &= \operatorname{sign} \left\{ \frac{(\omega_0^2 + m_0)(\omega_0^2 - m_0)}{\omega_0^2[(m_0 - \omega_0^2)^2 + (m_1 \omega_0)^2]} \right. \\ &\quad \left. + \frac{n_0^2}{\omega_0^2(n_0^2 + n_1^2 \omega_0^2)} \right\}. \end{aligned} \quad (15)$$

Since $(m_0 - \omega_0^2)^2 + (m_1 \omega_0)^2 = n_0^2 + n_1^2 \omega_0^2$, we have that

$$\begin{aligned} \operatorname{sign} \left\{ \frac{d(\operatorname{Re} y)}{d\tau} \right\}_{\tau=\tau_k} &= \operatorname{sign} \left\{ \frac{(\omega_0^2 + m_0)(\omega_0^2 - m_0)}{\omega_0^2[(m_0 - \omega_0^2)^2 + (m_1 \omega_0)^2]} + \frac{n_0^2}{\omega_0^2(n_0^2 + n_1^2 \omega_0^2)} \right\} \\ &= \operatorname{sign} \left\{ \frac{\omega_0^4 + n_0^2 - m_0^2}{\omega_0^2(n_0^2 + n_1^2 \omega_0^2)} \right\}. \end{aligned} \quad (16)$$

As $m_0^2 - n_0^2 < 0$, thus

$$\left. \frac{d(\operatorname{Re} y)}{d\tau} \right|_{\tau=\tau_0} > 0. \quad (17)$$

Therefore, the transversality condition holds and thus Hopf bifurcation occurs at $\tau = \tau_0$. The proof is complete. $\square$

3. Direction of the Hopf Bifurcation

In this section, we derive explicit formulae for computing the direction of the Hopf bifurcation and the stability of bifurcation periodic solution at critical value τ_0 by using the normal form theory and center manifold reduction.

Letting $t = s\tau$, $u_1 = L - L_*$, $u_2 = B - B_*$, $u_i(t) = u_i(\tau t)$, and $\tau = \tau_0 + \mu$, system (3) is transformed to an FDE as

$$\dot{u}(t) = L_v(u_t) + f(\mu, u_t), \quad (18)$$

where $u(t) = (u_1(t), u_2(t))^T \in R^2$, $u_t(\theta) = u(t + \theta)$, $\theta \in [0, 1]$, $L_v : C \rightarrow R$, $f : R \times C \rightarrow R$,

$$\begin{aligned} L_v(u_t) &= (\tau_0 + \mu) \\ &\times \begin{bmatrix} -\beta + \frac{2(\alpha + \delta)(\gamma + \delta)}{\alpha + \gamma + \delta} - \delta & -\beta + \frac{2(\alpha + \delta)(\gamma + \delta)}{\alpha + \gamma + \delta} \\ 0 & -\gamma - \delta \end{bmatrix} \\ &\times \begin{bmatrix} \varphi_1(0) \\ \varphi_2(0) \end{bmatrix} + (\tau_0 + \mu) \begin{bmatrix} -\alpha & 0 \\ \alpha & 0 \end{bmatrix} \begin{bmatrix} \varphi_1(-1) \\ \varphi_2(-1) \end{bmatrix} \\ &= (\tau_0 + \mu) A_1 \begin{bmatrix} \varphi_1(0) \\ \varphi_2(0) \end{bmatrix} + (\tau_0 + \mu) B_1 \begin{bmatrix} \varphi_1(-1) \\ \varphi_2(-1) \end{bmatrix}, \\ f(v, u_t) &= (\tau_0 + \mu) \begin{bmatrix} -\beta \varphi_1^2(0) - 2\beta \varphi_1(0) \varphi_2(0) - \beta \varphi_2^2(0) \\ 0 \end{bmatrix}. \end{aligned} \quad (19)$$

Using the Riesz representation theorem, there exists a function $\eta(\theta, \mu)$ of bounded variation for $\theta \in [0, 1]$, such that

$$L_v \varphi = \int_{-1}^0 d\eta(\theta, \mu) \varphi(\theta), \quad \varphi \in C. \quad (20)$$

In fact, we can choose

$$\eta(\theta, \mu) = (\tau_0 + \mu) A_2 \delta(\theta) + (\tau_0 + \mu) B_2 \delta(\theta + 1), \quad (21)$$

where $\delta(\theta)$ is Dirac delta function. In the following, for $\varphi \in [0, 1]$, we define

$$\begin{aligned} A(\mu) \varphi &= \begin{cases} \frac{d\varphi(\theta)}{d\theta}, & \theta \in [-1, 0), \\ \int_{-1}^0 d\eta(s, \mu) \varphi(s), & \theta = 0, \end{cases} \\ R(\mu) \varphi &= \begin{cases} 0, & \theta \in [-1, 0), \\ f(\mu, \varphi), & \theta = 0. \end{cases} \end{aligned} \quad (22)$$

Then system (18) can be rewritten as

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t, \quad (23)$$

where

$$\frac{du_t(\theta)}{d\theta} = \frac{du(t+\theta)}{d\theta} = \frac{du(t+\theta)}{dt} = \frac{du_t(\theta)}{dt}. \quad (24)$$

For $\phi \in C^* = C([0, 1], (R^2)^*)$, the adjoint operator A^* of A is defined by

$$A^*(0)\phi(s) = \begin{cases} -\frac{d\phi(s)}{ds}, & s \in (0, 1], \\ \int_{-1}^0 d\eta^T(t, 0)\phi(-t), & s = 0, \end{cases} \quad (25)$$

where η^T is the transpose of the matrix η . We define

$$\langle \phi, \varphi \rangle = \bar{\phi}(0)\varphi(0) - \int_{-1}^0 \int_{\xi=0}^{\theta} \bar{\phi}(\xi - \theta) d\eta(\theta)\varphi(\xi) d\xi, \quad (26)$$

where $\eta(\theta) = \eta(\theta, 0)$. We know that $i\omega_0\tau_0$ is an eigenvalue of $A(0)$, so $-i\omega_0\tau_0$ is also an eigenvalue of $A^*(0)$. We can get $q(\theta) = (1, q_1)^T e^{i\omega_0\tau_0\theta}$.

From the above discussion, it is easy to know that

$$\begin{aligned} Aq(\theta) &= i\omega_0\tau_0 q(\theta), \\ \tau_0 A_1 q(0) + \tau_0 B_1 q(-1) &= i\omega_0\tau_0 q(0). \end{aligned} \quad (27)$$

Hence, we obtain

$$q_1 = \frac{\alpha}{(i\omega_0 + \gamma + \delta)e^{i\omega_0\tau_0}}. \quad (28)$$

Suppose that the eigenvector

$$q^*(s) = \frac{1}{\rho}(1, q_1^*)^T e^{i\omega_0\tau_0 s}. \quad (29)$$

Then the following relationship is obtained:

$$A^*q^*(0) = -i\omega_0\tau_0 q^*(0). \quad (30)$$

Hence, we obtain

$$q_1^* = \frac{1}{(\gamma + \delta - i\omega_0)} \left(\frac{2(\alpha + \delta)(\gamma + \delta)}{\alpha + \gamma + \delta} - \beta \right). \quad (31)$$

Let

$$\begin{aligned} \langle q^*, q \rangle &= 1, \\ \langle q^*, q \rangle &= \bar{q}^*(0)q(0) - \int_{\theta=-1}^0 \int_{\xi=0}^{\theta} \bar{q}^{*T}(\xi - \theta) d\eta(\theta)\varphi(\xi) d\xi \\ &= \frac{1}{\rho}(1 + q_1\bar{q}_1^*) \\ &\quad - \int_{\theta=-1}^0 \int_{\xi=0}^{\theta} \tau_0 \frac{1}{\rho} [1, \bar{q}_1^*] [A_2\delta(\theta) + B_2\delta(\theta + 1)] \\ &\quad \times \begin{bmatrix} 1 \\ q_1 \end{bmatrix} e^{i\tau_0\omega_0\theta} d\xi d\theta \\ &= \frac{1}{\rho}(1 + q_1\bar{q}_1^*) + \tau_0 \frac{1}{\rho} \alpha (1 - \bar{q}_1^*) e^{-i\tau_0\omega_0} = 1. \end{aligned} \quad (32)$$

Hence, we obtain

$$\rho = (1 + q_1\bar{q}_1^*) + \tau_0\alpha(1 - \bar{q}_1^*)e^{-i\tau_0\omega_0}. \quad (33)$$

In the remainder of this section, by using the same notations as in the work by Hassard et al. [17], we first compute the coordinates f or describing the center manifold C_0 at $\mu = 0$. Letting u_t be the solution of (18) with $\mu = 0$, we define $z(t) = \langle q^*, u_t \rangle$, and

$$W(t, \theta) = u_t - 2 \operatorname{Re} \{z(t)q(\theta)\}. \quad (34)$$

On the center manifold C_0 we have

$$W(t, \theta) = W(z, \bar{z}, t), \quad (35)$$

where

$$W(z, \bar{z}, t) = W_{20}(\theta) \frac{z^2}{2} + W_{11}(\theta) \frac{z\bar{z}}{2} + W_{02}(\theta) \frac{\bar{z}^2}{2} + \dots. \quad (36)$$

In fact, z and $\bar{z}$ are local coordinates for C_0 in the direction of q and q^* . Note that if u_t is, we will deal with real solutions only. Since $\mu = 0$,

$$\begin{aligned} \dot{z}(t) &= \langle q^*, u_t \rangle = \langle q^*, A(\mu)u_t + R(\mu)u_t \rangle \\ &= \langle q^*, Au_t \rangle + \langle q^*, Ru_t \rangle \\ &= i\tau_0\omega_0 z + \bar{q}^*(0) \cdot f(0, W(t, 0) + 2 \operatorname{Re} [z(t)q(0)]). \end{aligned} \quad (37)$$

Rewrite (37) as

$$\dot{z}(t) = i\tau_0\omega_0 z + g(z, \bar{z}), \quad (38)$$

where

$$g(z, \bar{z}) = g_{20} \frac{z^2}{2} + g_{11} z\bar{z} + g_{02} \frac{\bar{z}^2}{2} + g_{21} \frac{z^2\bar{z}}{2} + \dots. \quad (39)$$

FIGURE 1: The positive equilibrium E_* of system (3) is asymptotically stable.

From (18) and (38), we have

$$\begin{aligned} \dot{W} &= \dot{u}_t - \dot{z}q - \dot{\bar{z}}\bar{q} \\ &= \begin{cases} AW - 2\operatorname{Re}[\bar{q}^*(0)f(z, \bar{z})q(\theta)], & -1 \leq \theta < 0, \\ AW - 2\operatorname{Re}[\bar{q}^*(0)f(z, \bar{z})q(\theta)] + f, & \theta = 0. \end{cases} \end{aligned} \quad (40)$$

Let

$$\dot{W} = AW + H(z, \bar{z}, \theta), \quad (41)$$

where

$$H(z, \bar{z}, \theta) = H_{20}(\theta) \frac{z^2}{2} + H_{11}(\theta) z\bar{z} + H_{02}(\theta) \frac{\bar{z}^2}{2} + \cdots. \quad (42)$$

Taking the derivative of W with respect to t in (36), we have

$$\dot{W} = W_z \dot{z} + W_{\bar{z}} \dot{\bar{z}}. \quad (43)$$

Substituting (36) and (38) into (43), we obtain

$$\begin{aligned} \dot{W} &= (W_{20}z + W_{11}\bar{z} + \cdots)(i\tau_0\omega_0 z + g) \\ &\quad + (W_{11}z + W_{02}\bar{z} + \cdots)(-i\tau_0\omega_0 \bar{z} + \bar{g}). \end{aligned} \quad (44)$$

Then substituting (36) and (41) into (42), we have the following results:

$$\begin{aligned} \dot{W} &= (AW_{20} + H_{20}) \frac{z^2}{2} + (AW_{11} + H_{11}) z\bar{z} \\ &\quad + (AW_{02} + H_{02}) \frac{\bar{z}^2}{2} + \cdots. \end{aligned} \quad (45)$$

Comparing the coefficients of (44) with (45), the following equations hold:

$$(A - 2i\tau_0\omega_0)W_{20}(\theta) = -H_{20}(\theta), \quad (46)$$

$$AW_{11}(\theta) = -H_{11}(\theta). \quad (47)$$

FIGURE 2: The bifurcation periodic solution is stable.

Since $u_t = u(t + \theta) = W(z, \bar{z}, \theta) + zq + \bar{z} \cdot \bar{q}$, we have

$$u_t = \begin{pmatrix} W^{(1)}(z, \bar{z}, \theta) \\ W^{(2)}(z, \bar{z}, \theta) \end{pmatrix} + z \begin{pmatrix} 1 \\ q_1 \end{pmatrix} e^{i\omega_0\theta} + \bar{z} \begin{pmatrix} 1 \\ \bar{q}_1 \end{pmatrix} e^{-i\omega_0\theta}. \quad (48)$$

Thus, we can obtain

$$\begin{aligned} \varphi_1(0) &= z + \bar{z} + W_{20}^{(1)}(0) \frac{z^2}{2} + W_{11}^{(1)}(0) z\bar{z} + W_{02}^{(1)}(0) \frac{\bar{z}^2}{2}, \\ \varphi_2(0) &= zq_1 + \bar{z}\bar{q}_1 + W_{20}^{(2)}(0) \frac{z^2}{2} + W_{11}^{(2)}(0) z\bar{z} + W_{02}^{(2)}(0) \frac{\bar{z}^2}{2}. \end{aligned} \quad (49)$$

So, we have

$$\begin{aligned} \varphi_1(0)\varphi_2(0) &= q_1 z^2 + \bar{q}_1 \bar{z}^2 + (q_1 + \bar{q}_1) z\bar{z} \\ &\quad + \left(W_{11}^{(2)} + \frac{1}{2} W_{20}^{(2)} + W_{11}^{(1)} q_1 + \frac{1}{2} W_{20}^{(1)} \bar{q}_1 \right) z^2 \bar{z}, \end{aligned}$$

$$\varphi_1^2(0) = z^2 + \bar{z}^2 + 2z\bar{z} + (W_{20}^{(1)} + 2W_{11}^{(1)}) z^2 \bar{z},$$

$$\varphi_2^2(0) = q_1^2 z^2 + \bar{q}_1^2 \bar{z}^2 + 2q_1 \bar{q}_1 z\bar{z} + (\bar{q}_1 W_{20}^{(2)} + 2q_1 W_{11}^{(2)}) z^2 \bar{z}. \quad (50)$$

It follows from (38) and (39) that

$$f(z, \bar{z}) = \begin{pmatrix} K_1 z^2 + K_2 z\bar{z} + K_3 \bar{z}^2 + K_4 z^2 \bar{z} \\ 0 \end{pmatrix}, \quad (51)$$

where

$$\begin{aligned} K_1 &= -\beta\tau_0 (1 + q_1^2 + 2q_1), \\ K_2 &= -\beta\tau_0 (2 + 2q_1 \bar{q}_1 + 2q_1 + 2\bar{q}_1), \\ K_3 &= -\beta\tau_0 (1 + \bar{q}_1^2 + 2\bar{q}_1), \\ K_4 &= -\beta\tau_0 (2W_{11}^{(2)} + W_{20}^{(2)} + 2W_{11}^{(1)} q_1 + W_{20}^{(1)} \bar{q}_1 \\ &\quad + W_{20}^{(1)} + 2W_{11}^{(1)} + \bar{q}_1 W_{20}^{(2)} + 2q_1 W_{11}^{(2)}). \end{aligned} \quad (52)$$

FIGURE 3: The positive equilibrium E_* of system is asymptotically stable.

Then we have

$$g(z, \bar{z}) = \frac{1}{\rho} (1, \bar{q}_1^*) \begin{pmatrix} K_1 z^2 + K_2 z \bar{z} + K_3 \bar{z}^2 + K_4 z^2 \bar{z} \\ 0 \end{pmatrix}. \quad (53)$$

Comparing the coefficients of the above equation with those in (41), we have

$$\begin{aligned} g_{20} &= \frac{1}{\rho} K_1, & g_{11} &= \frac{1}{\rho} K_2, \\ g_{02} &= \frac{1}{\rho} K_3, & g_{21} &= \frac{1}{\rho} K_4. \end{aligned} \quad (54)$$

In what follows, we focus on the computation of $W_{20}(\theta)$ and $W_{11}(\theta)$. For the expression of g_{21} , we have

$$\begin{aligned} H(z, \bar{z}, \theta) &= -2 \operatorname{Re}(\bar{q}^*(0) f(z, \bar{z}) q(\theta)) + Ru_t \\ &= -\left(\frac{1}{2} g_{20} z^2 + g_{11} z \bar{z} + \frac{1}{2} g_{02} \bar{z}^2 + \dots \right) q(\theta) \\ &\quad - \left(\frac{1}{2} \bar{g}_{20} \bar{z}^2 + \bar{g}_{11} z \bar{z} + \frac{1}{2} \bar{g}_{02} z^2 + \dots \right) \bar{q}(\theta) \\ &\quad + Ru_t. \end{aligned} \quad (55)$$

Comparing the coefficients of the above equation, we can obtain that

$$H_{20}(\theta) = -g_{20} q(\theta) - \bar{g}_{02} \bar{q}(\theta), \quad \theta \in [-1, 0), \quad (56)$$

$$H_{11}(\theta) = -g_{11} q(\theta) - \bar{g}_{11} \bar{q}(\theta), \quad \theta \in [-1, 0). \quad (57)$$

Substituting (56) into (46) and (57) into (47), respectively, we get

$$\dot{W}_{20}(\theta) = 2i\tau_0\omega_0 W_{20}(\theta) + g_{20} q(\theta) + \bar{g}_{02} \bar{q}(\theta), \quad (58)$$

$$\dot{W}_{11}(\theta) = g_{11} q(\theta) + \bar{g}_{11} \bar{q}(\theta).$$

We can easily obtain the solutions of (58) as

$$\begin{aligned} W_{20}(\theta) &= \frac{ig_{20}}{\tau_0\omega_0} q(0) e^{i\tau_0\omega_0\theta} - \frac{\bar{g}_{02}}{3i\tau_0\omega_0} \bar{q}(0) e^{-i\tau_0\omega_0\theta} \\ &\quad + E_1 e^{2i\tau_0\omega_0\theta}, \end{aligned} \quad (59)$$

$$\dot{W}_{11}(\theta) = \frac{g_{11}}{i\tau_0\omega_0} q(0) e^{i\tau_0\omega_0\theta} - \frac{\bar{g}_{11}}{i\tau_0\omega_0} \bar{q}(0) e^{-i\tau_0\omega_0\theta} + E_2.$$

FIGURE 4: The bifurcation periodic solution is stable.

We will determine E_1 and E_2 . Form the definition of A in (23), we have

$$\begin{aligned} \int_{-1}^0 d\eta(\theta) W_{20}(0) &= 2i\tau_0\omega_0 W_{20}(0) - H_{20}(0), \\ \int_{-1}^0 d\eta(\theta) W_{11}(0) &= -H_{11}(0). \end{aligned} \quad (60)$$

From (59), (56) and (57), we have

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta) + (K_1, 0)^T, \quad (61)$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta) + (K_2, 0)^T. \quad (62)$$

Substituting (59) and (61) into (62) and noticing that

$$\begin{aligned} \left(i\omega_0 I - \int_{-1}^0 e^{i\omega_0\theta} d\eta(\theta) \right) q(0) &= 0, \\ \left(-i\omega_0 I - \int_{-1}^0 e^{-i\omega_0\theta} d\eta(\theta) \right) \bar{q}(0) &= 0, \end{aligned} \quad (63)$$

we can obtain

$$\left(2i\omega_0 I - \int_{-1}^0 e^{2i\tau_0\omega_0\theta} d\eta(\theta) \right) E_1 = (K_1, 0)^T, \quad (64)$$

which leads to

$$\begin{aligned} &\begin{pmatrix} 2i\omega_0 - \beta - \frac{2(\alpha+\delta)(\gamma+\delta)}{\alpha+\gamma+\delta} + \delta + \alpha e^{-2i\omega_0\tau_0} & \beta - \frac{2(\alpha+\delta)(\gamma+\delta)}{\alpha+\gamma+\delta} \\ -\alpha e^{-2i\omega_0\tau_0} & 2i\omega_0 + \gamma + \delta \end{pmatrix} E_1 \\ &= \begin{pmatrix} K_1 \\ 0 \end{pmatrix}, \\ &\begin{pmatrix} -\beta - \frac{2(\alpha+\delta)(\gamma+\delta)}{\alpha+\gamma+\delta} + \delta + \alpha & \beta - \frac{2(\alpha+\delta)(\gamma+\delta)}{\alpha+\gamma+\delta} \\ -\alpha & \gamma + \delta \end{pmatrix} E_2 = \begin{pmatrix} K_2 \\ 0 \end{pmatrix}. \end{aligned} \quad (65)$$

It follows that

$$E_1 = \begin{pmatrix} 2i\omega_0 - \beta - \frac{2(\alpha + \delta)(\gamma + \delta)}{\alpha + \gamma + \delta} + \delta + \alpha e^{-2i\omega_0 \tau_0} & \beta - \frac{2(\alpha + \delta)(\gamma + \delta)}{\alpha + \gamma + \delta} \\ -\alpha e^{-2i\omega_0 \tau_0} & 2i\omega_0 + \gamma + \delta \end{pmatrix}^{-1} \times \begin{pmatrix} K_1 \\ 0 \end{pmatrix},$$

$$E_2 = \begin{pmatrix} -\beta - \frac{2(\alpha + \delta)(\gamma + \delta)}{\alpha + \gamma + \delta} + \delta + \alpha & \beta - \frac{2(\alpha + \delta)(\gamma + \delta)}{\alpha + \gamma + \delta} \\ -\alpha & \gamma + \delta \end{pmatrix}^{-1} \begin{pmatrix} K_2 \\ 0 \end{pmatrix}. \quad (66)$$

Hence, we know W_{20} and then we can obtain g_{21} . The following parameters can be calculated:

$$C_1(0) = \frac{i}{2\omega_0 \tau_0} \left(g_{20} g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2 \right) + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_0)\}}, \quad \beta_2 = 2 \operatorname{Re}\{C_1(0)\}, \quad (67)$$

$$T_2 = -\frac{\operatorname{Im}\{C_1(0)\} + \mu_2 \operatorname{Im}\{\lambda'(\tau_0)\}}{\tau_0 \omega_0}.$$

Theorem 3. Under the condition of Theorem 1, one has the following.

- (1) $\mu = 0$ is Hopf bifurcation value of system (18).
- (2) The direction of Hopf bifurcation is determined by the sign of μ_2 : if $\mu_2 > 0$, the Hopf bifurcation is supercritical; if $\mu_2 < 0$, the Hopf bifurcation is subcritical.
- (3) The stability of bifurcating periodic solutions is determined by β_2 : if $\beta_2 < 0$, the periodic solutions are stable; if $\beta_2 > 0$, they are unstable.

4. Numerical Examples

In this section, some numerical examples of system (3) are presented to justify the previous theorem above.

Example 1. Consider system (3) with parameters $\alpha = 0.8$, $\beta = 0.8$, $\gamma = 0.8$, and $\delta = 0.4$. Then $R_0 = 1.1111$, $E_* = (0.06, 0.04)$, and (7) has one positive real root $\omega = 0.3470$. It follows by (11) that $\tau_0 = 3.5705$. First, we choose $\tau = 3 < \tau_0$. For a set of initial conditions satisfying $L(0) = 0.1$ and $B(0) = 0.1$, Figure 1 demonstrates the evolutions from which it can be seen that the equilibrium is asymptotically stable. Second, we choose $\tau = 3.7 > \tau_0$. For a set of initial conditions satisfying, the corresponding wave form and phase plots are shown in Figure 2, from which it is easy to see that a Hopf bifurcation occurs.

Example 2. Consider system (3) with parameters $\alpha = 0.7$, $\beta = 0.8$, $\gamma = 0.5$, and $\delta = 0.4$. Then $R_0 = 1.2929$, $E_* = (0.1274, 0.0991)$, and (7) has one positive real root $\omega = 0.4463$. It follows by (11) that $\tau_0 = 4.0204$. First, we choose $\tau = 3.3 < \tau_0$. For a set of initial conditions satisfying

$L(0) = 0.1$ and $B(0) = 0.1$, Figure 3 demonstrates the evolutions from which it can be seen that the equilibrium is asymptotically stable. Second, we choose $\tau = 4.2 > \tau_0$. For a set of initial conditions satisfying $L(0) = 0.1$ and $B(0) = 0.1$, the corresponding wave form and phase plots are shown in Figure 4, from which it is easy to see that a Hopf bifurcation occurs.

5. Conclusions

In this paper, we have constructed a computer virus model with time delay depending on the SLBS model. The theoretical analyses for the computer virus models are given. Furthermore, it is proved that there exists a Hopf bifurcation when time crosses through the critical value. Finally, the numerical simulations illustrate our results.

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