Research Article

Existence Results for a Coupled System of Nonlinear Fractional Boundary Value Problems at Resonance

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Some new Banach spaces are established. Based on those new Banach spaces and by using the coincidence degree theory, we present the existence results for a coupled system of nonlinear fractional differential equations with multipoint boundary value conditions at resonance case.

1. Introduction

Fractional differential equations (FDEs) have been of great interest for the last three decades. It is caused both by the intensive development of the theory of fractional calculus itself and by the applications of such constructions in the modeling of many phenomena in various fields of science and engineering. Indeed, we can find numerous applications in viscoelasticity, electrochemistry, control, porous media, and so forth. Therefore, the theory of FDEs has been developed very quickly. Many qualitative theories of FDEs have been obtained. Many important results can be found in [1–13] and the references cited therein.

However, there are few articles which consider the boundary value problems at resonance for a coupled system of nonlinear fractional differential equations. In [14], the authors investigated the existence and uniqueness of solutions for the multipoint boundary value problems for fractional differential equations of the form

$$\begin{aligned} D_t^\alpha y(t) &= f(t, y(t), D_t^\beta y(t)), \quad t \in (0, 1), \\ y(0) &= 0, \quad D_t^\beta y(1) - \sum_{i=1}^{m-2} \zeta_i D_t^\beta y(\xi_i) = y_0, \end{aligned} \quad (1)$$

where $1 < \alpha \leq 2$, $0 < \beta < 1$, $0 < \xi_i < 1$ ($i = 1, 2, \dots, m-2$), $\zeta_i \geq 0$, with $\gamma = \sum_{i=1}^{m-2} \zeta_i \xi_i^{\alpha-\beta-1} < 1$ and D_t^α represents

the standard Riemann-Liouville fractional derivative. And the analysis relies on the Schauder fixed point theorem and the Banach contraction principle.

In [15], the following coupled system of nonlinear fractional differential equation boundary value problem:

$$\begin{aligned} D_{0+}^\alpha u(t) &= f(t, v(t), D_{0+}^{\beta-1} v(t)), \quad 0 < t < 1, \\ D_{0+}^\beta v(t) &= g(t, u(t), D_{0+}^{\alpha-1} u(t)), \quad 0 < t < 1, \\ u(0) &= v(0) = 0, \quad u(1) = \sigma_1 u(\eta_1), \\ v(1) &= \sigma_2 v(\eta_2) \end{aligned} \quad (2)$$

was considered. Where $1 < \alpha, \beta \leq 2$, $0 < \eta_1, \eta_2 < 1$, $\sigma_1, \sigma_2 > 0$, $\sigma_1 \eta_1^{\alpha-1} = \sigma_2 \eta_2^{\beta-1}$, D is the standard Riemann-Liouville fractional derivative and $f, g : [0, 1] \times R^2 \rightarrow R$ are continuous. For more works about coupled system involving fractional differential equations, for instance, see [16, 17].

In this paper, we investigate the existence of solutions for the boundary value problem with the following form:

$$\begin{aligned} D_{0+}^\alpha u(t) &= f(t, v(t), D_{0+}^\beta u(t)), \quad t \in (0, 1), \\ D_{0+}^\beta v(t) &= g(t, u(t), D_{0+}^\alpha v(t)), \quad t \in (0, 1), \end{aligned}$$

$$\begin{aligned} u(0) &= 0, & D_{0+}^p u(1) &= \sum_{i=1}^{m-2} a_i D_{0+}^p u(\xi_i), \\ v(0) &= 0, & D_{0+}^q v(1) &= \sum_{i=1}^{m-2} b_i D_{0+}^q v(\eta_i), \end{aligned} \quad (3)$$

where $1 < \alpha, \beta \leq 2$, $0 < p, q < 1$, $\alpha - p - 1, \beta - q - 1 \geq 0$, $a_i, b_i \geq 0$, $0 < \xi_i, \eta_i < 1$ ($i = 1, 2, \dots, m-2$). Moreover, at resonance case,

$$\sum_{i=1}^{m-2} a_i \xi_i^{\alpha-p-1} = 1, \quad \sum_{i=1}^{m-2} b_i \eta_i^{\beta-q-1} = 1, \quad (4)$$

a sufficient condition for the existence of solutions is established by the coincidence degree continuation theorem.

2. Background Materials and Preliminaries

For the convenience of the reader, we present here some necessary basic knowledge and definitions about fractional calculus theory. These definitions and lemmas can be found in [4–6].

Definition 1. The fractional integral of order $\alpha > 0$ of a function $f: (0, \infty) \rightarrow \mathbb{R}$ is given by

$$I_{0+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad (5)$$

provided the right side is pointwise defined on $(0, \infty)$.

Definition 2. The fractional derivative of order $\alpha > 0$ of a function $f: (0, \infty) \rightarrow \mathbb{R}$ is given by

$$D_{0+}^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds, \quad (6)$$

where $n = [\alpha] + 1$, provided the right side is pointwise defined on $(0, \infty)$.

It can be directly verified that the Riemann-Liouville fractional integration and fractional differentiation operators of the power functions t^β yield power functions of the same form. For $\alpha \geq 0$, $\beta > -1$, there are

$$I_{0+}^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta+\alpha+1)} t^{\beta+\alpha}, \quad D_{0+}^\alpha t^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}. \quad (7)$$

Lemma 3. Assume that $\alpha > 0$, $y \in C(0, 1) \cap L^1[0, 1]$, then the differential equation $D_{0+}^\alpha y(t) = 0$ has solutions

$$y(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n}, \quad (8)$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, $n = [\alpha] + 1$.

Lemma 4. Assume that $y \in C(0, 1) \cap L^1[0, 1]$ and $D_{0+}^\alpha y \in C(0, 1) \cap L^1[0, 1]$, then $I_{0+}^\alpha D_{0+}^\alpha y(t) = y(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_n t^{\alpha-n}$, where $c_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, $n = [\alpha] + 1$.

Lemma 5. Let $\alpha > \beta > 0$, $t \in [0, 1]$ and $y(t) \in L^1[0, 1]$, and then

$$\begin{aligned} I_{0+}^\alpha I_{0+}^\beta y(t) &= I_{0+}^{\alpha+\beta} y(t), & D_{0+}^\alpha I_{0+}^\alpha y(t) &= y(t), \\ D_{0+}^\beta I_{0+}^\alpha y(t) &= I_{0+}^{\alpha-\beta} y(t). \end{aligned} \quad (9)$$

The following definitions and coincidence degree theory are fundamental in the proofs of our main results.

Definition 6. Let Y and Z be Banach spaces. A linear mapping $L: \text{dom}(L) \subset Y \rightarrow Z$ is called a Fredholm mapping if the following two conditions hold:

- (1) $\text{Ker } L$ has a finite dimension;
- (2) $\text{Im } L$ is closed and has a finite codimension.

If L is a Fredholm mapping, its (Fredholm) index is the integer $\text{ind } L = \dim \text{Ker } L - \text{co dim Im } L$.

And if L is a Fredholm mapping of index zero, there exist linear continuous projectors $P: Y \rightarrow Y$ and $Q: Z \rightarrow Z$ such that $\text{Im } P = \text{Ker } L$, $\text{Ker } Q = \text{Im } L$, and $Y = \text{Ker } L \oplus \text{Ker } P$, $Z = \text{Im } L \oplus \text{Im } Q$. Then it follows that $L|_{\text{dom } L \cap \text{Ker } P}: \text{dom } L \cap \text{Ker } P \rightarrow \text{Im } L$ is invertible. We denote the inverse of this map by $K_P: \text{Im } L \rightarrow \text{dom } L \cap \text{Ker } P$. The generalized inverse of this map is denoted by $K_{P,Q}: Y \rightarrow \text{dom } L \cap \text{Ker } P$, $K_{P,Q} = K_P(I - Q)$.

Definition 7. Assume that $L: \text{dom } L \subset Y \rightarrow Z$ is a Fredholm mapping, and if Ω is an open bounded subset of Y , the map N will be called L -compact on $\overline{\Omega}$ if $QN: \Omega \rightarrow Z$ is bounded and $K_{P,Q}N: \Omega \rightarrow Y$ is compact.

If $\text{Im } Q$ is isomorphic to $\text{Ker } L$, there exists an isomorphism $J_{NL}: \text{Im } Q \rightarrow \text{Ker } L$. We have the coincidence degree continuation theorem which is proved in [18].

Theorem 8. Assume that L is a Fredholm mapping of index zero and let N be L -compact on $\overline{\Omega}$, where Ω is an open bounded subset of Y . Suppose that the following conditions are satisfied:

- (1) $Ly \neq \lambda Ny$ for each $(y, \lambda) \in [(\text{dom } L \setminus \text{Ker } L) \cap \partial Q] \times (0, 1)$;
- (2) $Ny \notin \text{Im } L$ for each $y \in \text{Ker } L \cap \partial Q$;
- (3) $\deg(J_{NL}QN|_{\text{Ker } L}, \Omega \cap \text{Ker } L, 0) \neq 0$, where $Q: Z \rightarrow Z$ is a continuous projection as mentioned above with $\text{Im } L = \text{Ker } Q$ and $J_{NL}: \text{Im } Q \rightarrow \text{Ker } L$ is any isomorphism.

Then the equation $Ly = Ny$ has at least one solution in $\text{dom } L \cap \overline{\Omega}$.

Let $Z_1 = L^1[0, 1]$ with the norm $\|x\|_1 = \int_0^1 |x(s)| ds$. For $0 < p, q < 1$, we define a linear space

$$C^p[0, 1], C^q[0, 1]:$$

$$Y_1 = C^p[0, 1]$$

$$= \{u(t) \mid u(t) = I_{0+}^p x(t), x \in C[0, 1], t \in [0, 1]\},$$

$$\begin{aligned}
Y_2 &= C^q [0, 1] \\
&= \{v(t) \mid v(t) = I_{0+}^q y(t), y \in C[0, 1], t \in [0, 1]\},
\end{aligned} \tag{10}$$

to be accompanied by

$$\begin{aligned}
\|u\|_{Y_1} &= \max_{t \in [0, 1]} |D_{0+}^p u(t)| + \max_{t \in [0, 1]} |u(t)|, \\
\|v\|_{Y_2} &= \max_{t \in [0, 1]} |D_{0+}^q v(t)| + \max_{t \in [0, 1]} |v(t)|.
\end{aligned} \tag{11}$$

We can prove that $Y = Y_1 \times Y_2$ is a Banach space with the norm $\|(u, v)\|_Y = \max\{\|u\|_{Y_1}, \|v\|_{Y_2}\}$, and $Z = Z_1 \times Z_1$ is a Banach space, with the norm $\|(x, y)\|_Z = \max\{\|x\|_1, \|y\|_1\}$.

Let L_1 to be the linear operator from $\text{dom } L_1 \cap Y_1$ to Z_1 with

$$\begin{aligned}
\text{dom } L_1 &= \left\{ u \in C^p [0, 1] \mid D_{0+}^\alpha u \in L^1 [0, 1], \right. \\
&\quad \left. u(0) = 0, D_{0+}^p u(1) = \sum_{i=1}^{m-2} a_i D_{0+}^p u(\xi_i) \right\}.
\end{aligned} \tag{12}$$

And $L_1 u = D_{0+}^\alpha u, u \in \text{dom } L_1$. Let L_2 be the linear operator from $\text{dom } L_2 \cap Y_2$ to Z_1 with

$$\begin{aligned}
\text{dom } L_2 &= \left\{ v \in C^q [0, 1] \mid D_{0+}^\beta v \in L^1 [0, 1], \right. \\
&\quad \left. v(0) = 0, D_{0+}^q v(1) = \sum_{j=1}^{m-2} b_j D_{0+}^q v(\eta_j) \right\}.
\end{aligned} \tag{13}$$

And $L_1 v = D_{0+}^\beta v, v \in \text{dom } L_2$. Define L to be the linear operator from $\text{dom } L \cap Y$ to Z with

$$\text{dom } L = \{(u, v) \in Y \mid u \in \text{dom } L_1, v \in \text{dom } L_2\}, \tag{14}$$

and $L(u, v) = (L_1 u, L_2 v)$. We define $N : Y \rightarrow Z$ by setting

$$N(u, v) = (N_1(u(t), v(t)), N_2(u(t), v(t))), \tag{15}$$

where $N_1(u(t), v(t)) = f(t, v(t), D_{0+}^p u(t))$, $N_2(u(t), v(t)) = g(t, u(t), D_{0+}^q v(t))$.

Then the coupled system of BVPs (3) can be written as $L(u, v) = N(u, v)$.

Lemma 9 (see [19]). $F \subset C^\mu[0, 1]$, $0 < \mu < 1$, is a sequentially compact set if and only if F is uniformly bounded and equicontinuous. Here uniformly bounded means that there exists $M > 0$ such that for every $u \in F$,

$$\|y\|_{C^\mu} = \max_{t \in J} |D_{0+}^\mu u(t)| + \max_{t \in J} |u(t)| < M, \tag{16}$$

and equicontinuous means that for all $\varepsilon > 0$, $\exists \delta > 0$, for all $t_1, t_2 \in [0, 1]$, $|t_1 - t_2| < \delta$, $u \in F$, there hold $|u(t_1) - u(t_2)| < \varepsilon$, $|D_{0+}^\beta u(t_1) - D_{0+}^\beta u(t_2)| < \varepsilon$.

Definition 10 (see [19]). One says that $f : [0, 1] \times R^2 \rightarrow R$ satisfies the Carathéodory condition if (i) for each $(x, y) \in R^2$, the function $t \in [0, 1] \rightarrow f(t, x, y) \in R$ is measurable on $[0, 1]$; (ii) for a.e. $t \in [0, 1]$, the function $(x, y) \in R^2 \rightarrow f(t, x, y) \in R$ is continuous on R^2 ; (iii) for each $r > 0$, there exists $\varphi_r(t) \in L^1[0, 1]$ such that $|f(t, x, y)| \leq \varphi_r(t)$ for a.e. $t \in [0, 1]$ and all $(x, y) \in R^2$ with $\sqrt{x^2 + y^2} \leq r$.

3. Main Results

In this section, we will give and prove our main results.

Lemma 11. Let $f, g \in Z_1$, $J = [0, 1]$, $\sum_{i=1}^{m-2} a_i \xi_i^{\alpha-p-1} = 1$, $\sum_{i=1}^{m-2} b_i \eta_i^{\beta-q-1} = 1$. Then $u, v \in Y$ are, respectively, the solutions of the following boundary value problems:

$$\begin{aligned}
D_{0+}^\alpha u(t) &= f(t), \quad t \in J, \\
u(0) &= 0, \quad D_{0+}^p u(1) = \sum_{i=1}^{m-2} a_i D_{0+}^p u(\xi_i),
\end{aligned} \tag{17}$$

$$\begin{aligned}
D_{0+}^\beta v(t) &= g(t), \quad t \in J, \\
v(0) &= 0, \quad D_{0+}^q v(1) = \sum_{i=1}^{m-2} b_i D_{0+}^q v(\eta_i)
\end{aligned} \tag{18}$$

if and only if u, v satisfied

$$\begin{aligned}
u(t) &= c_1 t^{\alpha-1} + I_{0+}^\alpha f(t), \\
&\sum_{i=1}^{m-2} a_i \xi_i^{\alpha-p-1} \int_0^1 (1-s)^{\alpha-p-1} f(s) ds \\
&= \int_0^{\xi_i} (\xi_i - s)^{\alpha-p-1} f(s) ds, \\
v(t) &= c_2 t^{\beta-1} + I_{0+}^\beta g(t), \\
&\sum_{i=1}^{m-2} b_i \eta_i^{\beta-q-1} \int_0^1 (1-s)^{\beta-q-1} g(s) ds \\
&= \int_0^{\eta_i} (\eta_i - s)^{\beta-q-1} g(s) ds.
\end{aligned} \tag{19}$$

Proof. (u, v) in $Y_1 \times Y_2$ are the solutions of the following boundary value problem:

$$\begin{aligned}
I_{0+}^\alpha D_{0+}^\alpha u(t) &= I_{0+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \\
I_{0+}^\beta D_{0+}^\beta v(t) &= I_{0+}^\beta g(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} g(s) ds.
\end{aligned} \tag{20}$$

According to Lemma 4, we have

$$\begin{aligned}
u(t) &= c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + I_{0+}^\alpha f(t), \\
v(t) &= c_3 t^{\beta-1} + c_4 t^{\beta-2} + I_{0+}^\beta g(t).
\end{aligned} \tag{21}$$

By $u(0) = v(0) = 0$, we get $c_2 = c_4 = 0$, and then $u(t) = c_1 t^{\alpha-1} + I_{0+}^{\alpha} f(t)$, $v(t) = c_3 t^{\beta-1} + I_{0+}^{\beta} g(t)$.

Using (7) and Lemma 5, we get that

$$\begin{aligned} D_{0+}^p u(t) &= D_{0+}^p I_{0+}^{\alpha} f(t) + \frac{c_1 \Gamma(\alpha)}{\Gamma(\alpha-p)} t^{\alpha-p-1} \\ &= I_{0+}^{\alpha-p} f(t) + \frac{c_1 \Gamma(\alpha)}{\Gamma(\alpha-p)} t^{\alpha-p-1}, \\ D_{0+}^q v(t) &= D_{0+}^q I_{0+}^{\beta} g(t) + \frac{c_3 \Gamma(\beta)}{\Gamma(\beta-q)} t^{\beta-q-1} \\ &= I_{0+}^{\beta-q} g(t) + \frac{c_3 \Gamma(\beta)}{\Gamma(\beta-q)} t^{\beta-q-1}. \end{aligned} \quad (22)$$

By the boundary conditions, $\sum_{i=1}^{m-2} a_i \xi_i^{\alpha-p-1} = 1$, $\sum_{i=1}^{m-2} b_i \eta_i^{\beta-q-1} = 1$, we have

$$\sum_{i=1}^{m-2} a_i \xi_i^{\alpha-p-1} \int_0^1 (1-s)^{\alpha-p-1} f(s) ds \quad (23)$$

$$= \int_0^{\xi_i} (\xi_i - s)^{\alpha-p-1} f(s) ds,$$

$$\sum_{i=1}^{m-2} b_i \eta_i^{\beta-q-1} \int_0^1 (1-s)^{\beta-q-1} g(s) ds \quad (24)$$

$$= \int_0^{\eta_i} (\eta_i - s)^{\beta-q-1} g(s) ds.$$

Conversely, suppose that (23) and (24) hold. In view of Lemma 5, we can easily verify that u, v are the solutions of (17) and (18), respectively. Then, we complete the proof. $\square$

Lemma 12. *The mapping $L : \text{dom}(L) \subset Y \rightarrow Z$ is a Fredholm operator of index zero.*

Proof. Since $\text{Ker } L_1 = \{u \mid D_{0+}^{\alpha} u(t) = 0\}$, $\text{Ker } L_2 = \{v \mid D_{0+}^{\beta} v(t) = 0\}$, it can be directly verified that $\text{Ker } L_1 = \{u \mid u = c_1 t^{\alpha-1}, c_1 \in \mathbb{R}\}$, $\text{Ker } L_2 = \{v \mid v = c_2 t^{\beta-1}, c_2 \in \mathbb{R}\}$. It is clear that $\text{Ker } L = \{c_1 t^{\alpha-1}, c_2 t^{\beta-1} \mid c_1, c_2 \in \mathbb{R}\}$.

Let $(x, y) \in \text{Im } L$; then there exists $(u, v) \in \text{dom } L$ such that $(x, y) = L(u, v)$, that is, $u \in Y_1$, $x = D_{0+}^{\alpha} u$ and $v \in Y_2$, $y = D_{0+}^{\beta} v$. From these results, we can easily deduce the results

$$\text{Im } L = \{(x, y) \in Z \mid x \text{ satisfies (23), } y \text{ satisfies (24)}\}. \quad (25)$$

Let $t \in (0, 1)$, $\sum_{i=1}^{m-2} a_i \xi_i^{\alpha-p-1} = 1$, $\sum_{i=1}^{m-2} b_i \eta_i^{\beta-q-1} = 1$. Assume that

$$\begin{aligned} \Lambda_1 &= \sum_{i=1}^{m-2} a_i \left(\xi_i^{\alpha-p-1} \int_0^1 (1-s)^{\alpha-p-1} s^k ds \right. \\ &\quad \left. - \int_0^{\xi_i} (\xi_i - s)^{\alpha-p-1} s^k ds \right) \neq 0, \end{aligned}$$

$$\begin{aligned} \Lambda_2 &= \sum_{i=1}^{m-2} b_i \left(\eta_i^{\beta-q-1} \int_0^1 (1-s)^{\beta-q-1} s^k ds \right. \\ &\quad \left. - \int_0^{\eta_i} (\eta_i - s)^{\beta-q-1} s^k ds \right) \neq 0. \end{aligned} \quad (26)$$

Consider the auxiliary mapping,

$$Q_1 : Z \rightarrow R : Q_1 x = \sum_{i=1}^{m-2} a_i \left(\xi_i^{\alpha-p-1} \int_0^1 (1-s)^{\alpha-p-1} x(s) ds \right. \\ \left. - \int_0^{\xi_i} (\xi_i - s)^{\alpha-p-1} x(s) ds \right),$$

$$Q_2 : Z \rightarrow R : Q_2 y = \sum_{i=1}^{m-2} b_i \left(\eta_i^{\beta-q-1} \int_0^1 (1-s)^{\beta-q-1} y(s) ds \right. \\ \left. - \int_0^{\eta_i} (\eta_i - s)^{\beta-q-1} y(s) ds \right). \quad (27)$$

It is obvious that Q_1, Q_2 are continuous linear mappings. Also we have that $Q_1(t^k) = \Lambda_1$, $Q_2(t^k) = \Lambda_2$. Take the mapping $Q : Z \rightarrow Z$ defined by $Q(x, y) = (Q_1' x, Q_2' y)$. $Q_1' x = (Q_1 x / \Lambda_1) t^k$, $Q_2' y = (Q_2 y / \Lambda_2) t^k$, where $x, y \in Z$. Note that $(x, y) \in \text{Im } L$ implies $Q(x, y) = (Q_1 x, Q_2 y) = (0, 0)$. Evidently,

$$\text{Im } Q = \{(x, y) \mid x = c_1 t^k, y = c_2 t^k, c_1, c_2 \in \mathbb{R}\}, \quad (28)$$

and $Q : Z \rightarrow Z$ is a continuous linear projector. In fact, for any $(x, y) \in Z$, we have

$$Q_1(Q_1' x) = Q_1 \left(\frac{Q_1 x}{\Lambda_1} t^k \right) = \frac{Q_1 x}{\Lambda_1} Q_1(t^k) = \frac{Q_1 x}{\Lambda_1} \Lambda_1 = Q_1 x,$$

$$Q_2(Q_2' y) = Q_2 \left(\frac{Q_2 y}{\Lambda_2} t^k \right) = \frac{Q_2 y}{\Lambda_2} Q_2(t^k) = \frac{Q_2 y}{\Lambda_2} \Lambda_2 = Q_2 y,$$

$$Q_1'^2 x = Q_1'(Q_1' x) = \frac{Q_1(Q_1' x)}{\Lambda_1} t^k = \frac{Q_1 x}{\Lambda_1} t^k = Q_1' x,$$

$$Q_2'^2 y = Q_2'(Q_2' y) = \frac{Q_2(Q_2' y)}{\Lambda_2} t^k = \frac{Q_2 y}{\Lambda_2} t^k = Q_2' y, \quad (29)$$

that is to say, $Q : Z \rightarrow Z$ is idempotent.

Observe that $(x, y) \in \text{Im } L$ leads to $Q(x, y) = (0, 0)$ and $(x, y) \in \text{Ker } Q$, where $(0, 0)$ is the element in Z . Conversely, if $(x, y) \in \text{Ker } Q$, we can have that $Q(x, y) = (0, 0)$, that is to say, $(x, y) \in \text{Im } L$. So, $\text{Ker } Q = \text{Im } L$.

Let $(x, y) = (x, y) - Q(x, y) + Q(x, y) = (I - Q)(x, y) + Q(x, y)$, where $(x, y) \in Z$ is an arbitrary element. Since $Q(x, y) \in \text{Im } Q$ and $(I - Q)(x, y) \in \text{Ker } Q$, we obtain that $Z = \text{Im } Q + \text{Ker } Q$. Take $(x_0, y_0) \in \text{Im } Q \cap \text{Ker } Q$, (x_0, y_0) can be written as $(x_0, y_0) = (c_1 t^k, c_2 t^k)$, $c_1, c_2 \in \mathbb{R}$. For

$(x_0, y_0) \in \text{Im } Q$ then since $(x_0, y_0) \in \text{Ker } Q = \text{Im } L$, by (27), we get that

$$\begin{aligned} Q(x_0, y_0) &= Q(c_1 t^k, c_2 t^k) = (c_1 Q_1(t^k), c_2 Q_2(t^k)) \\ &= (c_1 \Lambda_1, c_2 \Lambda_2) = (0, 0), \end{aligned} \quad (30)$$

which implies that $(x_0, y_0) = (0, 0)$. Therefore, $\text{Im } Q \cap \text{Ker } Q = \{0, 0\}$, and thus $Z = \text{Im } Q \oplus \text{Ker } Q = \text{Im } Q \oplus \text{Im } L$.

Now, $\dim \text{Ker } L = 1 = \dim \text{Im } Q = \text{co dim Ker } Q = \text{co dim Im } L < +\infty$, and observing that $\text{Im } L$ is closed in Z , so L is a Fredholm mapping of index zero.

Let $P : Y \rightarrow Y$ be defined by

$$\begin{aligned} (P_1 u)(t) &= D_{0+}^p u(t) |_{t=0} t^{\alpha-1}, \quad t \in J, \quad u \in Y_1, \\ (P_2 v)(t) &= D_{0+}^q v(t) |_{t=0} t^{\beta-1}, \quad t \in J, \quad v \in Y_2, \end{aligned} \quad (31)$$

and $P(u, v) = (P_1 u, P_2 v)$ it is clear that P_1, P_2, P are linear continuous projectors and $\text{Im } P = \text{Ker } L = \{c_1 t^{\alpha-1}, c_2 t^{\beta-1} \mid c_1, c_2 \in R\}$. Also, proceeding as the proof of Lemma 12, we can show that $Y = \text{Im } P \oplus \text{Ker } P = \text{Ker } L \oplus \text{Ker } P$. $\square$

Lemma 13. Assume that $f, g : [0, 1] \times R^2 \rightarrow R$ satisfies the Carathéodory conditions

$$\begin{aligned} \sum_{i=1}^{m-2} a_i \xi_i^{\alpha-p-1} &= 1, \quad \sum_{i=1}^{m-2} b_i \eta_i^{\beta-q-1} = 1, \quad \alpha - p - 1 \geq 0, \\ \beta - q - 1 &\geq 0, \quad \Lambda_1 \neq 0, \quad \Lambda_2 \neq 0. \end{aligned} \quad (32)$$

Then $K_P(I - Q)N : Y \rightarrow Y$ is completely continuous.

Here we omit reasoning process, and it is easy to prove.

Consider the mapping $K_P : \text{Im } L \rightarrow \text{dom } L \cap \text{Ker } P$, $K_P(x, y) = (I_{0+}^\alpha x, I_{0+}^\beta y)$,

$$\begin{aligned} (K_P x)(t) &= I_{0+}^\alpha x(t) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds, \quad x \in \text{Im } L_1, \\ (K_P y)(t) &= I_{0+}^\beta y(t) \\ &= \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s) ds, \quad y \in \text{Im } L_2. \end{aligned} \quad (33)$$

For $(x, y) \in \text{Im } L$, we have $(K_P L)(x, y) = K_P(L(x, y)) = (x, y)$.

$(LK_P)(x, y) = L(K_P(x, y)) = (x, y)$, for all $(x, y) \in \text{Im } L$. Thus, $K_P = (L_P)^{-1}$, where $L_P = L|_{\text{dom } L \cap \text{Ker } P} : \text{dom } L \cap$

$\text{Ker } P \rightarrow \text{Im } L$; hence, for each $(x, y) \in \text{Im } L$ and $t \in J$, we have

$$\begin{aligned} \|K_P(x, y)\|_Y &= \|(I_{0+}^\alpha x, I_{0+}^\beta y)\|_Y \\ &= \max_{t \in J} \left\{ \|I_{0+}^\alpha x\|_{Y_1}, \|I_{0+}^\beta y\|_{Y_2} \right\} \\ &= \max_{t \in J} \left\{ \|D_{0+}^p I_{0+}^\alpha x\| + \|I_{0+}^\alpha x\|, \right. \\ &\quad \left. \|D_{0+}^q I_{0+}^\beta y\| + \|I_{0+}^\beta y\| \right\} \\ &= \max_{t \in J} \left\{ \left| \frac{1}{\Gamma(\alpha-p)} \int_0^t (t-s)^{\alpha-p-1} x(s) ds \right| \right. \\ &\quad \left. + \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s) ds \right|, \right. \\ &\quad \left. \left| \frac{1}{\Gamma(\beta-q)} \int_0^t (t-s)^{\beta-q-1} y(s) ds \right| \right. \\ &\quad \left. + \left| \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} y(s) ds \right| \right\} \\ &\leq \max_{t \in J} \left\{ \left(\frac{1}{\Gamma(\alpha-p)} + \frac{1}{\Gamma(\alpha)} \right) \|x\|_1, \right. \\ &\quad \left. \left(\frac{1}{\Gamma(\beta-q)} + \frac{1}{\Gamma(\beta)} \right) \|y\|_1 \right\} \\ &= \max_{t \in J} \{ \eta_1 \|x\|_1, \eta_2 \|y\|_1 \}, \end{aligned} \quad (34)$$

where

$$\eta_1 = \frac{1}{\Gamma(\alpha-p)} + \frac{1}{\Gamma(\alpha)}, \quad \eta_2 = \frac{1}{\Gamma(\beta-q)} + \frac{1}{\Gamma(\beta)}. \quad (35)$$

The operator $K_P(I - Q)N : Y \rightarrow Y$ is continuous in view of the continuity of the function f and g . Let $\Omega \subset Y$ be bounded; that is to say, there exists a positive constant $M > 0$ such that $\|(x, y)\|_Y \leq M$ for all $(x, y) \in \Omega$.

We can derive that

$$\begin{aligned} \max_{(x,y) \in \Omega} |K_P(I - Q)N(x, y)| &= \max_{(x,y) \in \Omega} \left\{ |I_{0+}^\alpha \{Nx - QNx\}|, |I_{0+}^\beta \{Ny - QNy\}| \right\} \\ &= \max_{(x,y) \in \Omega} \left\{ \left| I_{0+}^\alpha \{f(t, x(t), D_{0+}^p x) - Q(f(t, x(t), D_{0+}^p x))\} \right|, \right. \\ &\quad \left. |I_{0+}^\beta \{g(t, y(t), D_{0+}^q y) - Q(g(t, y(t), D_{0+}^q y))\}| \right\} \end{aligned}$$

$$\begin{aligned}
&\leq \max_{(x,y) \in \Omega} \left\{ \frac{1}{\Gamma(\alpha)} \|g(t, y(t), D_{0+}^q y) - Q(g(t, y(t), D_{0+}^q y))\|_1, \right. \\
&\quad \left. \frac{1}{\Gamma(\beta)} \|g(t, y(t), D_{0+}^q y) - Q(g(t, y(t), D_{0+}^q y))\|_1 \right\} \\
&\leq \max_{(x,y) \in \Omega} \left\{ \frac{1}{\Gamma(\alpha)} M_1, \frac{1}{\Gamma(\beta)} M_2 \right\}.
\end{aligned} \tag{36}$$

Here,

$$\begin{aligned}
M_1 &= \max_{\|x\|_{C^p} \leq M} \|f(t, x(t), D_{0+}^p x) - Q(f(t, x(t), D_{0+}^p x))\|_1, \\
M_2 &= \max_{\|y\|_{C^q} \leq M} \|f(t, y(t), D_{0+}^q y) - Q(f(t, y(t), D_{0+}^q y))\|_1.
\end{aligned} \tag{37}$$

By Lemma 5,

$$\begin{aligned}
&\max_{x \in \Omega} |D_{0+}^p K_P (I - Q) N x| \\
&= \max_{x \in \Omega} |D_{0+}^p I_{0+}^\alpha \{N x - Q N x\}| \\
&= \max_{x \in \Omega} |I_{0+}^\alpha \{f(t, x(t), D_{0+}^p x) - Q(f(t, x(t), D_{0+}^p x))\}| \\
&\leq \frac{1}{\Gamma(\alpha - p)} \|f(t, x(t), D_{0+}^p x) - Q(f(t, x(t), D_{0+}^p x))\|_1 \\
&\leq \frac{1}{\Gamma(\alpha - p)} M_1.
\end{aligned} \tag{38}$$

Similarly, we can prove that $\max_{y \in \Omega} |D_{0+}^q K_P (I - Q) N y| \leq (1/\Gamma(\beta - q)) M_2$.

Hence,

$$\begin{aligned}
&\|K_P(I - Q)N(x, y)\|_Y \\
&\leq \max_{(x,y) \in \Omega} \left\{ \left(\frac{1}{\Gamma(\alpha - p)} + \frac{1}{\Gamma(\alpha)} \right) M_1, \right. \\
&\quad \left. \left(\frac{1}{\Gamma(\beta - q)} + \frac{1}{\Gamma(\beta)} \right) M_2 \right\},
\end{aligned} \tag{39}$$

that is to say, $K_P(I - Q)N(\Omega) \subset Y$ is bounded. On the other hand, let

$$\begin{aligned}
M_3 &= \max_{\|x\|_{Y_1} \leq M} \max_{t \in J} |f(t, x(t), D_{0+}^p x) - Q(f(t, x(t), D_{0+}^p x))| \\
&< +\infty,
\end{aligned}$$

$$\begin{aligned}
M_4 &= \max_{\|y\|_{Y_2} \leq M} \max_{t \in J} |f(t, y(t), D_{0+}^q y) - Q(f(t, y(t), D_{0+}^q y))| \\
&< +\infty.
\end{aligned} \tag{40}$$

For every $(x, y) \in \Omega$, $0 \leq t_1 \leq t_2 \leq 1$, for all $\varepsilon > 0$, there is $\delta = \min\{(\Gamma(\alpha - p)/(M_1 + 1))\varepsilon, (\Gamma(\beta - q)/(M_2 + 1))\varepsilon\}$, and $t_2 - t_1 < \delta$ such that

$$\begin{aligned}
&|D_{0+}^p K_P (I - Q) N x(t_2) - D_{0+}^p K_P (I - Q) N x(t_1)| \\
&= |I_{0+}^{\alpha-p} [f(t_2, x(t_2), D_{0+}^p x(t_2)) - Q(f(t_2, x(t_2), D_{0+}^p x(t_2)))] \\
&\quad - I_{0+}^{\alpha-p} [f(t_1, x(t_1), D_{0+}^p x(t_1)) - Q(f(t_1, x(t_1), D_{0+}^p x(t_1)))]| \\
&= \frac{1}{\Gamma(\alpha - p)} \left| \int_{t_1}^{t_2} (t - s)^{\alpha-p-1} \right. \\
&\quad \times [f(s, x(s), D_{0+}^p x(s)) - Q(f(s, x(s), D_{0+}^p x(s)))] ds \Big| \\
&\leq \frac{1}{\Gamma(\alpha - p)} M_1 \delta < \varepsilon,
\end{aligned} \tag{41}$$

$$|D_{0+}^q K_P (I - Q) N y(t_2) - D_{0+}^q K_P (I - Q) N y(t_1)| < \varepsilon. \tag{42}$$

This ensures that $K_P(I - Q)N(\Omega)$ is equicontinuous too. Then an application of Ascoli-Arezela theorem ensures that $K_P(I - Q)N : Y \rightarrow Y$ is completely continuous.

Theorem 14. Assume that $f, g : [0, 1] \times R^2 \rightarrow R$ satisfy the Carathéodory conditions, $t \in [0, 1]$, $\Lambda_1, \Lambda_2 \neq 0$, $1 < \alpha, \beta \leq 2$, $0 < p, q < 1$, $\alpha - p - 1 \geq 0$, $\beta - q - 1 \geq 0$, and $\sum_{i=1}^{m-2} a_i \xi_i^{\alpha-p-1} = 1$, $\sum_{i=1}^{m-2} b_i \eta_i^{\beta-q-1} = 1$. Assume that all of the following hypotheses hold.

(H₁) There exist nonnegative functions $a_i, b_i, c_i \in Z$, $i = 1, 2, t \in J$, such that for all $(x, y) \in R^2$,

$$\begin{aligned}
|f(t, x, y)| &\leq a_1(t) |x| + b_1(t) |y| + c_1(t), \\
|g(t, x, y)| &\leq a_2(t) |x| + b_2(t) |y| + c_2(t).
\end{aligned} \tag{43}$$

(H₂) $0 < \eta_1' \eta_1'' < 1$, $0 < \eta_2' \eta_2'' < 1$, where η_1, η_2 is defined by (35) and $\eta_1'' = \|a_1\|_1 + \|b_1\|_1$, $\eta_2'' = \|a_2\|_1 + \|b_2\|_1$, $\eta_1' = (\Gamma(p + 1) + 1)/\Gamma(\alpha - p) + \eta_1$, $\eta_2' = (\Gamma(q + 1) + 1)/\Gamma(\beta - q) + \eta_2$.

(H₃) There exist constants $T_1, T_2 > 0$ such that $Q_1(Nu) \neq 0$, $Q_2(Nv) \neq 0$, for each $u \in \text{dom } L_1$, $v \in \text{dom } L_2$ satisfying $|D_{0+}^p u(t)| > T_1$, $|D_{0+}^q v(t)| > T_2$, for all $t \in [0, 1]$.

(H₄) There exist constants $S_1, S_2 > 0$ such that for any $c_1, c_2 \in \mathbb{R}$, if $|c_1| > S_1, |c_2| > S_2$, either

$$c_1 \Lambda_1 Q_1 (N(c_1 t^{\alpha-1})) < 0, \quad c_2 \Lambda_2 Q_2 (N(c_2 t^{\beta-1})) < 0, \quad (44)$$

or else

$$c_1 \Lambda_1 Q_1 (N(c_1 t^{\alpha-1})) > 0, \quad c_2 \Lambda_2 Q_2 (N(c_2 t^{\beta-1})) > 0. \quad (45)$$

Then the problem (3) has at least one solution in $\text{dom } L$.

Proof. The proof consists of four main steps.

Step 1. Set $\Omega_1 = \{(u, v) \in \text{dom } L \setminus \text{Ker } L \mid L(u, v) = \lambda N(u, v), \lambda \in [0, 1]\}$, and prove Ω_1 is bounded. In fact, for $(u, v) \in \Omega_1$, then $(u, v) \in \text{dom } L \setminus \text{Ker } L$ and $L(u, v) = \lambda N(u, v)$, so $\lambda \neq 0$ and $N(u, v) \in \text{Im } L = \text{Ker } Q \subset Z$. Hence, $Q(N(u, v)) = 0$, that is, $Q_1(N_1 u(t)) = 0, Q_2(N_2 v(t)) = 0$. From (H₃), we have that $t_0 \in [0, 1]$ such that $|D_{0+}^p u(t_0)| \leq T_1, |D_{0+}^q v(t_0)| \leq T_2$.

Again, for $(u, v) \in \Omega_1$, we get

$$\begin{aligned} \|Pu\|_{Y_1} &= \max_{u \in \Omega_1} |D_{0+}^p Pu| + \max_{u \in \Omega_1} |Pu| \\ &= \max_{u \in \Omega_1} |D_{0+}^p ((D_{0+}^p u(t_0)) t^p)| + \max_{u \in \Omega_1} |(D_{0+}^p u(t_0)) t^p| \\ &\leq |D_{0+}^p u(t_0)| \Gamma(p+1) + |D_{0+}^p u(t_0)|, \\ \|Pv\|_{Y_2} &\leq |D_{0+}^q v(t_0)| \Gamma(q+1) + |D_{0+}^q v(t_0)|. \end{aligned} \quad (46)$$

In view of Lemma 5, we have

$$\begin{aligned} |D_{0+}^p u(t_0)| &= |D_{0+}^p u(t_0) - I_{0+}^{\alpha-p} (D_{0+}^p u(t_0))| \\ &\leq |D_{0+}^p u(t_0)| + \frac{1}{\Gamma(\alpha-p)} \|D_{0+}^p u(t)\|_1 \\ &\leq T_1 + \frac{1}{\Gamma(\alpha-p)} \|L_1 u\|_1 \\ &\leq T_1 + \frac{1}{\Gamma(\alpha-p)} \|Nu\|_1. \end{aligned} \quad (47)$$

Similarly, we can also obtain $|D_{0+}^q v(t_0)| \leq T_2 + (1/\Gamma(\beta-q)) \|L_2 v\|_1$.

Thus,

$$\begin{aligned} \|P(u, v)\|_Y &\leq \max \left\{ [\Gamma(p+1) + 1] \left[T_1 + \frac{1}{\Gamma(\alpha-p)} \|Nu\|_1 \right], \right. \\ &\quad \left. [\Gamma(q+1) + 1] \left[T_2 + \frac{1}{\Gamma(\beta-q)} \|Nv\|_1 \right] \right\}. \end{aligned} \quad (48)$$

According to Lemma 13, we have

$$\begin{aligned} \|(I-P)(u, v)\|_Y &= \|K_P L(I-P)(u, v)\|_Y \\ &\leq \max \{\eta_1 \|L(I-P)u\|_1, \eta_2 \|L(I-P)v\|_1\} \\ &= \max \{\eta_1 \|Lu\|_1, \eta_2 \|Lv\|_1\} \\ &\leq \max \{\eta_1 \|Nu\|_1, \eta_2 \|Nv\|_1\}. \end{aligned} \quad (49)$$

Combining (48) and (49), we can obtain

$$\begin{aligned} \|(u, v)\|_Y &= \|(I-P)u + Pu\|_Y \\ &\leq \|P(u, v)\|_Y + \|(I-P)(u, v)\|_Y \\ &= \max \{[\Gamma(p+1) + 1] T_1 + \eta_1 \|Nu\|_1, \\ &\quad [\Gamma(q+1) + 1] T_2 + \eta_2 \|Nv\|_1\}. \end{aligned} \quad (50)$$

From (H₁), for each $u \in \Omega_1$, we have

$$\begin{aligned} \|Nu\|_1 &= \int_0^1 |Nu(s)| ds \\ &= \int_0^1 |f(s, u(s), D_{0+}^p u(s))| ds \\ &\leq \int_0^1 |a_1(s)| |u(s)| ds \\ &\quad + \int_0^1 |b_1(s)| |D_{0+}^p u(s)| ds + \int_0^1 |c_1(s)| ds \\ &\leq \left(\int_0^1 |a_1(s)| ds + \int_0^1 |b_1(s)| ds \right) \|u\|_{Y_1} + \|c_1\|_1 \\ &= \eta_1'' \|u\|_{Y_1} + \eta_3. \end{aligned} \quad (51)$$

Also, we can obtain that

$$\|Nv\|_1 = \int_0^1 |Nv(s)| ds \leq \eta_2'' \|v\|_{Y_2} + \eta_4, \quad (52)$$

where $\eta_3 = \|c_1\|_1, \eta_4 = \|c_2\|_1$. Thus, by (H₂) and (50)–(52) we can derive that

$$\begin{aligned} \|(u, v)\|_Y &\leq \max \left\{ \frac{[\Gamma(p+1) + 1] T_1 + \eta_1 \eta_3}{1 - \eta_1 \eta_1''}, \right. \\ &\quad \left. \frac{[\Gamma(q+1) + 1] T_2 + \eta_2 \eta_4}{1 - \eta_2 \eta_2''} \right\}, \end{aligned} \quad (53)$$

which clearly states that Ω_1 is bounded.

Step 2. Set $\Omega_2 = \{(u, v) \in \text{Ker } L \mid N(u, v) \in \text{Im } L\}$ and prove that Ω_2 is bounded. Let $(u, v) \in \Omega_2$; then

$$\begin{aligned} (u, v) \in \text{Ker } L &= \{(u, v) \in \text{dom } L \mid u = c_1 t^{\alpha-1}, \\ &\quad v = c_2 t^{\beta-1}, \forall c_1, c_2 \in \mathbb{R}, t \in (0, 1)\}. \end{aligned} \quad (54)$$

Since $N(u, v) \in \text{Im } L = \text{Ker } Q$, we have $Q(N(u, v)) = (0, 0)$, that is, $Q_1(N(c_1 t^{\alpha-1})) = 0$, $Q_2(N(c_2 t^{\beta-1})) = 0$. Taking account of (H_4) , $|c_1| \leq S_1$, $|c_2| \leq S_2$, which implies that Ω_2 is bounded.

Step 3. Set $\Omega_3 = \{(u, v) \in \text{Ker } L \mid -\lambda(u, v) + (1 - \lambda)J_{NL}QN(u, v) = (0, 0), \lambda \in [0, 1]\}$; if (35) holds, we will set

$$\begin{aligned} \Omega_3 &= \{(u, v) \in \text{Ker } L \mid \lambda(u, v) + (1 - \lambda)J_{NL}QN(u, v) \\ &= (0, 0), \lambda \in [0, 1]\}, \end{aligned} \quad (55)$$

if (44) holds, where $J_{NL} : \text{Im } Q \rightarrow \text{Ker } L$ is a linear isomorphism defined as

$$\begin{aligned} J_{NL}(c_1 t^k) &= c_1 t^{\alpha-1}, \quad J_{NL}(c_2 t^k) = c_2 t^{\beta-1}, \\ \forall c_1, c_2 \in R, \quad t \in (0, 1). \end{aligned} \quad (56)$$

And then we will prove that Ω_3 is bounded. Without loss of generality, in the following part of the proof, we assume that (35) holds in (H_4) , for $(u, v) \in \Omega_3$; we have $u = c_1 t^{\alpha-1}$, $v = c_2 t^{\beta-1}$, $c_1, c_2 \in R$, $t \in [0, 1]$, and $\lambda(u, v) = (1 - \lambda)J_{NL}QN(u, v)$. Thus,

$$\begin{aligned} \lambda t^{\alpha-1} &= (1 - \lambda) \frac{Q_1(N(c_1 t^{\alpha-1}))}{c_1 \Lambda_1} t^{\alpha-1}, \\ \lambda t^{\beta-1} &= (1 - \lambda) \frac{Q_2(N(c_2 t^{\beta-1}))}{c_2 \Lambda_2} t^{\beta-1}. \end{aligned} \quad (57)$$

Therefore, via (H_4) and (35), we have $|c_1| \leq S_1$, $|c_2| \leq S_2$, which show that Ω_3 is bounded.

Step 4. Let Ω be an bounded open set such that $\Omega \supset \bigcup_{i=1}^3 \bar{\Omega}_i$ and prove that $\deg(J_{NL}QN|_{\text{Ker } L}, \Omega \cap \text{Ker } L, 0) \neq 0$. The operator N is L -compact on $\bar{\Omega}$ due to the fact that $QN(\bar{\Omega})$ is bounded and $K_{P,Q}N = K_P(I - P)N : \bar{\Omega} \rightarrow Y$ is compact by Lemma 13; then by Step 1 and Step 2, we have

- (i) $L(u, v) \neq \lambda N(u, v)$ for each $((u, v), \lambda) \in [(\text{dom } L \setminus \text{Ker } L) \cap \partial\Omega] \times (0, 1)$;
- (ii) $N(u, v) \notin \text{Im } L$ for each $(u, v) \in \text{Ker } L \cap \partial\Omega$.

Define $H((u, v), \lambda) = -\lambda I(u, v) + (1 - \lambda)J_{NL}QN(u, v)$, where I is the identity operator in Y . According to the arguments in Step 3, we have $H((u, v), \lambda) \neq 0$, for all $(u, v) \in \text{Ker } L \cap \partial\Omega$, and therefore, via the homotopy property of degree, we obtain that

$$\begin{aligned} \deg(J_{NL}QN|_{\text{Ker } L}, \Omega \cap \text{Ker } L, 0) &= \deg(H(\cdot, 0), \Omega \cap \text{Ker } L, 0) \\ &= \deg(H(\cdot, 1), \Omega \cap \text{Ker } L, 0) \\ &= \deg(-I, \Omega \cap \text{Ker } L, 0) = -1 \neq 0, \end{aligned} \quad (58)$$

which certifies the condition (3) of Theorem 8. Then, applying Theorem 8, we conclude that the problem (3) has at least one solution in $\text{dom } L \cap \bar{\Omega}$. The proof is complete. $\square$

4. Conclusion

In this paper, we obtain the existence results of multi-point boundary value problem at resonance for a coupled system of nonlinear fractional differential equations by means of the coincidence degree theory.

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

Ruijuan Liu completed the proof and wrote the initial draft. Chunhai Kou provided the problem and gave some suggestions of amendment. Ruijuan Liu then finalized the paper. All authors read and approved the final paper.

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