Research Article

Robust Stability and $H_{\infty}$ Stabilization of Switched Systems with Time-Varying Delays Using Delta Operator Approach

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1. Introduction

Switched systems are a kind of hybrid systems consisting of a set of discrete event dynamic subsystems or continuous variable dynamic subsystems and a switching rule which defines a particular subsystem working during a certain interval of time. Switched systems have numerous applications in network control systems [1], robot control systems [2], intelligent traffic control systems [3], chemical industry control systems [4], and many other areas [5, 6]. Many important achievements on stability and stabilization of switched systems have been developed [7–10]. It was shown in the literature that the average dwell time (ADT) method is a powerful tool to deal with the stability of switched systems.

The delta operator which is a novel approach with good finite word length performance under fast sampling rates has been investigated by many researchers due to their extensive applications [11–13], for instance, optimal filtering [14], signal processing [15], robust control [16], system identification [17], and so forth. As stated in [15], the standard shift operator was mostly adopted in the study of control theories for discrete-time systems. However, the dynamic response of a discrete system does not converge smoothly to its continuous counterpart when the sampling period tends to zero; namely, data are taken at high sampling rates. The delta operator method can solve the above problem. In addition, it was shown in [15] that delta operator requires smaller word length when implemented in fixed-point digital control processors than shift operator does. So far, some useful results on delta operator systems have been formulated in [18–21]. As is well known, time delay phenomena which often cause instability or undesirable performance in control systems are involved in a variety of real systems, such as chaotic systems, and hydraulic pressure systems [22]. In the past years, a mass of results on delta operator systems with time delay have appeared [23–27]. The delta operator is defined by

$$\delta x(t) = \begin{cases} \frac{dx(t)}{dt}, & T = 0, \\ \frac{(x(t + T) - x(t))}{T}, & T \neq 0, \end{cases}$$

(1)

where $T$ is a sampling period. When $T \to 0$, the delta operator model will approach the continuous system before discretization and reflect a quasicontinuous performance [28].

It should be noted that external disturbances are generally inevitable, and the output will be subsequently affected by disturbances in the system. Some results on $H_{\infty}$ control were...
developed by many researchers to restrain the external disturbances [29–33]. The \( H_{\infty} \) control problem for a class of discrete systems was solved by using delta operator approach [34]. Low order sampled data \( H_{\infty} \) control using the delta operator was reported in the literature [35]. Robust \( H_{\infty} \) control for a class of uncertain switched systems using delta operator was investigated [36]. However, few results on the issues of robust stability and \( H_{\infty} \) controller design for delta operator switched systems with time-varying delay are presented, which motivates the present investigations.

In this paper, we concentrate our interest on investigating the stability and \( H_{\infty} \) controller design problems for delta operator switched systems with time-varying delay. The main contributions of this paper can be summarized as follows: (1) by constructing a new Lyapunov-Krasovskii functional candidate and using the average dwell time approach, an exponential stability criterion for the considered system is proposed and (2) a state feedback controller design scheme is developed such that the corresponding closed-loop system is exponentially stable with a guaranteed \( H_{\infty} \) performance.

The remainder of the paper is organized as follows. The formulation of the considered systems and some corresponding definitions and lemmas is given in Section 2. In Section 3, the exponential stability analysis and \( H_{\infty} \) control for the underlying system are developed. A numerical example is given to illustrate the feasibility and effectiveness of the proposed method in Section 4. Finally, concluding remarks are presented in Section 5.

**Notations.** \( \| \cdot \| \) denotes the Euclidean norm. \( \lambda_{\min}(\cdot) \) and \( \lambda_{\max}(\cdot) \) denote the minimum and maximum eigenvalues of a matrix, respectively; \( A^T \) means the transpose of matrix \( A \); \( R \) denotes the set of all real numbers; \( R^n \) represents the \( n \)-dimensional real vector space; \( R^{m \times n} \) is the set of all \( (m \times n) \)-dimensional real matrices. The notation \( A > 0(\geq 0) \) means that the matrix \( A \) is positive (nonnegative) definite; \( \text{diag}(\cdots) \) refers to the block-diagonal matrix; \( I \) is the identity matrix of appropriate dimension. \( l_2[k_0, \infty) \) stands for the space of square summable functions on \([k_0, \infty)\).

### 2. Problem Formulation

Consider the following delta operator switched system with time-varying delay:

\[
\begin{align*}
\delta x(k) &= \tilde{A}_{\sigma(k)} x(k) + \tilde{A}_{d\sigma(k)} x(k - \tau(k)) + \tilde{D}_{d\sigma(k)} w(k), \\
z(k) &= C_{\sigma(k)} x(k) + G_{\sigma(k)} w(k), \\
x(k_0 + \theta) &= \phi(\theta), \quad \theta = -\tau, -\tau + 1, \ldots, 0,
\end{align*}
\]

where \( x(k) \in R^n \) is the state vector, \( z(k) \in R^l \) denotes the controlled output, and \( w(k) \in R^w \) represents the disturbance input belonging to \( l_2[k_0, \infty) \). \( k \) means the time \( t = kT \) and \( T > 0 \) is the sampling period; \( k_0 \) is the initial instant. \( \sigma(k) : [k_0, \infty) \rightarrow N = \{1, 2, \ldots, N\} \) is the switching signal with \( N \) being the number of subsystems. \( \tau(k) \) is the time-varying delay satisfying \( 0 \leq \tau \leq \tau(k) \leq \bar{\tau} \) for known constants \( \tau \) and \( \bar{\tau} \). \( \phi(\theta) \) is the discrete vector-valued initial function. \( C_i, \) \( D_i, \) and \( G_i \) are constant matrices with proper dimensions. \( \tilde{A}_i \) and \( \tilde{A}_{di} \) are uncertain real-valued matrices with appropriate dimensions and have the following form:

\[
[\tilde{A}_i \; \tilde{A}_{di}] = [A_i \; A_{di}] + H F_i(k) [E_{ai} \; E_{adi}],
\]

where \( A_i, \; A_{di}, \; H, \; E_{ai}, \) and \( E_{adi} \) are known real constant matrices of suitable dimensions and \( F_i(k) \) is an unknown time-varying matrix which satisfies

\[
F_i^T(k) F_i(k) \leq I.
\]

To obtain the main results, we first give some definitions and lemmas which will be essential in our later development.

**Definition 1** (see [36]). Consider system (2) with \( w(k) = 0 \).

It is said to be exponentially stable under a switching signal \( \sigma(k) \) if, for the initial condition \( x(k_0 + \theta) = \phi(\theta) \), \( \theta = -\tau, -\tau + 1, \ldots, 0 \), there exist constants \( \alpha > 0 \) and \( \beta > 0 \) such that the solution \( x(k) \) satisfies

\[
\| x(k) \| \leq e^{\alpha(k-k_0)} e^{-\beta(k-k_0)}, \quad \forall k \geq k_0,
\]

where \( \| x(k_0) \|_c = \sup_{T \leq \theta \leq 0} \| x(k_0 + \theta) \| \).

**Definition 2.** For given \( 0 < \alpha < 1/T \) and \( \gamma > 0 \), system (2) is said to have an \( H_{\infty} \) performance level \( \gamma \) if there exists a switching signal \( \sigma(k) \) such that the following conditions are satisfied:

1. system (2) is exponentially stable when \( w(k) = 0 \);
2. under the zero-initial condition, that is, \( \phi(\theta) = 0, \theta = -\tau, -\tau + 1, \ldots, 0 \), system (2) satisfies

\[
\sum_{k = k_0}^{\infty} (1 - T\alpha)^{k-k_0} \| z(k) \|^2 \leq \gamma^2 \sum_{k = k_0}^{\infty} \| w(k) \|^2,
\]

\( \forall \omega(k) \in l_2[k_0, \infty) \).

**Definition 3** (see [37]). For any switching signal \( \sigma(k) \) and any \( k_2 > k_1 \geq 0 \), let \( N_{\sigma(k)}(k_1, k_2) \) denote the number of switching of \( \sigma(k) \) over the interval \([k_1, k_2]\). For given \( \tau_a > 0 \) and \( N_0 \geq 0 \), if the inequality

\[
N_{\sigma(k)}(k_1, k_2) \leq N_0 + \frac{k_2 - k_1}{\tau_a}
\]

holds, then the positive constant \( \tau_a \) is called the average dwell time and \( N_0 \) is called the chattering bound. As commonly used in the literature, we choose \( N_0 = 0 \) in this paper.

**Lemma 4** (see [20]). For a given matrix \( S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \) where \( S_{11} \) and \( S_{22} \) are square matrices, the following conditions are equivalent:

1. \( S < 0 \);
2. \( S_{11} < 0, S_{22} - S_{21}^T S_{11}^{-1} S_{21} < 0 \);
3. \( S_{22} < 0, S_{11} - S_{12} S_{22}^{-1} S_{12}^T < 0 \).
Lemma 5 (see [20]). Let $U, V, W,$ and $X$ be real matrices of appropriate dimensions with $X$ satisfying $X = X^T$; then, for all $V^T V \leq I$,

$$X + UVW + W^TV^TU^T < 0$$

(8)

if and only if there exists a scalar $\varepsilon$ such that

$$X + \varepsilon UV^T + V^T \varepsilon^2 W^TW < 0. \tag{9}$$

Lemma 6 (see [28]). For any time function $x(t)$ and $y(t)$, the following equation holds:

$$\delta(x(t), y(t)) = \delta(x(t)) y(t) + x(t) \delta(y(t)) + T \delta(x(t)) \delta(y(t)), \tag{10}$$

where $T$ is the sampling period.

The objectives of the paper are (1) to find a class of switching signal $\sigma(k)$ such that system (2) is exponentially stable with a guaranteed $H_\infty$ performance and (2) to determine a class of switching signal and design a state feedback controller $u(k) = K_{\sigma(k)} x(k)$ for the following delta operator switched system with time-varying delay:

$$\delta x(k) = \tilde{A}_{\sigma(k)} x(k) + \tilde{A}_{d\sigma(k)} x(k - \tau(k)) + \tilde{B}_{\sigma(k)} u(k) + D_{\sigma(k)} w(k), \tag{11}$$

$$z(k) = C_{\sigma(k)} x(k) + G_{\sigma(k)} w(k),$$

$$x(k_0 + \theta) = \phi(\theta), \quad \theta = -\tau, -\tau + 1, \ldots, 0,$$

such that the corresponding closed-loop system is exponentially stable with a guaranteed $H_\infty$ performance.

3. Main Results

3.1. Robust Stability Analysis. In this section, we will focus on the stability of system (2) with $w(k) = 0$.

Theorem 7. For a given positive constant $0 < \alpha < 1/T$, if there exist scalars $\varepsilon_i$ and positive definite symmetric matrices $X_i$ and $Q_i$, $i \in \mathbb{N}$, with appropriate dimensions, such that

$$\begin{bmatrix}
E_i & A_{di} X_i & TX_i A_{di}^T & e_i H_i & X_{ji} E_{ai}^T \\
X_i A_{di}^T & -(1 - T\alpha)^{(T+1)} Q_i & TX_i A_{di}^T & 0 & X_{ji} E_{ai}^T \\
T A_{di} X_i & T A_{di} X_i & -T X_i & e_i T H_i & 0 \\
e_i H_i^T & 0 & e_i T H_i^T & -e_i I & 0 \\
E_{ai} X_i & E_{ai} X_i & 0 & 0 & -e_i I
\end{bmatrix} < 0, \tag{12}$$

where $E_i = A_i X_i + A_i^T X_i + \alpha X_i + (1 - T\alpha)(\tau - 1 + 1)Q_i$, then system (2) with $w(k) = 0$ is exponentially stable for any switching signal $\sigma(k)$ with the following average dwell time scheme:

$$\tau_a > \tau^*_a = \frac{\ln \mu}{\ln(1 - T\alpha)}, \tag{13}$$

where $\mu \geq 1$ satisfies

$$X_i \leq \mu X_j, \quad Q_i \leq \mu Q_j, \quad \forall i, j \in \mathbb{N}. \tag{14}$$

Proof. Choose the following Lyapunov-Krasovskii functional candidate for the $i$th subsystem

$$V_i(k) = V_{i1}(k) + V_{i2}(k) + V_{i3}(k), \quad \forall i \in \mathbb{N}, \tag{15}$$

where

$$V_{i1}(k) = x^T(k) P_i x(k),$$

$$V_{i2}(k) = T \sum_{s=k-\tau(k)}^{k-1} (1 - T\alpha)^{(s-\tau)} x^T(s) S_i x(s), \tag{16}$$

$$V_{i3}(k) = T \sum_{b=-\tau+1}^{s} \sum_{s=b+\tau+1}^{k-1} (1 - T\alpha)^{(s-\tau)} x^T(s) S_i x(s).$$

Taking the delta operator manipulations of Lyapunov functional candidate $V_i(k)$ along the trajectory of system (2), by Lemma 6, we have

$$\delta V_{i1}(k) = \delta x^T(k) P_i x(k)$$

$$= \delta x^T(k) P_i x(k) + x^T(k) P_i \delta x(k) + T \delta x^T(k) P_i \delta x(k)$$

$$\begin{bmatrix}
x(k) \\
x(k - \tau(k))
\end{bmatrix}^T P_i \begin{bmatrix}
x(k) \\
x(k - \tau(k))
\end{bmatrix} \tag{17}$$
\[
\delta V_i (k) = \frac{1}{T} (V_i (k + 1) - V_i (k)) \\
\leq - T \alpha \sum_{s = k - \tau (k)}^{k - \tau (k) + 1} \left( 1 - T \alpha \right)^{(k - s)} x^T (s) S_i x (s) \\
+ \sum_{s = k + 1 - \tau}^{k - \tau + 1} \left( 1 - T \alpha \right)^{(k - s)} x^T (s) S_i x (s),
\]

\[
\delta V_\Omega (k) = \frac{1}{T} (V_\Omega (k + 1) - V_\Omega (k)) \\
\leq - T \alpha \sum_{s = k - \tau (k)}^{k - \tau (k) + 1} \left( 1 - T \alpha \right)^{(k - s)} x^T (s) S_i x (s) \\
+ \sum_{s = k + 1 - \tau}^{k - \tau + 1} \left( 1 - T \alpha \right)^{(k - s)} x^T (s) S_i x (s),
\]

Combining (17)–(19), we have

\[
\delta V_\Omega (k) + \alpha V_i (k) = \left[ \begin{array}{c} x (k) \\ x (k - \tau (k)) \end{array} \right]^T \\
\times \left[ P_i \tilde{A}_i + A_i^T P_i + \alpha P_i + (1 - T \alpha) (\bar{\tau} - \tau + 1) S_i + T \tilde{A}_i^T P_i \tilde{A}_i + T \tilde{A}_i^T P_i \tilde{A}_i \\
\tilde{A}_i^T P_i + T \tilde{A}_i^T P_i \tilde{A}_i \right]^{-1} \left[ P_i \tilde{A}_i + A_i^T P_i + \alpha P_i + (1 - T \alpha) (\bar{\tau} - \tau + 1) S_i + T \tilde{A}_i^T P_i \tilde{A}_i + T \tilde{A}_i^T P_i \tilde{A}_i \right] \\
\times \left[ \begin{array}{c} x (k) \\ x (k - \tau (k)) \end{array} \right]
\]

where

\[
\Omega_i = \left[ \begin{array}{ccc} P_i \tilde{A}_i + A_i^T P_i + \alpha P_i + (1 - T \alpha) (\bar{\tau} - \tau + 1) S_i + T \tilde{A}_i^T P_i \tilde{A}_i + T \tilde{A}_i^T P_i \tilde{A}_i & P_i \tilde{A}_i & T \tilde{A}_i^T \\
\tilde{A}_i^T P_i & - (1 - T \alpha) (\bar{\tau} + 1) S_i & T \tilde{A}_i^T \end{array} \right].
\]

Applying Lemma 4, we can obtain that \( \Omega_i < 0 \) is equivalent to

\[
\left[ \begin{array}{ccc} P_i \tilde{A}_i + A_i^T P_i + \alpha P_i + (1 - T \alpha) (\bar{\tau} - \tau + 1) S_i + T \tilde{A}_i^T P_i \tilde{A}_i & P_i \tilde{A}_i & T \tilde{A}_i^T \\
\tilde{A}_i^T P_i & - (1 - T \alpha) (\bar{\tau} + 1) S_i & T \tilde{A}_i^T \end{array} \right] < 0.
\]

where

\[
E_i = \tilde{A}_i P_i + A_i^T P_i + \alpha P_i + (1 - T \alpha) (\bar{\tau} - \tau + 1) P_i S_i P_i^{-1}.
\]

Denote \( Q_i = P_i S_i P_i^{-1} \) and \( X_i = P_i^{-1} \); then, substituting (3) into (23) and applying Lemmas 4 and 5, we obtain that (23) is equivalent to (12). Thus, from (12), we can easily obtain

\[
\delta V_i (k) + \alpha V_i (k) \leq 0.
\]
It follows from (24) that
\[ \delta V_i(k) = \frac{V_i(k+1) - V_i(k)}{T} \leq -\alpha V_i(k), \]
\[ V_i(k+1) - V_i(k) \leq -\alpha T V_i(k), \]
\[ V_i(k+1) \leq (1 - \alpha T) V_i(k). \] 

(25)

Let \( k_1 < \cdots < k_q \) denote the switching instants of \( \sigma(k) \) over the interval \([k_0, k)\). Consider the following piecewise Lyapunov functional for system (2):

\[ V(k) = V_{\sigma(k)}(k) = V_{\sigma(k_p)}(k), \]
\[ \forall k \in [k_p, k_{p+1}), \ p = 0, 1, \ldots, q. \] 

(26)

From (14), we obtain

\[ V_{\sigma(k_p)}(k_p) \leq \mu V_{\sigma(k_{p-1})}(k_{p-1}), \ p = 0, 1, \ldots, q. \] 

(27)

It can be obtained from (24), (27), and Definition 3 that

\[ V_{\sigma(k)}(k) \leq (1 - T\alpha)^{(k-k_0)}V_{\sigma(k_0)}(k_0) \]
\[ \leq \mu (1 - T\alpha)^{(k-k_0)}V_{\sigma(k_{q-1})}(k_{q-1}) \]
\[ \leq \mu^2 (1 - T\alpha)^{(k-k_0)}V_{\sigma(k_{q-2})}(k_{q-2}) \]
\[ \leq \cdots \]
\[ \leq \mu^{N_{\sigma(k_0,k)}}(1 - T\alpha)^{(k-k_0)}V_{\sigma(k_0)}(k_0). \]

Considering the definition of \( V_{\sigma(k)}(k) \), it yields that

\[ V_{\sigma(k)}(k) \leq a\|x(k)\|^2, \]
\[ V_{\sigma(k)}(k_0) \leq b\|x(k_0)\|^2, \]

where

\[ a = \min_{i \in \mathbb{N}} \lambda_{\min}(P_i), \]
\[ b = \max_{i \in \mathbb{N}} \{\lambda_{\max}(P_i) + T(\bar{\tau}^2 - \bar{\tau} + \bar{T})\lambda_{\max}(S_i)\}, \]
\[ \|x(k_0)\| = \sup_{-\tau_0 \leq \theta \leq 0} \|x(k_0 + \theta)\|. \]

(31)

Combining (29) and (30), we have

\[ \|x(k)\|^2 \leq \frac{b}{a} (\mu^{1/T}(1 - T\alpha))^{(k-k_0)}\|x(k_0)\|^2. \] 

(32)

Therefore, system (2) with \( u(k) = 0 \) is exponentially stable under the average dwell time scheme (13).

The proof is completed.

\[ \square \]

Remark 8. When \( \mu = 1 \) in (14), which leads to \( X_{i} = X_{j}, Q_{i} = Q_{j}, \forall i, j \in \mathbb{N} \), and \( r_{\sigma}^{*} = 0 \) by (13), system (2) has a common Lyapunov-Krasovskii functional and the switching signal can be arbitrary.

When \( r(k) = 0 \), system (2) with \( u(k) = 0 \) becomes the following system:

\[ \begin{align*}
\delta x(k) &= (\bar{A}_{\sigma(k)} + \bar{A}_{dot(k)}) x(k), \\
z(k) &= C_{\sigma(k)} x(k).
\end{align*} \]

(33)

Then we have the following corollary.

Corollary 9. For a given positive constant \( 0 < \alpha < 1/T \), if there exist scalars \( \epsilon_i \) and positive definite symmetric matrices \( X_i, \forall i \in \mathbb{N} \), of appropriate dimensions, such that

\[ \epsilon_i H_i X_i (E_{ai}^T + E_{adi}^T) \begin{bmatrix}
(A_i + A_{di}) X_i + X_i (A_i^T + A_{di}^T) + \alpha X_i & TX_i & X_i (E_{ai}^T + E_{adi}^T) \\
T (A_i + A_{di}) X_i & -TX_i & \epsilon_i H_i \\
\epsilon_i H_i^T & -\epsilon_i I & 0 \\
(E_{ai} + E_{adi}) X_i & \epsilon_i TH_i & 0 \end{bmatrix} < 0, \] 

(34)

then system (33) is exponentially stable for any switching signal \( \sigma(k) \) with average dwell time scheme (13), where \( \mu \geq 1 \) satisfies

\[ X_i \leq \mu X_j, \ \forall i, j \in \mathbb{N}. \]

(35)

3.2. \( H_{\infty} \) Performance Analysis. The following theorem gives sufficient conditions for the existence of an \( H_{\infty} \) performance level for system (2).
Theorem 10. For given positive constants \( \gamma \) and \( 0 < \alpha < 1/T \), if there exist scalars \( \epsilon_i \) and positive definite symmetric matrices \( X_i \) and \( Q_i \), \( i \in N \), of appropriate dimensions, such that

\[
\begin{bmatrix}
\Delta_i & A_{di}X_i & D_i & TX_iA_i^T & C_i & \epsilon_iH_i & X_iF_i^T \\
X_iA_{di} & -(1-T\alpha)^{\tau+1}Q_i & 0 & TX_iA_{di} & 0 & 0 & X_iP_{adi}^T \\
D_i^T & 0 & -\gamma^2I & TD_i & G_i & 0 & 0 \\
TA_iX_i & TA_{di}X_i & TD_i & -TX_i & 0 & \epsilon_iTH_i & 0 \\
C_iX_i & 0 & G_i & 0 & -I & 0 & 0 \\
\epsilon_iH_i^T & 0 & 0 & \epsilon_iTH_i^T & 0 & -\epsilon_iI & 0 \\
E_{adi}X_i & E_{adi}X_i & 0 & 0 & 0 & 0 & 0
\end{bmatrix} < 0,
\]

(36)

where \( \Delta_i = A_iX_i + X_iA_i^T + \alpha X_i + (1 - T\alpha)(T - \tau + 1)Q_i \), then system (2) is exponentially stable with an \( H_\infty \) performance level \( \gamma \) for any switching signal \( \sigma(k) \) with average dwell time scheme (13), where \( \mu \geq 1 \) satisfies (14).

Proof. Equation (12) in Theorem 7 can be directly derived from (36). Thus, system (2) is exponentially stable. We are now in a position to show the \( H_\infty \) performance of system (2).

Choosing the Lyapunov-Krasovskii functional candidate (15) and following the proof line of Theorem 7, we get

\[
\delta V_{i1}(k) = \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^T \Theta_i \begin{bmatrix} x(k) \\ w(k) \end{bmatrix},
\]

where

\[
\begin{bmatrix}
P_iA_i + \bar{A}_i^TP_iA_i & P_iD_i + T\bar{A}^TP_iD_i + C_i^TG_i \\
P_iA_{di} + T\bar{A}_{di}^TP_iA_{di} & P_iD_{di} + T\bar{A}_{di}^TP_iD_{di} + C_i^TG_{di} \\
D_i^TP_i + TD_i^TP_iA_i + C_i^TG_i & TD_i^TP_iA_{di} \\
D_{di}^TP_i + TD_{di}^TP_iA_{di} + C_{di}^TG_{di} & TD_{di}^TP_iD_i + G_{di}^TG_{di} - \gamma^2I
\end{bmatrix},
\]

(40)

\[
\delta V_{i2}(k) \leq \sum_{s=1}^{k-1} (1 - Ta)^{k-s} x^T(s) S_i x(s)
\]

\[
+ (1 - Ta)x^T(k)S_i x(k) + (1 - aT)(T - \tau) x^T(k - \tau(k)) S_i x(k - \tau(k))
\]

\[
+ \sum_{s=1}^{k-1} \sum_{l=1}^{k-s} (1 - Ta)^{k-s} x^T(s) S_i x(s),
\]

(38)

It follows from (37)-(38) that

\[
\delta V_i(k) + \alpha V_i(k) + z^T(k) \gamma^2 \omega^T(k) \omega(k) = \delta V_i(k) + \alpha V_i(k)
\]

\[
+ (C^T x(k) + G^T w(k))^T (C^T x(k) + G^T w(k)) - \gamma^2 \omega^T(k) \omega(k) = \begin{bmatrix} x(k) \\ w(k) \end{bmatrix}^T \Theta_i \begin{bmatrix} x(k) \\ w(k) \end{bmatrix},
\]

where
Applying Lemma 4, we can obtain that $\Theta_i < 0$ is equivalent to the following inequality:

$$
\begin{bmatrix}
P_i \tilde{A}_i + \tilde{A}_i^T P_i + \alpha P_i + (1 - T\alpha) (\bar{\tau} - \tau + 1) S_i \\
\tilde{A}_i^T P_i \\
D_i^T P_i \\
T \tilde{A}_i \\
C_i
\end{bmatrix}
\begin{bmatrix}
P_i \tilde{A}_i \\
0 \\
0 \\
T \tilde{A}_i \\
0
\end{bmatrix}
\begin{bmatrix}
P_i \\
\tilde{A}_i^T \\
D_i \\
C_i
\end{bmatrix}
< 0.
$$

(41)

Using $\text{diag} \{ P_i^{-1} P_i \}$ to pre- and post-multiply both sides of (41), respectively, we have

$$
\begin{bmatrix}
M_i \\
P_i^{-1} \tilde{A}_i \\
D_i \\
T \tilde{A}_i P_i^{-1} \\
C_i P_i^{-1}
\end{bmatrix}
\begin{bmatrix}
\tilde{A}_i P_i^{-1} \\
D_i T P_i^{-1} \\
D_i T P_i^{-1} \\
0 \\
0
\end{bmatrix}
< 0,
$$

(42)

where $M_i = \tilde{A}_i P_i^{-1} + P_i^{-1} \tilde{A}_i^T + \alpha P_i^{-1} + (1 - T\alpha) (\bar{\tau} - \tau + 1) S_i P_i^{-1}$.

Set $Q_i = P_i^{-1} S_i P_i^{-1}$ and $X_i = P_i^{-1}$; then, substituting (3) into (42) and applying Lemmas 4 and 5, we can obtain that (42) is equivalent to (36).

Therefore, one has, for $k \in [k_p, k_{p+1})$,

$$
V(k) \leq (1 - T\alpha)^{(k-k_p)} V_{\sigma(k_p)}(k_p) - \sum_{s=k_p}^{k-1} (1 - T\alpha)^{(k-1-s)} \Lambda(s),
$$

(43)

where $\Lambda(s) = \|z(s)\|^2 - \nu^2 T \|w(s)\|^2$.

Following the proof line of (28), we obtain

$$
V_{\sigma(k)}(k) \leq \mu (1 - T\alpha)^{(k-k_p)} V_{\sigma(k_p)}(k_p) - \sum_{s=k_p}^{k-1} (1 - T\alpha)^{(k-1-s)} \Lambda(s)
$$

$$
\leq \mu (1 - T\alpha)^{(k-k_p)} V_{\sigma(k_{q-1})}(k_{q-1})
$$

(44)

Under the zero initial condition, we get

$$
0 \leq - \sum_{s=k_p}^{k-1} \mu (1 - T\alpha)^{(k-1-s)} \Lambda(s).
$$

(45)
Namely,
\[
\sum_{s=k_0}^{k-1} \mu_N^{s,k}(1 - T \alpha)^{(k-s)} \|z(s)\|^2 
\leq \gamma^2 \sum_{s=k_0}^{k-1} \mu_N^{s,k}(1 - T \alpha)^{(k-s)} \|w(s)\|^2. 
\]  
(46)

Multiplying both sides of (46) by \(\mu^{-N_{\alpha}(k_0,k)}\) leads to
\[
\sum_{s=k_0}^{k-1} \mu_N^{s,k}(1 - T \alpha)^{(k-s)} \|z(s)\|^2 
\leq \gamma^2 \sum_{s=k_0}^{k-1} \mu_N^{s,k}(1 - T \alpha)^{(k-s)} \|w(s)\|^2. 
\]  
(47)

From Definition 3 and (13), we have
\[
\mu^{-N_{\alpha}(k_0,k)} \leq (1 - T \alpha)^{-k_0}. 
\]  
(48)

Combining (47) and (48) leads to
\[
\sum_{s=k_0}^{k-1} (1 - T \alpha)^{(s-k_0)} (1 - T \alpha)^{(k-s)} \|z(s)\|^2 
\leq \gamma^2 \sum_{s=k_0}^{k-1} (1 - T \alpha)^{(k-s)} \|w(s)\|^2. 
\]  
(49)

Then, summing both sides of (49) from \(k_0\) to co leads to
\[
\sum_{k=k_0}^{\infty} (1 - T \alpha)^{(k-k_0)} \|z(k)\|^2 \leq \gamma^2 \sum_{k=k_0}^{\infty} \|w(k)\|^2. 
\]  
(50)

According to Definition 2, we can conclude that the theorem is true.

The proof is completed. \(\blacksquare\)

3.3. \(H_{\infty}\) Controller Design. In this section, a state feedback controller \(u(k) = K_{\alpha(k)}x(k)\) will be designed for system (II) such that the corresponding closed-loop system (51) is exponentially stable and satisfies an \(H_{\infty}\) performance.

Consider
\[
\delta x(k) = \left( \tilde{A}_{\alpha(k)} + \tilde{B}_{\alpha(k)}K_{\alpha(k)} \right) x(k) + \tilde{A}_{d\alpha(k)}x(k - \tau(k)) + D_{\alpha(k)}w(k), \quad z(k) = C_{\alpha(k)}x(k) + E_{\alpha(k)}w(k), 
\]  
(51)

where \(K_i, i \in \mathbb{N}_0\), are the controller gains to be determined. \(\tilde{B}_i\) are uncertain real-valued matrices with appropriate dimensions and have the following form
\[
\tilde{B}_i = B_i + H_i F_i(k) E_{bi}. 
\]  
(52)

**Theorem 11.** Consider system (II). For given positive constants \(\gamma\) and \(0 < \alpha < 1/T\), if there exist scalars \(\epsilon_i\), positive definite symmetric matrices \(Q_i\) and \(X_i\), and any matrices \(W_i, i \in \mathbb{N}_0\), of appropriate dimensions, such that

\[
\begin{bmatrix}
Y_i & A_{di}X_i & D_i & T(A_iX_i + B_iW_i)^T & X_iC_i^T & \epsilon_iH_i & (E_{ai}X_i + E_{bi}W_i)^T \\
X_iA_{di}^T & - (1 - T \alpha)^{\tau-1}Q_i & 0 & TX_iA_{di}^T & 0 & 0 & X_iE_{adi}^T \\
D_i^T & 0 & -\gamma^2 I & TD_i^T & G_i^T & 0 & 0 \\
T(A_iX_i + B_iW_i) & TA_{di}X_i & TD_i & -TX_i & \epsilon_i & 0 & 0 \\
C_iX_i & 0 & G_i & 0 & -I & 0 & 0 \\
\epsilon_iH_i^T & 0 & 0 & \epsilon_i & 0 & 0 & -\epsilon_i \\
(E_{ai}X_i + E_{bi}W_i)^T & E_{adi}X_i & 0 & 0 & 0 & 0 & -\epsilon_i \\
\end{bmatrix} < 0, 
\]  
(53)

where \(Y_i = (A_iX_i + B_iW_i) + (A_iX_i + B_iW_i)^T + \alpha X_i + (1 - T \alpha)(\tau - \tau + 1)Q_i\), then under the state feedback controller
\[
u(k) = K_{\alpha(k)}x(k), \quad K_i = W_iX_i^{-1} \]  
(54)

Proof. Replacing \(\tilde{A}_i\) in (36) with \(\tilde{A}_i + \tilde{B}_iK_i\), we get
\[
\Theta_i = \begin{bmatrix}
\phi_i & A_{di}X_i & D_i & TX_i(A_i + B_iK_i)^T & X_iC_i^T & \epsilon_iH_i & X_i(E_{ai} + E_{bi}K_i)^T \\
X_iA_{di}^T & -(1 - Ta)^{(T+1)Q_i} & 0 & TX_iA_{di}^T & 0 & 0 & X_iE_{adi}^T \\
D_i^T & 0 & -\gamma^2 I & TD_i^T & G_i^T & 0 & 0 \\
T(A_i + B_iK_i)X_i & TA_{di}X_i & TD_i & -TX_i & 0 & \epsilon_iTH_i & 0 \\
C_iX_i & 0 & G_i & 0 & -I & 0 & 0 \\
(\epsilon_iH_i^T & 0 & 0 & \epsilon_iTH_i^T & 0 & -\epsilon I & 0 \\
(\epsilon_iH_i^T)^T & 0 & 0 & (\epsilon_iH_i^T)^T & 0 & -\epsilon I & 0 \\
\end{bmatrix} < 0, \quad (55)
\]

where \( \phi_i = (A_i + B_iK_i)X_i + X_i(A_i + B_iK_i)^T + \alpha X_i + (1 - Ta)(T - 1)Q_i \).

Denoting \( W_i = K_iX_i \), (53) is directly obtained.

The proof is completed. \( \square \)

We are now in a position to give an algorithm for determining \( K_i \) and \( \tau^*_a \).

**Algorithm 12.** Step 1. Input the system matrices.

Step 2. Choose the parameters \( 0 < \alpha < 1 / T \) and \( \gamma > 0 \). By solving (53), one can obtain the solutions of \( \epsilon_i, W_i, X_i, \) and \( Q_i \).

Step 3. By (54), with the obtained \( W_i \) and \( X_i \), one can compute the gain matrices \( K_i \).

Step 4. Compute \( \mu \) and \( \tau^*_a \) by (13)-(14).

## 4. Numerical Example

Consider system (11) with parameters as follows:

\[
A_1 = \begin{bmatrix} 0.5 & -0.7 \\ 0 & 0.4 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0.2 & 0 \\ 0 & -0.4 \end{bmatrix}, \\
B_1 = \begin{bmatrix} 0.12 & 0 \\ 0.4 & -1 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.2 & -0.18 \end{bmatrix}, \\
D_1 = \begin{bmatrix} 0.01 \\ 0 \end{bmatrix}, \quad G_1 = 0.01, \\
H_1 = \begin{bmatrix} 0.05 \\ -0.05 \end{bmatrix}, \quad E_{a1} = \begin{bmatrix} -0.11 \\ 0.03 \end{bmatrix}^T, \\
E_{ad1} = \begin{bmatrix} 0.03 \\ -0.1 \end{bmatrix}^T, \quad E_{b1} = \begin{bmatrix} 0.02 \\ -0.01 \end{bmatrix}^T, \\
A_2 = \begin{bmatrix} 1.2 & -1.3 \\ 1.2 & -0.8 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0 & 0.3 \\ 0.1 & 0 \end{bmatrix}, \\
\]

and the state feedback gain matrices are as follows:

\[
K_1 = \begin{bmatrix} -25.2606 & 14.3786 \\ -11.3117 & 7.9375 \end{bmatrix}, \\
K_2 = \begin{bmatrix} 24.9665 & -8.2606 \\ 16.0614 & -12.3235 \end{bmatrix}, \quad (58)
\]

According to (14), we have \( \mu = 6.5134 \). Then, from (13), we get \( \tau_a > \tau^*_a = 7.3516 \). Choosing \( \tau_a = 7.5 \), the simulation results are shown in Figures 1 and 2, where the initial conditions are \( x(0) = [-1 \ 1]^T \), \( x(k) = [0 \ 0]^T \), and \( k \in [-1, 0] \) and the exogenous disturbance input is \( \nu(k) = 0.05e^{-0.5k} \). The switching signal with average dwell time \( \tau_a = 7.5 \) is shown in.
Finally, a numerical example is given to illustrate the feasibility of the proposed approach. In our future work, we will study the problem of robust $H_{\infty}$ filtering for delta operator switched systems with uncertainties and time-varying delays.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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