Research Article

Application of the RBF Method to the Estimation of Temperature on the External Surface in Laminar Pipe Flow

Shiqin Lyu, Can Wu, and Sufang Zhang

College of Mathematics, Taiyuan University of Technology, Taiyuan 030024, China

Correspondence should be addressed to Shiqin Lyu; 33248688@qq.com

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The inverse heat conduction problem on the heat transfer characteristics of cooled/heated laminar flows through finite length thick-walled circular tubes is studied, using temperature measurements taken at several different locations within the fluid in this paper. The method of radial basis functions is coupled with the boundary control technique to estimate the unknown temperature on the external surface of the circular pipe. The main idea of the proposed method is to solve the direct problem instead of solving the inverse problem directly. The temperature data obtained from the direct problem are used to simulate the temperature measurement for the inverse problem, and during the calculation, the Tikhonov regularization and L-curve methods are employed to solve the inverse problem. Therefore, this study also considers the influence of errors in these measurements upon the precision of the estimated results as well as the influence of the locations and number of sensors used upon the accuracy of the estimated results. The results indicate that the accuracy of the estimated results is improved by taking temperature measurements in locations close to the unknown boundary. Finally, the results confirm that the proposed method is capable of yielding accurate results even when errors in the temperature measurements are present.

1. Introduction

The study of the inverse heat conduction problem on the heat transfer characteristics of cooled/heated laminar flows through a finite length thick-walled circular tubes has great practical significance in industrial production. It can provide a systemic and referential method to solve some common problems in industrial production such as the estimation of the fluid heat consumption and the choice of materials for tubes. So this kind of inverse heat conduction problem has been extensively investigated by many scholars. In 2009, Wei and Li [1] have presented the regularization method to identify the moving boundary in a multilayer domain based on Cauchy data under one-dimensional heat equations. Lu et al. [2] developed a conjugate gradient method to estimate the unknown temperature of fluid near the inner wall of pipe in the two-dimensional inverse heat conduction problem in 2010. In 2011, the conjugate gradient method combined with a finite element method was applied by Lu et al. [3] to study three-dimensional inverse heat conduction problem.

A solution was proposed to investigate inverse problem by using a finite difference method and a least square method in literature [4]. Some other methods, such as the method of fundamental solutions [5], were also used to solve this kind of inverse heat conduction problem.

In this work, the inverse heat conduction problem we study is a typical Cauchy problem which is based on Laplace's equation. This class of problem is typically ill-posed. Some numerical methods such as the method of fundamental solutions (MFS) [6], the boundary element method (BEM) [7], and the finite difference method (FDM) [5, 8–10] were commonly used for solving ill-posed problems. All these methods rely heavily on the grid. Tiny changes to the grid structure could have significant impacts on calculations. Therefore, in recent years, many scholars have focussed their attention on meshless methods, especially meshless methods based on radial basis functions (RBF). It is a new and rapidly developing numerical method that follows traditional methods. The most prominent advantage of a meshless method is that it overcomes the reliance on grid, eliminating the

division and reconstruction of a grid completely or partially, so the meshless method, like the RBF method, is widely used in the field of numerical calculation. It was proven that the RBF method was a great method for solving the Cauchy problem by Yao et al. [11]. Lai [12] introduced the application of the RBF method in computational electromagnetics, which reflected the wide applicability of the RBF method. Cheng and Cabral [13] applied the RBF method to get a direct solution to the Cauchy ill-conditioned boundary value problem, but the heat source has not been accounted for. Through these studies, it has been found that the RBF method is easier to implement than other methods and the initial conditions were not needed. In addition, compared with BEM, FDM, and finite element methods (FEM) which depended on the grid [8], the RBF method is faster in calculation and higher in accuracy.

This paper is organized as follows. In Section 2, the physical model is briefly introduced and developed. Based on this model, a numerical method is presented in Section 3 to solve the linear system. The temperature attained in the direct problem is used as a known condition for calculating the temperature on external surface of the tube, combining regularization method in inverse problem. In Section 4, two examples are presented to consider the influence of the measurement error, the locations of measurement points, and the number of measurement points upon the accuracy of the estimated results by analysing the temperature distributions and the relative root mean square error of two examples.

2. Physical Model and Governing Equations

The physical model under consideration consists of constant property flow with steady laminar motion in a circular tube. In this system, the pipe wall is assumed to be homogeneous with a constant thermal conductivity k_s , and the fluid is assumed to be incompressible and homogeneous with a constant thermal conductivity k_f . The fluid temperature at the entrance of the pipe is assumed to be uniform and equal to T_{in} , and $u(r)$ is the fluid velocity distribution. An adiabatic condition is applied in the upstream region and the downstream. As the fluid moves through the pipe, some of its heat escapes through the wall of the pipe via heat conduction. This leads to the uncertain and spatially nonuniform distribution of temperature $q(x)$ along the surface of the outer wall. The pipe flow and conjugate heat transfer system are shown in Figure 1. Because of the symmetrical characteristics of our model, the domain can be considered one half of the pipe flow, and this problem can be assumed to be two-dimensional.

In the wall region, the governing equations for the temperature field can be described by the following partial differential equations:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T(x, r)}{\partial r} \right) + \frac{\partial^2 T(x, r)}{\partial x^2} = 0. \quad (1)$$

According to our assumptions, the boundary conditions can be expressed as follows:

$$\begin{aligned} T(x, r) &= q(x), \quad r = r_{ow}, \quad 0 \leq x \leq L, \\ \frac{\partial T(x, r)}{\partial x} &= 0, \quad x = 0, \quad r_{iw} \leq r \leq r_{ow}, \\ \frac{\partial T(x, r)}{\partial x} &= 0, \quad x = L, \quad r_{iw} \leq r \leq r_{ow}, \end{aligned} \quad (2)$$

where r_{iw} and r_{ow} are the inner and outer radii of the pipe, respectively.

In the fluid region, the governing equations for the temperature field can be described by the following partial differential equations:

$$\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T(x, r)}{\partial r} \right) + \frac{\partial^2 T(x, r)}{\partial x^2} - \frac{\rho_f C_{pf}}{k_f} u(r) \frac{\partial T(x, r)}{\partial x} = 0, \quad (3)$$

where ρ_f is the fluid density and C_{pf} is the specific heat at constant pressure. In this model, $u(r)$ is given by the following conventional Hagen-Poiseuille expression:

$$u(r) = 2u_{mean} \left(1 - \left(\frac{r}{r_{iw}} \right)^2 \right), \quad (4)$$

where u_{mean} is the average velocity, and it can be determined from the measured flow rate.

Similarly, according to our assumptions, we have

$$\begin{aligned} T(0, r) &= T_{in}, \quad x = 0, \quad 0 \leq r \leq r_{iw}, \\ \frac{\partial T(x, r)}{\partial r} &= 0, \quad 0 \leq x \leq L, \quad r = 0. \end{aligned} \quad (5)$$

Notice that since the length of the downstream section is long enough to account for the wall axial conduction effects, at $x = L$, the following boundary condition is appropriate:

$$\frac{\partial T(x, r)}{\partial x} = 0, \quad 0 \leq r \leq r_{iw}, \quad x = L. \quad (6)$$

In addition, at $x = r_{iw}$, the interface between the pipe wall and the fluid, the following interface conditions are given

$$\begin{aligned} T_s(x, r) &= T_f(x, r), \\ k_s \frac{\partial T_s(x, r)}{\partial r} &= k_f \frac{\partial T_f(x, r)}{\partial r}. \end{aligned} \quad (7)$$

In the heat conduction problem for this system, every dependent parameter would, more or less, affect the heat conduction. A parametric study of the individual parameters would require a vast set of results; however, they are not the principal objective of this work; hence, we assume that these parameters were kept constant in our work:

$$\begin{aligned} Pe &= 2u_{mean} r_{iw} \frac{\rho_f C_{pf}}{k_f} = 24.8, \\ k_{sf} &= \frac{k_s}{k_f} = 5.035. \end{aligned} \quad (8)$$

FIGURE 1: Physical model.

3. Numerical Method

3.1. Direct Problem. In order to reduce the amount of computation and simplify representation, the following scale parameters are introduced to reduce the governing equations and the boundary conditions into the dimensionless forms:

$$\bar{T} = \frac{T}{T_{in}}, \quad \bar{r} = \frac{r}{r_{ow}}, \quad \bar{x} = \frac{x}{L}. \quad (9)$$

The RBF method is employed to analyze the direct problem. The main aim is to determine the temperature at some points when all the boundary conditions and thermal properties are known. This temperature data with added noise will then be used as measured data for the inverse problem.

For simplicity, the adopted notations are shown in Table 1.

Firstly, we collocate some points to ensure that the governing equations and boundary conditions are satisfied; that is, take interior points $\{\bar{\mathbf{x}}_j^{(s)}, j = 1, 2, \dots, m_s\}$ on Ω_s satisfying the governing equation (1), take interior points $\{\bar{\mathbf{x}}_j^{(f)}, j = 1, 2, \dots, m_f\}$ on Ω_f satisfying the governing equation (3), and take collocation points $\{\bar{\mathbf{x}}_k^{(k)}, k = 1, 2, \dots, m_k\}$ on Γ_k ($k = 1, 2, \dots, 7$) satisfying the boundary conditions (2) and (5)–(7), respectively.

Next, we assume that the approximate solution $\hat{T}$ of $\bar{T}$ is given by the series

$$\hat{T}(\bar{\mathbf{x}}) = \begin{cases} \sum_{i=1}^{N_s} \alpha_i^{(s)} \varphi_i^{(s)}(\bar{\mathbf{x}}), & \bar{\mathbf{x}} \in \Omega_s \\ \sum_{i=1}^{N_f} \alpha_i^{(f)} \varphi_i^{(f)}(\bar{\mathbf{x}}), & \bar{\mathbf{x}} \in \Omega_f, \end{cases} \quad (10)$$

where $N_s = m_1 + m_2 + m_3 + m_4 + m_s$, $N_f = m_4 + m_5 + m_6 + m_7 + m_f$, $\alpha_i^{(s)}$ and $\alpha_i^{(f)}$ are constant coefficients to be determined by collocation, $\varphi_i^{(s)}(\cdot)$ and $\varphi_i^{(f)}(\cdot)$ are basis functions. The multiquadric basis function (MQ)

$$\varphi(r) = \sqrt{r^2 + c^2}, \quad (11)$$

is used throughout this paper, in which c is called the shape parameter. It is well known that the value of c leads to a

method with exponential convergence properties for solving the PDE. Let

$$\begin{aligned} \alpha &= (\alpha_1^{(s)}, \alpha_2^{(s)}, \dots, \alpha_{N_s}^{(s)}, \alpha_1^{(f)}, \alpha_2^{(f)}, \dots, \alpha_{N_f}^{(f)})^T, \\ \mu_i^{(s)} &= \left(\frac{\partial^2}{\partial \bar{r}^2} + \frac{1}{\bar{L}^2} \frac{\partial^2}{\partial \bar{x}^2} + \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} \right) \varphi_i^{(s)}, \\ \nu_i^{(f)} &= \left(\frac{\partial^2}{\partial \bar{r}^2} + \frac{1}{\bar{L}^2} \frac{\partial^2}{\partial \bar{x}^2} + \frac{1}{\bar{r}} \frac{\partial}{\partial \bar{r}} - \text{Pe} \frac{1}{\bar{L}} \frac{1}{\bar{r}_{iw}} \right. \\ &\quad \times \left[1 - \left(\frac{\bar{r}}{\bar{r}_{iw}} \right)^2 \right] \frac{\partial}{\partial \bar{x}} \right) \varphi_i^{(f)}, \end{aligned} \quad (12)$$

where $\bar{L} = L/r_{ow}$. By fitting (1)–(7), it is easy to deduce a linear system of equations

$$\mathbf{A}\alpha = \mathbf{b}, \quad (13)$$

where

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{0} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \\ \mathbf{0} & \mathbf{A}_{32} \end{pmatrix}, \quad (14)$$

is a $(N_s + N_f) \times (N_s + N_f)$ coefficient matrix, in which

$$\begin{aligned} \mathbf{A}_{11} &= \begin{pmatrix} \varphi_j^{(s)}(\bar{\mathbf{x}}_i^{(1)}) \\ \frac{\partial \varphi_j^{(s)}(\bar{\mathbf{x}}_i^{(2)})}{\partial \bar{x}} \\ \frac{\partial \varphi_j^{(s)}(\bar{\mathbf{x}}_i^{(3)})}{\partial \bar{r}} \\ \mu_j^{(s)}(\bar{\mathbf{x}}_i^{(s)}) \end{pmatrix}, & \mathbf{A}_{32} &= \begin{pmatrix} \nu_j^{(f)}(\bar{\mathbf{x}}_i^{(f)}) \\ \frac{\partial \varphi_j^{(f)}(\bar{\mathbf{x}}_i^{(5)})}{\partial \bar{r}} \\ \frac{\partial \varphi_j^{(f)}(\bar{\mathbf{x}}_i^{(6)})}{\partial \bar{x}} \\ \varphi_j^{(f)}(\bar{\mathbf{x}}_i^{(7)}) \end{pmatrix}, \\ \mathbf{A}_{21} &= \begin{pmatrix} \varphi_j^{(s)}(\bar{\mathbf{x}}_i^{(4)}) \\ \frac{\partial \varphi_j^{(s)}(\bar{\mathbf{x}}_i^{(4)})}{\partial \bar{r}} \\ k_{sf} \frac{\partial \varphi_j^{(s)}(\bar{\mathbf{x}}_i^{(4)})}{\partial \bar{r}} \end{pmatrix}, & \mathbf{A}_{22} &= \begin{pmatrix} -\varphi_j^{(f)}(\bar{\mathbf{x}}_i^{(4)}) \\ \frac{\partial \varphi_j^{(f)}(\bar{\mathbf{x}}_i^{(4)})}{\partial \bar{r}} \\ -\frac{\partial \varphi_j^{(f)}(\bar{\mathbf{x}}_i^{(4)})}{\partial \bar{r}} \end{pmatrix}. \end{aligned} \quad (15)$$

$\mathbf{b} = (\mathbf{q}^T, \mathbf{0}, \mathbf{p}^T)^T$ is a $(N_s + N_f) \times 1$ vector with $\mathbf{p}$ as an $m_7 \times 1$ vector of all the elements of 1. Consider the following:

$$\mathbf{q} = (\bar{q}(\bar{\mathbf{x}}_1^{(1)}), \bar{q}(\bar{\mathbf{x}}_2^{(1)}), \dots, \bar{q}(\bar{\mathbf{x}}_{m_1}^{(1)}))^T. \quad (16)$$

Finally, we can determine the vector of unknown coefficients α by the Gauss elimination method. Then the temperature at point $\bar{\mathbf{x}}$ can be approximated by (10).

3.2. Inverse Problem. The main purpose of the inverse problem is to determine the unknown outer wall temperature $q(x)$ by using the temperature measurement value obtained within the fluid in the pipe.

Let $\{\bar{\mathbf{x}}_j^{(m)}, j = 1, 2, \dots, N_m\}$ be the measuring points in the fluid region, and let $\tilde{T}(\bar{\mathbf{x}}_j^{(m)})$ be the temperature observation

TABLE 1: The adopted notations.

$\Gamma_1 = \{0 \leq \bar{x} \leq 1, \bar{r} = 1\}$	$\Gamma_2 = \{\bar{x} = 0, \bar{r}_{iw} \leq \bar{r} \leq 1\}$	$\Gamma_3 = \{\bar{x} = 1, \bar{r}_{iw} \leq \bar{r} \leq 1\}$
$\Gamma_4 = \{0 \leq \bar{x} \leq 1, \bar{r} = \bar{r}_{iw}\}$	$\Gamma_5 = \{0 \leq \bar{x} \leq 1, \bar{r} = 0\}$	$\Gamma_6 = \{\bar{x} = 1, 0 \leq \bar{r} \leq \bar{r}_{iw}\}$
$\Gamma_7 = \{\bar{x} = 0, 0 \leq \bar{r} \leq \bar{r}_{iw}\}$	$\Omega_s = \{0 \leq \bar{x} \leq 1, 0 \leq \bar{r} \leq \bar{r}_{iw}\}$	$\Omega_f = \{0 \leq \bar{x} \leq 1, \bar{r}_{iw} \leq \bar{r} \leq 1\}$

with random noise added at these points. Although higher accuracy can be obtained by selecting a larger number of measurement points the computational and experimental costs increase as N_m increases, so the number of measurement points N_m is usually small. Hence, it is necessary to specify a number of measuring locations which yields an acceptable compromise between the precision and the associated process costs.

Suppose that the function $\bar{q}(\bar{x})$ can be approximated by

$$\bar{q}(\bar{x}) = \sum_{k=1}^n \lambda_k \phi_k(\bar{x}), \quad (17)$$

where λ_k ($k = 1, 2, \dots, n$) are unknown coefficients $\phi_k(\bar{x})$ ($k = 1, 2, \dots, n$) are basis functions, usually taken to be triangular basis functions or power basis functions. At points $\{\bar{\mathbf{x}}_j^{(1)}, j = 1, 2, \dots, m_1\}$, we can obtain

$$\bar{q}(\bar{\mathbf{x}}_j) = \bar{T}(\bar{\mathbf{x}}_j^{(1)}) = \sum_{k=1}^n \lambda_k \phi_k(\bar{\mathbf{x}}_j), \quad j = 1, 2, \dots, m_1. \quad (18)$$

In the matrix notation, (18) can be rewritten as

$$\mathbf{q} = \Phi \lambda, \quad (19)$$

where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)^T$ and Φ is an $m_1 \times n$ matrix with $\phi_j(\bar{\mathbf{x}}_i^{(1)})$ as its (i, j) th element.

From (13), we have $\alpha = \mathbf{A}^{(-1)} \mathbf{b}$, where $\mathbf{A}^{(-1)}$ represents the generalized inverse of $\mathbf{A}$, thus we can get

$$\begin{pmatrix} \alpha_s \\ \alpha_f \end{pmatrix} = \mathbf{A}^{(-1)} \begin{pmatrix} q^T \\ 0 \\ p^T \end{pmatrix}. \quad (20)$$

Let

$$\mathbf{A}^{(-1)} = \begin{pmatrix} * & * \\ \mathbf{B} & \mathbf{C} \end{pmatrix}; \quad (21)$$

then

$$\begin{aligned} \begin{pmatrix} \alpha_s \\ \alpha_f \end{pmatrix} &= \begin{pmatrix} * & * \\ \mathbf{B} & \mathbf{C} \end{pmatrix} \begin{pmatrix} q^T \\ 0 \\ p^T \end{pmatrix}, \\ \alpha_f &= (\mathbf{B} \ \mathbf{C}) \begin{pmatrix} q^T \\ 0 \\ p^T \end{pmatrix} = (\mathbf{B} \ \mathbf{C}) \mathbf{b}; \end{aligned} \quad (22)$$

that is, $(\mathbf{B} \ \mathbf{C})_{N_f \times (m_1 + m_7)}$ is the last N_f rows of $\mathbf{A}^{(-1)}$, and

$$\begin{aligned} \mathbf{B} &= (\mathbf{0} \ \mathbf{I}_{N_f}) \mathbf{A}^{-1} \begin{pmatrix} \mathbf{I}_{m_1} \\ \mathbf{0} \end{pmatrix}, \\ \mathbf{C} &= (\mathbf{0} \ \mathbf{I}_{N_f}) \mathbf{A}^{-1} \begin{pmatrix} \mathbf{0} \\ \mathbf{I}_{m_7} \end{pmatrix}, \end{aligned} \quad (23)$$

where $\mathbf{I}_k$ is k by k unit matrix; according to (17) and (19), $\alpha^{(f)} = (\alpha_1^{(f)}, \alpha_2^{(f)}, \dots, \alpha_{N_f}^{(f)})^T$ which can be expressed in terms of λ as

$$\alpha^{(f)} = \mathbf{B}\mathbf{q} + \mathbf{C}\mathbf{p} = \mathbf{B}\Phi\lambda + \mathbf{C}\mathbf{p}. \quad (24)$$

To satisfy the conditions at the measurement points $\{\bar{\mathbf{x}}_j^{(m)}, j = 1, 2, \dots, N_m\}$, we require

$$\sum_{i=1}^{N_f} \alpha_i^{(f)} \varphi_i^{(f)}(\bar{\mathbf{x}}_j^{(m)}) = \tilde{T}(\bar{\mathbf{x}}_j^{(m)}), \quad j = 1, 2, \dots, N_m. \quad (25)$$

In matrix form, the values of the unknown coefficients λ can be obtained from solving the following matrix equation:

$$\mathbf{D}\alpha^{(f)} = \tilde{\mathbf{T}}, \quad (26)$$

where $\mathbf{D}$ is a N_m by N_f matrix with $\varphi_j^{(f)}(\bar{\mathbf{x}}_i^{(m)})$ as its (i, j) th element and

$$\tilde{\mathbf{T}} = (\tilde{T}(\bar{\mathbf{x}}_1^{(m)}), \tilde{T}(\bar{\mathbf{x}}_2^{(m)}), \dots, \tilde{T}(\bar{\mathbf{x}}_{N_m}^{(m)}))^T. \quad (27)$$

Let $\mathbf{E} = \mathbf{D}\mathbf{B}\Phi$ and $\mathbf{f} = \tilde{\mathbf{T}} - \mathbf{D}\mathbf{C}\mathbf{p}$. Then by substituting (24) into (26), and using some simple calculation, we can obtain that

$$\mathbf{E}\lambda = \mathbf{f}. \quad (28)$$

Since the original inverse problem belongs to this class of ill-posed problems, any small error in measured data can lead to a dramatic change to the solution. Although it is transformed into the form of direct problem, the ill-conditioning of the matrix $\mathbf{E}$ in (28) still persists. Due to the large condition number of the matrix $\mathbf{E}$, most standard numerical methods are unfit to solve the matrix equation (28). Many regularization methods have been developed for solving these kinds of ill-conditioned problems. In our case, the Tikhonov regularization [14] is employed to solve the matrix equation (28). The Tikhonov regularized solution λ is defined to be the solution to the following penalized least square problem:

$$\min_{\lambda} \{ \|\mathbf{E}\lambda - \mathbf{f}\|^2 + \gamma^2 \|\lambda\|^2 \}, \quad (29)$$

where $\|\cdot\|$ denotes the usual Euclidean norm and γ is called a regularization parameter. The determination of a suitable value for the regularization parameter is crucial to the accuracy of the method; the L-curve [15] criterion is used in our computation to determine a suitable value of γ .

Once the coefficient vector λ is determined, the outer wall temperature of pipe $q(x)$ can be evaluated by using (17).

FIGURE 2: RBF collocation nodes.

4. Result and Discussion

To demonstrate the effectiveness and stability of our method for this kind of inverse problem, we present some examples. Without loss of generality, in our illustration, the value of the dimensionless length of the pipe, $\bar{L}$, is specified as equal to 20, which is sufficiently long to observe the thermal development of the flow. The dimensionless radius of the interface between the pipe wall and the fluid inside the pipe $\bar{r}$ is set equal to 0.6, and the dimensionless temperature on the external surface of pipe $\bar{q}(\bar{x})$ is given as follows

Example 1. Consider the following:

$$\bar{q}(\bar{x}) = e^{-1.5\bar{x}}, \quad 0 \leq \bar{x} \leq 1. \quad (30)$$

Example 2. Consider the following:

$$\bar{q}(\bar{x}) = 0.7e^{2\bar{x}} - 0.5\bar{x} + 0.35, \quad 0 \leq \bar{x} \leq 1. \quad (31)$$

In the present RBF collocation solution, the multiquadric given in (11) is used as the basis functions. The collocation points form a 11×11 grid as shown in Figure 2. Different types of collocation points are marked in different symbols. On the interior points, shown in • symbol, the governing equation is collocated. On the four sides of the boundary and the interface, shown in * symbol, the Dirichlet and the Neumann boundary conditions are collocated.

The dimensionless temperature distribution and error distribution for the outer wall are shown in Figures 3 and 4, the left for Example 1 with the shape parameters $c_s = 0.16$ and $c_f = 0.23$, and the right for Example 2 with the shape parameters $c_s = 0.09$ and $c_f = 0.04$. The shape parameters are chosen by minimizing the sum of residual on the given Dirichlet boundary Γ_1 and Γ_7 . In Figure 3, The solid line in black represents the exact solution and the dotted line with a plus at each data point represents the approximate solution by the method outlined in Section 3.1. It can be seen that there is almost no difference between the two lines. In Figure 4, the error of $\bar{q}(\bar{x})$ at each point is defined as the difference between the approximate solution and the exact solution. From the numerical results demonstrated in Figures 3 and 4, we can observe that the proposed scheme in Section 3.1 gives a reasonable approximation to the exact solution $\bar{q}(\bar{x})$. The

maximum values for the errors in Examples 1 and 2 are less than 8×10^{-15} and 6×10^{-14} , respectively.

For the inverse problem, generally speaking, the results are mainly affected by the following three factors: measurement error, the location of measured points, and the number of measured points. In order to validate the effectiveness and stability of the method discussed in this paper, the influence of these three factors for calculation results is analysed as follows.

For numerical error estimation, we define the relative root mean square error of the boundary Γ_1 on which the temperature data are unknown, by the following formula

$$\varepsilon(\bar{q}) = \left(\frac{1}{m_1} \sum_{j=1}^{m_1} \left| \frac{\bar{q}_j - q(\bar{x}_j)}{q(\bar{x}_j)} \right|^2 \right)^{1/2}. \quad (32)$$

There are no real temperature measurements taken in advance, so the temperature which is used for calculation in the inverse problem is obtained from direct problem. In fact, due to the different degree of accuracy of the measuring instruments and the use of different measured methods, the errors are produced more or less; therefore, the simulated temperature tend to consider the influence of the measurement error in the inverse problem. The temperature obtained in the direct problem is taken to be the exact temperature. We add a random error disturbance to these exact values and use this noisy data in the inverse problem. The temperature observation values with random noises at measurement points can be expressed as

$$\tilde{T}(\cdot) = \hat{T}(\cdot) (1 + \sigma \xi), \quad (33)$$

where ξ is a random number between -1 and 1 with zero mean and σ indicates the error level. In our computations, the random variable ξ has been calculated by employing the unifrnd function in MATLAB.

4.1. The Influence of the Measured Error for the Results. To illustrate the impact on the estimated results of different measured error σ on the estimated results, the $\bar{r}$ value of measuring points is fixed at $\bar{r}$, where $\bar{r}$ is the dimensionless radius of the interface between the pipe wall and the fluid inside the pipe. In our calculation, we take 9 measuring points on the line of $\bar{r} = 0.5$; that is,

$$\begin{aligned} & \{\bar{x}_j^{(m)}, j = 1, 2, \dots, N_m\} \\ &= \left\{ \left(\bar{x}_j, \bar{r}_j \right) = \left(\frac{j}{10}, 0.5 \right), j = 1, 2, \dots, 9 \right\}. \end{aligned} \quad (34)$$

The basis functions $\{\phi_k(x), k = 1, 2, \dots, n\}$ are taken as power basis function $\{x^{k-1}, k = 1, 2, \dots, n\}$ with $n = 5$.

Figures 5 and 6 demonstrate the influence of different measured error level σ for Examples 1 and 2, respectively. The solid lines with no symbol represent the exact distribution of $\bar{q}(\bar{x})$, the lines with a plus symbol represent the approximate distribution of $\bar{q}(\bar{x})$ without noise, the lines with a star symbol represent the approximate distribution on the level

FIGURE 3: The exact and approximate dimensionless temperature distribution for outer wall.

FIGURE 4: The error distribution for the outer wall.

of $\sigma = 0.01$, and the lines with a circle symbol represent the approximate distribution on the level of $\sigma = 0.05$.

From Figures 5 and 6, we observe that the estimated lines are consistent with the actual shape. These show that this method can obtain accurate results even with an error rate of up to 5%. Furthermore, we can easily see that the deviation

from the exact solution increased with the increasing of measurement error.

The relative root mean square error on the boundary Γ_1 , obtained for Examples 1 and 2 using various levels of noisy data, is presented in Figure 7, on the left for Example 1 and on the right for Example 2. It can be seen that the values

FIGURE 5: Estimated dimensionless outer-wall temperature distributions for Example 1.

FIGURE 6: Estimated dimensionless outer-wall temperature distributions for Example 2.

for $\varepsilon(\bar{q})$ are very small, even for a relatively high amount of noise ($\sigma = 10\%$) added into the measurement data. In other words, the numerical result for the temperature on the outer wall represents a good approximation when compared to its analytical value. Hence our method provides stable numerical solutions to this kind of inverse boundary value problem.

4.2. The Influence of the Location of Measured Points on the Results. In this subsection, to investigate the relationship between the locations of the measurement point and the accuracy of the corresponding estimated results, in our simulation, the error level is fixed at $\sigma = 1\%$, and the number of measurement points is fixed at $N_m = 9$.

FIGURE 7: The relative root mean square error of the outer wall for different noise level.

FIGURE 8: The relationship between the radial distance from the measuring points to the centerline of the pipe and the accuracy of the estimated results for Example 1.

The following three different types of locations of sensors are considered:

- (i) $\bar{x}_j$ is uniformly distributed in $[0.1, 0.9]$ at intervals of 0.1, and $\bar{r}_j = 0.5$;
- (ii) $\bar{x}_j$ is uniformly distributed in $[0.1, 0.9]$ at intervals of 0.1, and $\bar{r}_j = 0.3$;

- (iii) $\bar{x}_j$ is uniformly distributed in $[0.1, 0.9]$ at intervals of 0.1, and $\bar{r}_j = 0$.

Similar to the previous subsection, the basis functions $\{\phi_k(x), k = 1, 2, \dots, n\}$ are taken also as power basis function $\{x^{k-1}, k = 1, 2, \dots, n\}$ with $n = 5$.

Figures 8 and 9 indicate that under Examples 1 and 2, respectively, the different values of measuring points

FIGURE 9: The relationship between the radial distance from the measuring points to the centerline of the pipe and the accuracy of the estimated results for Example 2.

TABLE 2: The relationship between the relative root mean square error and the value of $\bar{r}$.

	$\bar{r} = 0$	$\bar{r} = 0.3$	$\bar{r} = 0.5$
Example 1	3.1168×10^{-2}	2.2554×10^{-2}	6.7477×10^{-3}
Example 2	1.2103×10^{-2}	5.2828×10^{-3}	2.4151×10^{-3}

have an influence on the deviation of the estimated results and the exact values; that is, the position of the measured points has an influence on the estimated results. In the left figure, the solid lines with no symbol represent the exact distribution of $\bar{q}(\bar{x})$, the lines with a plus symbol represent the approximate distribution of $\bar{q}(\bar{x})$, which was obtained by the first type of locations of the measurement points, the lines with a star symbol represent the approximate distribution on the level of $\bar{r} = 0.3$, and the lines with a circle symbol represent the approximate distribution on the level of $\bar{r} = 0$.

The relative error distribution of the three types of locations of the measurement points is given on the right of Figures 8 and 9. In addition, the relative root mean square error on the boundary Γ_1 , which was obtained for Examples 1 and 2 using each of the three types of locations of measuring points is presented in Table 2. From this table, we can also find that the variations of the deterministic bias are influenced by the radial distance from the measuring points to the center line of the pipe, and we can conclude that the accuracy of the inverse method is improved when the sensors are located closer to the interface between solid and fluid. As the locations of the sensors approach the outer wall

whose unknown boundary conditions are to be predicted, the accuracy of the estimated results increases correspondingly.

4.3. The Influence of the Number of Measured Points on Results. During the process of estimating the temperature on the external surface of the laminar pipe flow, the number of selected points also affected our results. To analyse the influence of the number of measurement points on the estimated results, the error level is fixed at $\sigma = 0.01$ and the basis functions $\{\phi_k(x), k = 1, 2, \dots, n\}$ are still taken as power basis function $\{x^{k-1}, k = 1, 2, \dots, n\}$ with $n = 5$. Let N_m denotes the number of measuring points. The following situations regarding the number of sensors are considered:

- (i) $N_m = 18$, $\bar{x}_j$ is uniformly distributed in $[0.1, 0.9]$ at intervals of 0.1, and $\bar{r}_j = 0.5$ or 0.3;
- (ii) $N_m = 9$, $\bar{x}_j$ is uniformly distributed in $[0.1, 0.9]$ at intervals of 0.1, and $\bar{r}_j = 0.5$;
- (iii) $N_m = 5$, $\bar{x}_j$ is uniformly distributed in $[0.1, 0.9]$ at intervals of 0.2, and $\bar{r}_j = 0.5$.

Figures 10 and 11 present a comparison between the estimated results and the exact solutions of the temperature distributions obtained using the three situations above for a measurement error of $\sigma = 1\%$. In the left figure, the solid lines with no symbol represent the exact distribution of $\bar{q}(\bar{x})$, the lines with a plus symbol represent the approximate distribution of $\bar{q}(\bar{x})$, which was obtained by 18 measurement points, the lines with a star symbol represent the approximate distribution on the level of $N_m = 9$, and the lines with a circle symbol represent the approximate distribution on the level of

FIGURE 10: The relationship between the number of the measuring points and the accuracy of the estimated results for Example 1.

FIGURE 11: The relationship between the number of the measuring points and the accuracy of the estimated results for Example 2.

$N_m = 5$. The corresponding relative root mean square errors for different number of measuring points are given on the right of Figures 10 and 11.

We observe that estimations are accurate and robust for the first type (i.e., 18 measuring points) and that the

accuracy of the results using the second type temperature measurements (i.e., 9 measuring points) is superior to that obtained using the third type (i.e., 5 measuring points). This indicates that the accuracy of the estimated results increases as the amount of input information (number of sensors) is

TABLE 3: The relationship between relative root mean square error and the number of the measuring points.

	$N_m = 18$	$N_m = 9$	$N_m = 5 (N_m = n)$	$N_m = 4 (N_m < n)$
Example 1	6.7640×10^{-3}	9.7693×10^{-3}	1.6578×10^{-2}	0.1154
Example 2	1.9879×10^{-3}	5.9696×10^{-3}	1.8819×10^{-2}	0.1247

increased. It also can be concluded that the proposed inverse method is capable of yielding accurate results despite the fewer measurement points.

From Table 3, it is again confirmed that the accuracy of the inverse method is improved with an increase in the number of measuring points. Even if the number of sensors is small, provided that $N_m \geq n$ (where n denotes the number of unknown coefficients), we also can get very good accuracy. Thus, in practical applications, we can install fewer sensors to save costs without sacrificing accuracy. When the number of sensors is less than the number of unknown coefficients (i.e. $N_m < n$), the estimation accuracy drops dramatically; hence, the number of sensors installed must be at least n .

5. Conclusions

This paper has successfully studied the use of RBF collocation method and boundary control technique in estimating the unknown outer surface temperature of thick-walled circular tubes with cooled/heated laminar flows.

Experiments indicate that when the measured points are closer to the unknown boundary, the accuracy of our estimation is greater. For a larger measurement error, the increasing number of measured points can improve the accuracy of the estimated value. Even if the measurement error is as large as 10%, we can still get very accurate results. One advantage of this approach is that it does not require the initial temperature. Computationally, this method is more simple and efficient than traditional methods. The unknown boundary can be directly obtained by only one operation, and the repeated iteration process is also avoided. The experiment suggests that the RBF method is an accurate, stable, and effective method for estimating the unknown outer surface temperature of circular tubes.

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