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Research Article

Adaptive Modified Function Projective Lag Synchronization of Uncertain Hyperchaotic Dynamical Systems with the Same or Different Dimension and Structure

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Modified function projective lag synchronization (MFPLS) of uncertain hyperchaotic dynamical systems with the same or different dimensions and structures is studied. Based on Lyapunov stability theory, a general theorem for controller designing, parameter update rule designing, and control gain strength adapt law designing is introduced by using adaptive control method. Furthermore, the scheme is applied to four typical examples: MFPLS between two five-dimensional hyperchaotic systems with the same structures, MFPLS between two four-dimensional hyperchaotic systems with different structures, MFPLS between a four-dimensional hyperchaotic system and a three-dimensional chaotic system and MFPLS between a novel three-dimensional chaotic system, and a five-dimensional hyperchaotic system. And the system parameters are all uncertain. Corresponding numerical simulations are performed to verify and illustrate the analytical results.

1. Introduction

During the past three decades, chaos synchronization has been a hot topic in nonlinear science due to its various applications [1–3]. A variety of synchronization approaches have been revealed, such as complete synchronization (CS) [4], antisynchronization (AS) [5], phase synchronization [6], lag synchronization (LS) [7], projective synchronization (PS) [8], function projective synchronization (FPS) [9–11], and others [12, 13]. Recently, modified function projective synchronization (MFPS) is proposed, in which the drive system and the response system could be synchronized up to a scaling function matrix [14]. MFPS can enhance the security of communication, because the unpredictability of the scaling functions increases the complexity of the systems. So, MFPS attracts the interests of researchers in many fields [15–20].

In the past, many theoretical results focused on the systems as identical, similar. But in a great many practical situations, the parameters of systems cannot be known or certain entirely, and sometimes the synchronization is carried out even though the drive system and the response system have different dimensions, especially the systems in biological science and social science. Recently, some researchers carried out works related to this.

Sun et al. [21] proposed the MFPS of uncertain hyperchaotic systems with identical or nonidentical structures, gave a general formula for designing the controllers and parameter update rules, and applied the theoretical results to three typical cases. Zheng [22] investigated the MFPS between two different dimensional chaotic systems with fully unknown or partially unknown parameters via increased order method, designed a unified adaptive controller and parameter update laws, and the control strength of the controller can adaptively be identified. Reference [23] studied the MFPS between four-dimensional Lorenz and Chen hyperchaotic dynamical systems with fully unknown parameters; scaling function matrix had the form of $\mathbf{M}h(t)$, $\mathbf{M}$ is a constant diagonal matrix and $h(t)$ is a continuous differentiable function with $h(t) \neq 0$, and the scaling function matrix is more flexible and variable through choosing different $\mathbf{M}$ and $h(t)$. So, it has broad application prospects in practical situations [24].

Time delay is frequently encountered in the controlling process of many nonlinear systems, the existence of which makes the controller design and analysis much more complex. Some existing papers consider the influence of time delay on MFPS of chaotic systems. Du et al. [25] introduced modified function projective lag synchronization (MFPLS) and investigated a general method of MFPLS based on Lyapunov stability theory. But in the analysis, they suppose that the system parameters were known and the dimensions of the drive system and response system were equal. Recently, Cai et al. [26] studied the modified function lag projective synchronization of a financial hyperchaotic system, when the parameters are known and unknown, respectively. And their work aimed for MFPLS of the concrete system and could not give the general method. More recently, Liu et al. [27] analyzed the MFPLS between two nonidentical multiscroll chaotic systems with unknown disturbances. Du [28] researched the MFPS of complex dynamical networks with different nodes through adaptive open-plus-closed-loop control method and investigated MFPS in drive-response dynamical networks with time-varying coupling delay.

Inspired by the previous discussion, in this paper, we study the modified function projective lag synchronization (MFPLS) of hyperchaotic systems; the parameters of the systems are all uncertain, dimensions and structures of the drive system and the response system are the same or different. Based on the Lyapunov stability theory and adaptive control technique, an adaptive controller and corresponding parameter update rule are constructed. By applying the method, MFPLS between a five-dimensional hyperchaotic system and itself with uncertain parameters, a four-dimensional hyperchaotic system and another four-dimensional hyperchaotic system with unknown parameters and different structures, a four-dimensional hyperchaotic system and a three-dimensional chaotic system with uncertain parameters, and a three-dimensional chaotic system and a five-dimensional hyperchaotic system with unknown parameters is achieved. Corresponding simulation results show the effectiveness of the proposed scheme.

The outline of the paper is as follows. In Section 2, the definition of MFPLS is introduced. In Section 3, a general method for MFPLS between two uncertain hyperchaotic systems with the same or different dimension and structure is given. Based on the Lyapunov stability theory and adaptive control technique, an adaptive controller, corresponding parameter update rule, and control gain strength adapt law are designed. In Section 4, we give four typical examples to verify the effectiveness of the proposed scheme by numerical simulation. Finally, the conclusions are drawn in Section 5.

2. The Definition of MFPLS

Consider the drive (master) chaotic system and the response (slave) chaotic system given in the following form:

$$\dot{\mathbf{x}}(t) = f(\mathbf{x}(t)), \quad (1)$$

$$\dot{\mathbf{y}}(t) = g(\mathbf{y}(t)) + \mathbf{U}(t), \quad (2)$$

where $\mathbf{x}(t)$, $\mathbf{y}(t)$ are the state vectors of systems (1) and (2), respectively; f, g are two continuous vector functions, and $\mathbf{U}$ is a controller to be designed for synchronization between systems (1) and (2). The error dynamical system is defined as follows:

$$\mathbf{e}(t) = \mathbf{y}(t) - \mathbf{M}h(t)\mathbf{x}(t-\boldsymbol{\tau}), \quad (3)$$

where $\mathbf{M}$ is a constant diagonal matrix, $h(t)$ is a nonzero continuous differentiable function, and the time delay vector is $\boldsymbol{\tau}$.

Definition 1. If there exists a constant diagonal matrix $\mathbf{M}$ and function $h(t)$, such that $\lim_{t\to\infty}\|\mathbf{e}(t)\| = 0$, then the synchronization between the system (1) and system (2) is modified function projective lag synchronization (MFPLS).

Remark 2. If $h(t) = 1$ and $\boldsymbol{\tau} = 0$, MFPLS becomes modified projective synchronization. If $\mathbf{M} = 1$ and $\boldsymbol{\tau} = 0$, MFPLS turns out to be the function projective synchronization. If $\boldsymbol{\tau} = 0$, MFPLS is MFPS. If $\mathbf{M} = 1$ and $h(t) = 1$, then lag synchronization appears.

3. The Modified Function Projective Lag Synchronization Scheme

Suppose that the parameters in the drive and response system are uncertain. Rewrite them as follows:

$$\dot{\mathbf{x}}(t) = f_1(\mathbf{x}(t))\mathbf{A} + f_2(\mathbf{x}(t)), \quad (4)$$

$$\dot{\mathbf{y}}(t) = g_1(\mathbf{y}(t))\mathbf{B} + g_2(\mathbf{y}(t)) + \mathbf{U}(t), \quad (5)$$

where $\mathbf{x}(t) = (x_1, x_2, \ldots, x_m)^{\mathrm{T}} \in R^m$, $\mathbf{y}(t) = (y_1, y_2, \ldots, y_n)^{\mathrm{T}} \in R^n$ are the state vectors of systems (4) and (5), respectively; $f : R^m \to R^m$, $g : R^n \to R^n$ are two continuous vector functions; $\mathbf{A}$, $\mathbf{B}$ are the uncertain parameter vectors of the drive and response system, respectively; $\mathbf{U}$ is a controller to be designed for synchronization between systems (4) and (5). The error system is defined as (3).

The dimension of the drive system is m and the order of the response system is n; they are not mostly the same. So, in the following, we discuss three cases: $m = n$, $m > n$, and $n > m$ and give a general scheme for the synchronization controller design and parameter update rule design.

Case 1 ($m = n$). At this time, the drive and response systems are of the same order.

Theorem 3. *For the given diagonal matrix* $\mathbf{M} = \mathrm{diag}\{m_1, m_2, \ldots, m_n\} \in R^{n\times n}$, *nonzero continuous differentiable function* $\mathbf{h}(t)$, *and time delay vector* $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_n)^T$, *the modified function projective lag synchronization (MFPLS) between system (4) and system (5) is achieved by the following controller (6), the control gain matrix* $\mathbf{K} = \mathrm{diag}(k_1, k_2, \ldots, k_n)$, *(*$k_i$ *is* *the control gain strength)* (7), *the parameter update laws* (8) *and* (9) *as below:*

$$\mathbf{U}(t) = -g_1(\mathbf{y}(t))\widehat{\mathbf{B}} - g_2(\mathbf{y}(t)) + \mathbf{M}\dot{h}(t)\mathbf{x}(t-\boldsymbol{\tau}) + \mathbf{M}h(t)\left(f_1(\mathbf{x}(t))\widehat{\mathbf{A}} + f_2(\mathbf{x}(t))\right) - \mathbf{K}\mathbf{e}, \tag{6}$$

$$\dot{k}_i = \varepsilon_i e_i^2, \quad \varepsilon_i > 0, \ i = 1, 2, \dots, n, \tag{7}$$

$$\dot{\mathbf{e}}_A(t) = -f_1^T(\mathbf{x}(t))h^T(t)\mathbf{M}\mathbf{e}(t), \tag{8}$$

$$\dot{\mathbf{e}}_B(t) = g_1(\mathbf{y}(t))\mathbf{e}(t), \tag{9}$$

where $\mathbf{e}_i(t) = y_i(t) - m_i h(t) x_i(t-\tau_i)$; $\widehat{\mathbf{A}}$, $\widehat{\mathbf{B}}$ *are the estimated values of unknown parameters* $\mathbf{A}$ *and* $\mathbf{B}$, *respectively;* $\mathbf{e}_A(t) = \widehat{\mathbf{A}} - \mathbf{A}$ *and* $\mathbf{e}_B(t) = \widehat{\mathbf{B}} - \mathbf{B}$ *are parameter error vector matrices.*

Proof. According to the definition of MFPLS, the error vector has the following form: $\mathbf{e}(t) = \mathbf{y}(t) - \mathbf{M}h(t)\mathbf{x}(t-\boldsymbol{\tau})$. And the error dynamical system is

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \dot{\mathbf{y}}(t) - \mathbf{M}\dot{h}(t)\mathbf{x}(t-\boldsymbol{\tau}) - \mathbf{M}h(t)\dot{\mathbf{x}}(t-\boldsymbol{\tau}) \\ &= g_1(\mathbf{y}(t))\mathbf{B} + g_2(\mathbf{y}(t)) + \mathbf{U}(t) - \mathbf{M}\dot{h}(t)\mathbf{x}(t-\boldsymbol{\tau}) \\ &\quad - \mathbf{M}h(t)\left(f_1(\mathbf{x}(t))\mathbf{A} + f_2(\mathbf{x}(t))\right).\end{aligned} \tag{10}$$

By substituting (6) into (10), one can obtain the following:

$$\begin{aligned}\dot{\mathbf{e}}(t) &= g_1(\mathbf{y}(t))\left(\mathbf{B} - \widehat{\mathbf{B}}\right) + \mathbf{M}h(t)f_1(\mathbf{x}(t))\left(\widehat{\mathbf{A}} - \mathbf{A}\right) - \mathbf{K}\mathbf{e} \\ &= -g_1(\mathbf{y}(t))\mathbf{e}_B(t) + \mathbf{M}h(t)f_1(\mathbf{x}(t))\mathbf{e}_A(t) - \mathbf{K}\mathbf{e}.\end{aligned} \tag{11}$$

The Lyapunov function is chosen as

$$V(t) = \frac{1}{2}\left(\mathbf{e}^T(t)\mathbf{e}(t) + \mathbf{e}_A^T(t)\mathbf{e}_A(t) + \mathbf{e}_B^T(t)\mathbf{e}_B(t)\right) + \frac{1}{2}\sum_{i=1}^{n}\frac{1}{\varepsilon_i}(k_i - k)^2, \tag{12}$$

where $k > 0$ is a positive constant.

Then the time derivation of the Lyapunov function along the trajectory of error systems (10) is

$$\dot{V}(t) = \mathbf{e}^T(t)\dot{\mathbf{e}}(t) + \mathbf{e}_A^T(t)\dot{\mathbf{e}}_A(t) + \mathbf{e}_B^T(t)\dot{\mathbf{e}}_B(t) + \dot{k}_i\sum_{i=1}^{n}\frac{1}{\varepsilon_i}(k_i - k)^2. \tag{13}$$

Constituting (7), (8), and (9) into (13), we get the following:

$$\begin{aligned}\dot{V}(t) &= \mathbf{e}^T(t)\dot{\mathbf{e}}(t) + \mathbf{e}_A^T(t)\dot{\mathbf{e}}_A(t) + \mathbf{e}_B^T(t)\dot{\mathbf{e}}_B(t) \\ &\quad + \dot{k}_i\sum_{i=1}^{n}\frac{1}{\varepsilon_i}(k_i - k)^2 \\ &= -\mathbf{e}^T(t)g_1(\mathbf{y}(t))\mathbf{e}_B(t) + \mathbf{e}^T(t)\mathbf{M}h(t)f_1(\mathbf{x}(t))\mathbf{e}_A(t) \\ &\quad - \mathbf{e}^T(t)\mathbf{K}\mathbf{e} + \sum_{i=1}^{n}(k_i - k)^2 e_i^2 \\ &= -k\mathbf{e}^T(t)\mathbf{e} < 0.\end{aligned} \tag{14}$$

$V(t)$ is positive definite and $\dot{V}(t) < 0$; thus, according to Barbalat's Lemma, MFPLS between the drive system (4) and the response system (5) is achieved. However, we cannot conclude that the unknown parameters can be estimated to their true values. □

Linear Independence Condition. To achieve synchronization-based parameter identification, the nonlinear vector functions $-f_1^T(\mathbf{x}(t))h^T(t)\mathbf{M}$ and $g_1^T(\mathbf{y}(t))$ must be linearly independent of the synchronization manifold, and then the unknown parameters can be identified [29, 30].

Remark 4. Suppose that the parameter $\mathbf{A}$ in the drive system (4) are known in priori, then the controller and parameter update rule can be designed as follows:

$$\begin{aligned}\mathbf{U}(t) &= -g_1(\mathbf{y}(t))\widehat{\mathbf{B}} - g_2(\mathbf{y}(t)) + \mathbf{M}\dot{h}(t)\mathbf{x}(t-\boldsymbol{\tau}) \\ &\quad + \mathbf{M}h(t)\left(f_1(\mathbf{x}(t))A + f_2(\mathbf{x}(t))\right) - \mathbf{K}\mathbf{e}, \\ \dot{\mathbf{e}}_B(t) &= g_1(\mathbf{y}(t))\mathbf{e}(t).\end{aligned} \tag{15}$$

Control gain strength adapt rule is as (7).

Remark 5. Suppose that the parameter $\mathbf{B}$ in the response system (5) is known in priori, then the controller and parameter update rule can be modified as follows:

$$\begin{aligned}\mathbf{U}(t) &= -g_1(\mathbf{y}(t))\mathbf{B} - g_2(\mathbf{y}(t)) + \mathbf{M}\dot{h}(t)\mathbf{x}(t-\boldsymbol{\tau}) \\ &\quad + \mathbf{M}h(t)\left(f_1(\mathbf{x}(t))\widehat{\mathbf{A}} + f_2(\mathbf{x}(t))\right) - \mathbf{K}\mathbf{e}, \\ \dot{\mathbf{e}}_A(t) &= -f_1^T(\mathbf{x}(t))h^T(t)\mathbf{M}\mathbf{e}(t).\end{aligned} \tag{16}$$

Control gain strength adapt law is as (7).

Remark 6. If the drive and response system have the identical structures, then the controller and parameter update rule is as follows:

$$\begin{aligned}\mathbf{U}(t) &= \left(-f_1(\mathbf{y}(t)) + \mathbf{M}h(t)f_1(\mathbf{x}(t))\right)\widehat{\mathbf{A}} \\ &\quad + \left(-f_2(\mathbf{y}(t)) + \mathbf{M}h(t)f_2(\mathbf{x}(t))\right) \\ &\quad + \mathbf{M}\dot{h}(t)\mathbf{x}(t-\boldsymbol{\tau}) - \mathbf{K}\mathbf{e}, \\ \dot{\mathbf{e}}_A(t) &= -f_1^T(\mathbf{x}(t))h^T(t)\mathbf{M}\mathbf{e}(t).\end{aligned} \tag{17}$$

Control gain strength adapt rule is as (7). When the drive system and the response system have the concrete structure, the controller can be largely simplified.

Remark 7. In many references [25–28], the control gain strength is fixed, and sometimes it may be the maximal; thus, it can give a kind of energy waste. The method of our paper is different from them. The control gain strength k_i can be automatically adapted to a suitable value depending on the initial values.

Case 2 ($m > n$). The dimension of the drive system is higher than that of the response system. And we can achieve MFPLS through adding extra-auxiliary states to the response system, so that the dimensions of the drive and response system are equal.

Denote the auxiliary states as $y' \in R^{m-n}$, and the auxiliary vector is as follow:

$$\begin{aligned}\dot{y}' &= (\phi_1(\mathbf{y}), \phi_2(\mathbf{y}), \dots, \phi_{m-n}(\mathbf{y})) + U'(t) \\ &= g_1'(\mathbf{y}(t))\mathbf{B} + g_2'(\mathbf{y}(t)) + U'(t).\end{aligned} \tag{18}$$

Then, the response system is modified as follows:

$$\dot{\tilde{\mathbf{y}}}(t) = g_1(\tilde{\mathbf{y}}(t))\mathbf{B} + g_2(\tilde{\mathbf{y}}(t)) + \widetilde{\mathbf{U}}(t), \tag{19}$$

where $\tilde{\mathbf{y}} = \begin{pmatrix}\mathbf{y} \\ \mathbf{y}'\end{pmatrix}$, $g_1(\tilde{\mathbf{y}}(t)) = \begin{pmatrix}g_1(\mathbf{y}(t)) \\ g_1'(\mathbf{y}(t))\end{pmatrix}$, $g_2(\tilde{\mathbf{y}}(t)) = \begin{pmatrix}g_2(\mathbf{y}(t)) \\ g_2'(\mathbf{y}(t))\end{pmatrix}$, $\widetilde{\mathbf{U}}(t) = \begin{pmatrix}\mathbf{U}(t) \\ \mathbf{U}'(t)\end{pmatrix}$.

Thus the controller, parameter update law, and control gain strength adapt rule are designed as follows:

$$\begin{aligned}\widetilde{\mathbf{U}}(t) &= -g_1(\tilde{\mathbf{y}}(t))\widehat{\mathbf{B}} - g_2(\tilde{\mathbf{y}}(t)) + \mathbf{M}\dot{h}(t)\mathbf{x}(t-\boldsymbol{\tau}) \\ &\quad + \mathbf{M}h(t)\left(f_1(\mathbf{x}(t))\widehat{\mathbf{A}} + f_2(\mathbf{x}(t))\right) - \mathbf{Ke}, \\ \dot{k}_i &= \varepsilon_i e_i^2, \quad \varepsilon_i > 0,\ i = 1, 2, \dots, m, \\ \dot{\mathbf{e}}_A(t) &= -f_1^{\mathrm{T}}(\mathbf{x}(t))h^{\mathrm{T}}(t)\mathbf{Me}(t), \\ \dot{\mathbf{e}}_B(t) &= g_1(\tilde{\mathbf{y}}(t))\mathbf{e}(t).\end{aligned} \tag{20}$$

The proof is similar to Case 1 and is omitted here.

Case 3 ($n > m$). The dimension of the drive system is lower than that of the response system. We can achieve the MFPLS in increased order method. This time, we add some auxiliary states which is the function of drive state $\mathbf{x}$ to the drive system, and in the final, the order of the drive system is equal to that of the response system.

Denote the auxiliary states as $\mathbf{x}' \in R^{n-m}$, and the auxiliary function vector is

$$\begin{aligned}\dot{\mathbf{x}}' &= \boldsymbol{\varphi}(\mathbf{x}) = (\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x}), \dots, \varphi_{n-m}(\mathbf{x})) \\ &= f_1'(\mathbf{x}(t))\mathbf{A} + f_2'(\mathbf{x}(t)) \in R^{n-m}.\end{aligned} \tag{21}$$

Then, the drive system is composed as

$$\tilde{\mathbf{x}}(t) = f_1(\tilde{\mathbf{x}}(t))\mathbf{A} + f_2(\tilde{\mathbf{x}}(t)), \tag{22}$$

where $\tilde{\mathbf{x}} = \begin{pmatrix}\mathbf{x} \\ \mathbf{x}'\end{pmatrix}$, $f_1(\tilde{\mathbf{x}}(t)) = \begin{pmatrix}f_1(\mathbf{x}(t)) \\ f_1'(\mathbf{x}(t))\end{pmatrix}$, $f_2(\tilde{\mathbf{x}}(t)) = \begin{pmatrix}f_2(\mathbf{x}(t)) \\ f_2'(\mathbf{x}(t))\end{pmatrix}$.

The controller, parameter update law, and control gain strength adapt rule are illustrated as follows:

$$\begin{aligned}\mathbf{U}(t) &= -g_1(\mathbf{y}(t))\widehat{\mathbf{B}} - g_2(\mathbf{y}(t)) + \mathbf{M}\dot{h}(t)\mathbf{x}(t-\boldsymbol{\tau}) \\ &\quad + \mathbf{M}h(t)\left(f_1(\tilde{\mathbf{x}}(t))\widehat{A} + f_2(\tilde{\mathbf{x}}(t))\right) - \mathbf{Ke}, \\ \dot{k}_i &= \varepsilon_i e_i^2, \quad \varepsilon_i > 0,\ i = 1, 2, \dots, n, \\ \dot{\mathbf{e}}_A(t) &= -f_1^{\mathrm{T}}(\tilde{\mathbf{x}}(t))h^{\mathrm{T}}(t)\mathbf{Me}(t), \\ \dot{\mathbf{e}}_B(t) &= g_1(\mathbf{y}(t))\mathbf{e}(t).\end{aligned} \tag{23}$$

The proof is similar to Case 1 and is omitted here.

4. Numerical Simulations

In this section, four typical cases are provided to verify and show the effectiveness of the controller, parameter update rule, and control gain strength adapt law. The solver DDE23 in Matlab is used to integrate the delay differential equations.

4.1. Case 1: MFPLS between Two Five-Dimensional Hyperchaotic Systems with Identical Structures. In this subsection, we consider the case that $m = n$ and the drive and response system have the identical dimensions and structures. Recently, Hu [31] proposed a new five-dimensional hyperchaotic Lorenz system by introducing two state feedback controllers to the classical three-dimensional Lorenz system, which is described by

$$\begin{aligned}\dot{x}_1 &= -\sigma x_1 + \sigma x_2 + x_4, \\ \dot{x}_2 &= r x_1 - x_2 - x_1 x_3 - x_5, \\ \dot{x}_3 &= -\beta x_3 + x_1 x_2, \\ \dot{x}_4 &= -x_1 x_3 + d_1 x_4, \\ \dot{x}_5 &= d_2 x_2,\end{aligned} \tag{24}$$

where x_1, x_2, x_3, x_4, and x_5 are state variables and σ, β, r, d_1, and d_2 are the unknown system parameters to be identified. When $\sigma = 10$, $\beta = 8/3$, $r = 28$, $d_1 = 2$, and $d_2 \in (2, 12)$, the system is hyperchaotic with three positive LEs. u_1, u_2, u_3, u_4, and u_5 are the controllers to be designed.

The five-dimensional Lorenz hyperchaotic system, as the response system, is described as

$$\begin{aligned}\dot{y}_1 &= -\sigma y_1 + \sigma y_2 + y_4 + u_1, \\ \dot{y}_2 &= r y_1 - y_2 - y_1 y_3 - y_5 + u_2, \\ \dot{y}_3 &= -\beta y_3 + y_1 y_2 + u_3, \\ \dot{y}_4 &= -y_1 y_3 + d_1 y_4 + u_4, \\ \dot{y}_5 &= d_2 y_2 + u_5,\end{aligned} \tag{25}$$

where y_1, y_2, y_3, y_4, and y_5 are state variables, σ, β, r, d_1, and d_2 are the unknown system parameters to be identified. u_1, u_2, u_3, u_4, and u_5 are the controllers to be constructed so that the drive system (24) and the response system (25) can be synchronized in the sense of MFPLS.

We define the MFPLS error as follows:

$$\begin{aligned} e_1(t) &= y_1(t) - m_1 h(t) x_1(t-\tau_1), \\ e_2(t) &= y_2(t) - m_2 h(t) x_2(t-\tau_2), \\ e_3(t) &= y_3(t) - m_3 h(t) x_3(t-\tau_3), \\ e_4(t) &= y_4(t) - m_4 h(t) x_4(t-\tau_4), \\ e_5(t) &= y_5(t) - m_5 h(t) x_5(t-\tau_5), \end{aligned} \quad (26)$$

where m_i $(i = 1, 2, 3, 4, 5)$ is a scaling constant, $h(t)$ is a nonzero continuous differentiable function, and the time delay vector is $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)^{\mathrm{T}}$.

The time derivative of the error system (26) is as follows:

$$\begin{aligned} \dot{e}_1(t) &= \dot{y}_1(t) - m_1 \dot{h}(t) x_1(t-\tau_1) - m_1 h(t) \dot{x}_1(t-\tau_1), \\ \dot{e}_2(t) &= \dot{y}_2(t) - m_2 \dot{h}(t) x_2(t-\tau_2) - m_2 h(t) \dot{x}_2(t-\tau_2), \\ \dot{e}_3(t) &= \dot{y}_3(t) - m_3 \dot{h}(t) x_3(t-\tau_3) - m_3 h(t) \dot{x}_3(t-\tau_3), \\ \dot{e}_4(t) &= \dot{y}_4(t) - m_4 \dot{h}(t) x_4(t-\tau_4) - m_4 h(t) \dot{x}_4(t-\tau_4), \\ \dot{e}_5(t) &= \dot{y}_5(t) - m_5 \dot{h}(t) x_5(t-\tau_5) - m_5 h(t) \dot{x}_5(t-\tau_5). \end{aligned} \quad (27)$$

Substituting (24) and (25) into (27), we can get the following:

$$\begin{aligned} \dot{e}_1(t) &= -\sigma y_1(t) + \sigma y_2(t) + y_4(t) + u_1(t) \\ &\quad - m_1 \dot{h}(t) x_1(t-\tau_1) - m_1 h(t) \\ &\quad \times (-\sigma x_1(t-\tau_1) + \sigma x_2(t-\tau_1) + x_4(t-\tau_1)), \\ \dot{e}_2(t) &= r y_1(t) - y_2(t) - y_1(t) y_3(t) - y_5(t) + u_2(t) \\ &\quad - m_2 \dot{h}(t) x_2(t-\tau_2) - m_2 h(t) \\ &\quad \times (r x_1(t-\tau_2) - x_2(t-\tau_2) \\ &\qquad - x_1(t-\tau_2) x_3(t-\tau_2) - x_5(t-\tau_2)), \\ \dot{e}_3(t) &= -\beta y_3(t) + y_1(t) y_2(t) + u_3(t) \\ &\quad - m_3 \dot{h}(t) x_3(t-\tau_3) - m_3 h(t) \\ &\quad \times (-\beta x_3(t-\tau_3) + x_1(t-\tau_3) x_2(t-\tau_3)), \\ \dot{e}_4(t) &= -y_1(t) y_3(t) + d_1 y_4(t) + u_4(t) \\ &\quad - m_4 \dot{h}(t) x_4(t-\tau_4) - m_4 h(t) \\ &\quad \times (-x_1(t-\tau_4) x_3(t-\tau_4) + d_1 x_4(t-\tau_4)), \\ \dot{e}_5(t) &= d_2 y_2(t) + u_5(t) - m_5 \dot{h}(t) x_5(t-\tau_5) \\ &\quad - m_5 h(t) d_2 x_2(t-\tau_5). \end{aligned} \quad (28)$$

Based on Theorem 3, the controller, parameter update rule, and control gain strength adapt law are described as follows:

$$\begin{aligned} u_1(t) &= \hat{\sigma}(y_1(t) - y_2(t)) - y_4(t) + m_1 \dot{h}(t) x_1(t-\tau_1) \\ &\quad - m_1 h(t)(\hat{\sigma} x_1(t-\tau_1) - \hat{\sigma} x_2(t-\tau_1) - x_4(t-\tau_1)) \\ &\quad - k_1 e_1, \\ u_2(t) &= -\hat{r} y_1(t) + y_2(t) + y_1(t) y_3(t) + y_5(t) \\ &\quad + m_2 \dot{h}(t) x_2(t-\tau_2) - m_2 h(t) \\ &\quad \times (-\hat{r} x_1(t-\tau_2) + x_2(t-\tau_2) + x_1(t-\tau_2) \\ &\qquad \times x_3(t-\tau_2) + x_5(t-\tau_2)) - k_2 e_2, \\ u_3(t) &= \hat{\beta} y_3(t) - y_1(t) y_2(t) + m_3 \dot{h}(t) x_3(t-\tau_3) \\ &\quad - m_3 h(t)(\hat{\beta} x_3(t-\tau_3) - x_1(t-\tau_3) x_2(t-\tau_3)) \\ &\quad - k_3 e_3, \\ u_4(t) &= y_1(t) y_3(t) - \hat{d}_1 y_4(t) + m_4 \dot{h}(t) x_4(t-\tau_4) \\ &\quad - m_4 h(t)(x_1(t-\tau_4) x_3(t-\tau_4) - \hat{d}_1 x_4(t-\tau_4)) \\ &\quad - k_4 e_4, \\ u_5(t) &= -\hat{d}_2 y_2(t) + m_5 \dot{h}(t) x_5(t-\tau_5) \\ &\quad + m_5 h(t) \hat{d}_2 x_2(t-\tau_5) - k_5 e_5, \end{aligned} \quad (29)$$

$$\begin{aligned} \dot{e}_\sigma &= \dot{\hat{\sigma}} = -e_1(y_1(t) - y_2(t)) \\ &\quad + m_1 h(t) e_1 (x_1(t-\tau_1) - x_2(t-\tau_1)), \\ \dot{e}_\beta &= \dot{\hat{\beta}} = -e_3 y_3(t) + m_3 h(t) e_3 x_3(t-\tau_3), \\ \dot{e}_r &= \dot{\hat{r}} = e_2 y_1(t) - m_2 h(t) e_2 x_1(t-\tau_2), \\ \dot{e}_{d_1} &= \dot{\hat{d}}_1 = e_4 y_4(t) - m_4 h(t) e_4 x_4(t-\tau_4), \\ \dot{e}_{d_2} &= \dot{\hat{d}}_2 = e_5 y_2(t) - m_5 h(t) e_5 x_2(t-\tau_5), \end{aligned} \quad (30)$$

$$\dot{k}_i = \varepsilon_i e_i^2, \quad \varepsilon_i > 0, \ i = 1, 2, \ldots, 5, \quad (31)$$

where $\hat{\sigma}$, $\hat{\beta}$, $\hat{r}$, $\hat{d}_1$, and $\hat{d}_2$ are the estimation of uncertain parameters σ, β, r, d_1, and d_2, respectively, $e_\sigma = \hat{\sigma} - \sigma$, $e_\beta = \hat{\beta} - \beta$, $e_r = \hat{r} - r$, $e_{d_1} = \hat{d}_1 - d_1$, and $e_{d_2} = \hat{d}_2 - d_2$ are the corresponding parameter errors.

Proof. The Lyapunov function is chosen as

$$\begin{aligned} V(t) &= \frac{1}{2}\left(e_1^{\mathrm{T}}(t) e_1(t) + e_2^{\mathrm{T}}(t) e_2(t) + e_3^{\mathrm{T}}(t) e_3(t) \right. \\ &\quad + e_4^{\mathrm{T}}(t) e_4(t) + e_5^{\mathrm{T}}(t) e_5(t) + e_\sigma^{\mathrm{T}}(t) e_\sigma(t) \end{aligned}$$

$$+ e_{\beta}^{\mathrm{T}}(t)\, e_{\beta}(t) + e_{r}^{\mathrm{T}}(t)\, e_{r}(t) + e_{d_1}^{\mathrm{T}}(t)\, e_{d_1}(t) + e_{d_2}^{\mathrm{T}}(t)\, e_{d_2}(t)\Big) + \frac{1}{2}\sum_{i=1}^{5}\frac{1}{\varepsilon_i}(k_i - k)^2, \qquad (32)$$

where $k > 0$ is a positive constant.

Then the time derivation of the Lyapunov function along the trajectory of error systems (27) is as follows:

$$\begin{aligned}\dot{V}(t) = {} & e_1^{\mathrm{T}}(t)\,\dot{e}_1(t) + e_2^{\mathrm{T}}(t)\,\dot{e}_2(t) + e_3^{\mathrm{T}}(t)\,\dot{e}_3(t) \\ & + e_4^{\mathrm{T}}(t)\,\dot{e}_4(t) + e_5^{\mathrm{T}}(t)\,\dot{e}_5(t) + e_{\sigma}^{\mathrm{T}}(t)\,\dot{e}_{\sigma}(t) \\ & + e_{\beta}^{\mathrm{T}}(t)\,\dot{e}_{\beta}(t) + e_{r}^{\mathrm{T}}(t)\,\dot{e}_{r}(t) + e_{d_1}^{\mathrm{T}}(t)\,\dot{e}_{d_1}(t) \\ & + e_{d_2}^{\mathrm{T}}(t)\,\dot{e}_{d_2}(t) + \dot{k}_i\sum_{i=1}^{5}\frac{1}{\varepsilon_i}(k_i - k)^2. \end{aligned} \qquad (33)$$

Constituting (28), (29), (30), and (31) into (33), we get the following:

$$\dot{V}(t) = -k\left(e_1^{\mathrm{T}}(t)\,e_1 + e_2^{\mathrm{T}}(t)\,e_2 + e_3^{\mathrm{T}}(t)\,e_3 + e_4^{\mathrm{T}}(t)\,e_4 + e_5^{\mathrm{T}}(t)\,e_5\right) < 0. \qquad (34)$$

$V(t)$ is positive definite and $\dot{V}(t) < 0$; thus, according to Barbalat's Lemma, MFPLS between the drive system (24) and the response system (25) is achieved. The uncertain parameters and control gain strengths can be identified and defined as well. □

To verify the effectiveness of the proposed synchronization method, the numerical simulation is performed. The system parameters are chosen as $\sigma = 10$, $\beta = 8/3$, $r = 28$, $d_1 = 2$, and $d_2 = 8$, such that the drive system (24) and the response system (25) can exhibit hyperchaotic behaviors without control. The initial conditions of the drive system (24) and the response system (25) are taken as $\mathbf{x}(0) = (-2, 1, 4, 2, -3)^{\mathrm{T}}$, $\mathbf{y}(0) = (-2, -2, 3, 4, 1)^{\mathrm{T}}$, respectively. The initial values of the control gain strengths are selected as $k_1(0) = k_2(0) = k_3(0) = k_4(0) = k_5(0) = 6$, and the constants are $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 10$. The initial values of the unknown parameters are set as $\widehat{\sigma}(0) = \widehat{\beta}(0) = \widehat{r}(0) = \widehat{d}_1(0) = \widehat{d}_2(0) = 0.001$. The time delays are arbitrarily chosen as $(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5) = (0.2, 0.1, 0.3, 0.1, 0.4)$. The scaling constants are randomly taken as $(m_1, m_2, m_3, m_4, m_5) = (2, -3, -2, 1, -4)$, and nonzero continuous differentiable function is selected as $h(t) = 2\sin(t) + 1$.

The corresponding simulation results are illustrated in Figures 1, 2, and 3. Figure 1 displays that the synchronization error variables e_1, e_2, e_3, e_4, and e_5 converge to zero after a transient time, respectively. Figure 2 shows that the estimated values of the uncertain parameters approach to the true values; that is, $\widehat{\sigma} \to 10$, $\widehat{\beta} \to 8/3$, $\widehat{r} \to 28$, $\widehat{d}_1 \to 2$, and $\widehat{d}_2 \to 8$ as $t \to \infty$. Figure 3 depicts the control gain strengths adapt themselves to a certain value, that is, $k_1 = 32.34$, $k_2 = 88.97$, $k_3 = 10.82$, $k_4 = 10.77$, and $k_5 = 27.67$ as $t \to \infty$. In Figure 3, the upper right small figure is the magnified drawing of the circling part. These results show that the MFPLS has been achieved with the adaptive controller (29) and the parameter update rule (30), and control gain strengths can be identified automatically.

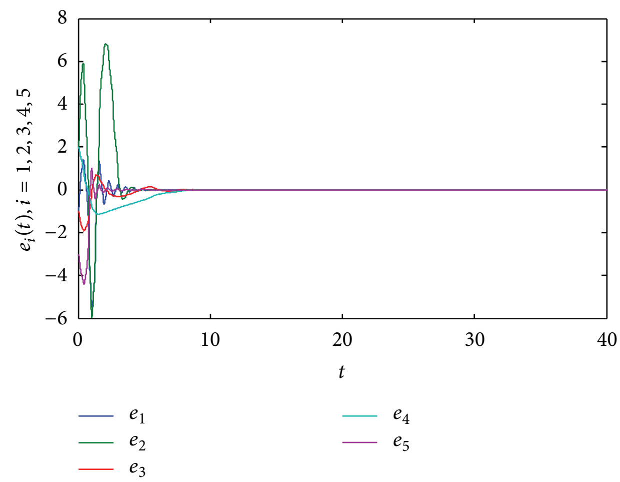

Figure 1: Time evolution of MFPLS errors between systems (24) and (25).

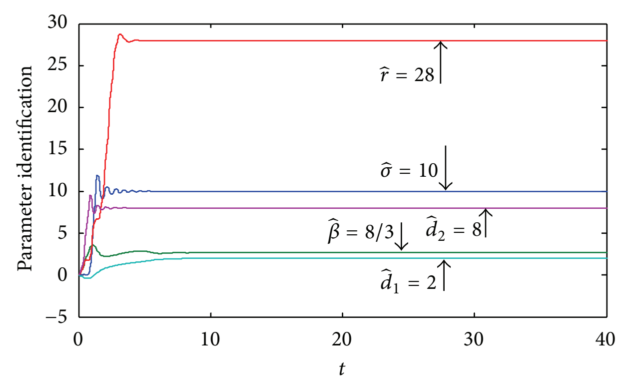

Figure 2: Time evolution of parameter estimation for the systems (24) and (25).

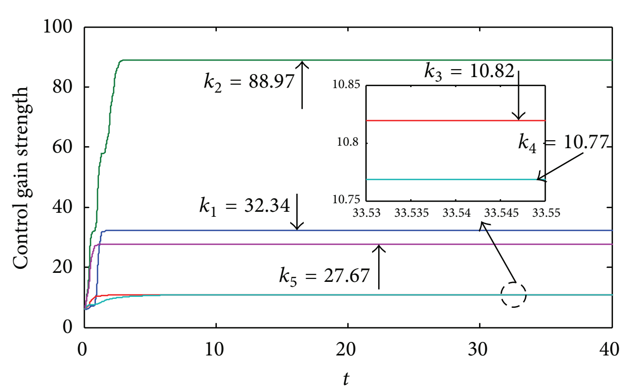

Figure 3: Time evolution of the control gain strength.

4.2. Case 2: MFPLS between Two Four-Dimensional Hyperchaotic Systems with Different Structures. In this subsection, we consider the case that $m = n$ and the drive and response system have the same dimensions and different structures. Furthermore, we select the four-dimensional Lü hyperchaotic system [32] and the four-dimensional Lorenz-Stenflo (LS) hyperchaotic system [33] as the drive system and the response system, respectively.

The four-dimensional Lü hyperchaotic system [32] is given by the following equations:

$$\begin{aligned}\dot{x}_1 &= a_1 (x_2 - x_1) + x_4,\\ \dot{x}_2 &= -x_1 x_3 + c_1 x_2,\\ \dot{x}_3 &= x_1 x_2 - b_1 x_3,\\ \dot{x}_4 &= x_1 x_3 + d_1 x_4,\end{aligned} \tag{35}$$

where x_1, x_2, x_3, and x_4 are state variables and a_1, b_1, c_1, and d_1 are uncertain system parameters. When $a_1 = 36$, $b_1 = 3$, $c_1 = 20$, and $d_1 = 1$, system (35) is hyperchaotic.

Lorenz-Stenflo (LS) hyperchaotic system [33] is described as

$$\begin{aligned}\dot{y}_1 &= \alpha (y_2 - y_1) + \beta y_4 + u_1,\\ \dot{y}_2 &= \gamma y_1 - y_1 y_3 - y_2 + u_2,\\ \dot{y}_3 &= y_1 y_2 - \theta y_3 + u_3,\\ \dot{y}_4 &= -y_1 - \alpha y_4 + u_4,\end{aligned} \tag{36}$$

where y_1, y_2, y_3, and y_4 are state variables, α, β, γ, and θ are the unknown system parameters. When $\alpha = 1$, $\beta = 1.5$, $\gamma = 26$, and $\theta = 0.7$, system (36) exhibits hyperchaotic behavior. And u_1, u_2, u_3, and u_4 are the controllers such that the two hyperchaotic systems can be synchronized in the sense of MFPLS.

We define the error as

$$\begin{aligned}e_1(t) &= y_1(t) - m_1 h(t) x_1(t-\tau_1),\\ e_2(t) &= y_2(t) - m_2 h(t) x_2(t-\tau_2),\\ e_3(t) &= y_3(t) - m_3 h(t) x_3(t-\tau_3),\\ e_4(t) &= y_4(t) - m_4 h(t) x_4(t-\tau_4),\end{aligned} \tag{37}$$

where m_i $(i = 1,2,3,4)$ is a scaling constant, $h(t)$ is a nonzero continuous differentiable function, and $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3, \tau_4)^{\mathrm{T}}$ are the time delays.

The time derivative of the error system (37) is

$$\begin{aligned}\dot{e}_1(t) &= \dot{y}_1(t) - m_1 \dot{h}(t) x_1(t-\tau_1) - m_1 h(t) \dot{x}_1(t-\tau_1),\\ \dot{e}_2(t) &= \dot{y}_2(t) - m_2 \dot{h}(t) x_2(t-\tau_2) - m_2 h(t) \dot{x}_2(t-\tau_2),\\ \dot{e}_3(t) &= \dot{y}_3(t) - m_3 \dot{h}(t) x_3(t-\tau_3) - m_3 h(t) \dot{x}_3(t-\tau_3),\\ \dot{e}_4(t) &= \dot{y}_4(t) - m_4 \dot{h}(t) x_4(t-\tau_4) - m_4 h(t) \dot{x}_4(t-\tau_4).\end{aligned} \tag{38}$$

Substituting (35) and (36) into (38), the error dynamical system is obtained as follows:

$$\begin{aligned}\dot{e}_1(t) &= \alpha (y_2(t) - y_1(t)) + \beta y_4(t) + u_1(t)\\ &\quad - m_1 \dot{h}(t) x_1(t-\tau_1) - m_1 h(t)\\ &\quad \times (a_1 (x_2(t-\tau_1) - x_1(t-\tau_1)) + x_4(t-\tau_1))\\ &\quad - k_1 e_1,\\ \dot{e}_2(t) &= \gamma y_1(t) - y_1(t) y_3(t) - y_2(t) + u_2(t)\\ &\quad - m_2 \dot{h}(t) x_2(t-\tau_2) - m_2 h(t)\\ &\quad \times (-x_1(t-\tau_2) x_3(t-\tau_2) + c_1 x_2(t-\tau_2)) - k_2 e_2,\\ \dot{e}_3(t) &= y_1(t) y_2(t) - \theta y_3(t) + u_3(t)\\ &\quad - m_3 \dot{h}(t) x_3(t-\tau_3) - m_3 h(t)\\ &\quad \times (x_1(t-\tau_3) x_2(t-\tau_3) - b_1 x_3(t-\tau_3)) - k_3 e_3,\\ \dot{e}_4(t) &= -y_1(t) - \alpha y_4(t) + u_4(t)\\ &\quad - m_4 \dot{h}(t) x_4(t-\tau_4) - m_4 h(t)\\ &\quad \times (x_1(t-\tau_4) x_3(t-\tau_4) + d_1 x_4(t-\tau_4)) - k_4 e_4.\end{aligned} \tag{39}$$

Based on Theorem 3, the controller, parameter update rule, and control gain strength adapt law are constructed as follows:

$$\begin{aligned}u_1(t) &= -\widehat{\alpha} (y_2(t) - y_1(t)) - \widehat{\beta} y_4(t)\\ &\quad + m_1 \dot{h}(t) x_1(t-\tau_1) - m_1 h(t)\\ &\quad \times (-\widehat{a}_1 (x_2(t-\tau_1) - x_1(t-\tau_1)) - x_4(t-\tau_1))\\ &\quad - k_1 e_1,\\ u_2(t) &= -\widehat{\gamma} y_1(t) + y_1(t) y_3(t) + y_2(t)\\ &\quad + m_2 \dot{h}(t) x_2(t-\tau_2) - m_2 h(t)\\ &\quad \times (x_1(t-\tau_2) x_3(t-\tau_2) - \widehat{c}_1 x_2(t-\tau_2))\\ &\quad - k_2 e_2,\\ u_3(t) &= -y_1(t) y_2(t) + \widehat{\theta} y_3(t) + m_3 \dot{h}(t) x_3(t-\tau_3)\\ &\quad - m_3 h(t) \left(-x_1(t-\tau_3) x_2(t-\tau_3) + \widehat{b}_1 x_3(t-\tau_3)\right)\\ &\quad - k_3 e_3,\\ u_4(t) &= y_1(t) + \widehat{\alpha} y_4(t) + m_4 \dot{h}(t) x_4(t-\tau_4)\\ &\quad - m_4 h(t) \left(-x_1(t-\tau_4) x_3(t-\tau_4) - \widehat{d}_1 x_4(t-\tau_4)\right)\\ &\quad - k_4 e_4,\end{aligned} \tag{40}$$

$$\dot{e}_{a_1} = \dot{\hat{a}}_1 = -m_1 h(t) e_1 (x_2(t-\tau_1) - x_1(t-\tau_1)),$$
$$\dot{e}_{b_1} = \dot{\hat{b}}_1 = m_3 h(t) e_3 x_3(t-\tau_3),$$
$$\dot{e}_{c_1} = \dot{\hat{c}}_1 = -m_2 h(t) e_2 x_2(t-\tau_2),$$
$$\dot{e}_{d_1} = \dot{\hat{d}}_1 = -m_4 h(t) e_4 x_4(t-\tau_4), \quad (41)$$
$$\dot{e}_{\alpha} = \dot{\hat{\alpha}} = -e_4 y_4(t) + e_1(y_2(t) - y_1(t)),$$
$$\dot{e}_{\beta} = \dot{\hat{\beta}} = e_1 y_4(t),$$
$$\dot{e}_{\gamma} = \dot{\hat{\gamma}} = e_2 y_1(t),$$
$$\dot{e}_{\theta} = \dot{\hat{\theta}} = -e_3 y_3(t),$$

$$\dot{k}_i = \varepsilon_i e_i^2, \quad \varepsilon_i > 0, \ i = 1, 2, \ldots, 4, \quad (42)$$

where $\hat{a}_1, \hat{b}_1, \hat{c}_1, \hat{d}_1, \hat{\alpha}, \hat{\beta}, \hat{\gamma}$, and $\hat{\theta}$ are estimated values of the unknown parameters $a_1, b_1, c_1, d_1, \alpha, \beta, \gamma$, and θ, respectively, and $e_{a_1} = \hat{a}_1 - a_1$, $e_{b_1} = \hat{b}_1 - b_1$, $e_{c_1} = \hat{c}_1 - c_1$, $e_{d_1} = \hat{d}_1 - d_1$, $e_{\alpha} = \hat{\alpha} - \alpha$, $e_{\beta} = \hat{\beta} - \beta$, $e_{\gamma} = \hat{\gamma} - \gamma$, and $e_{\theta} = \hat{\theta} - \theta$ are the parameter errors.

Proof. The Lyapunov function is chosen as

$$\begin{aligned} V(t) = \frac{1}{2} \big(& e_1^{\mathrm{T}}(t) e_1(t) + e_2^{\mathrm{T}}(t) e_2(t) + e_3^{\mathrm{T}}(t) e_3(t) \\ & + e_4^{\mathrm{T}}(t) e_4(t) + e_{a_1}^{\mathrm{T}}(t) e_{a_1}(t) + e_{b_1}^{\mathrm{T}}(t) e_{b_1}(t) \\ & + e_{c_1}^{\mathrm{T}}(t) e_{c_1}(t) + e_{d_1}^{\mathrm{T}}(t) e_{d_1}(t) + e_{\alpha}^{\mathrm{T}}(t) e_{\alpha}(t) \\ & + e_{\beta}^{\mathrm{T}}(t) e_{\beta}(t) + e_{\gamma}^{\mathrm{T}}(t) e_{\gamma}(t) + e_{\theta}^{\mathrm{T}}(t) e_{\theta}(t) \big) \\ & + \frac{1}{2} \sum_{i=1}^{4} \frac{1}{\varepsilon_i} (k_i - k)^2, \end{aligned} \quad (43)$$

where $k > 0$ is a positive constant.

Then the time derivation of the Lyapunov function along the trajectory of error systems (39) is

$$\begin{aligned} \dot{V}(t) = & e_1^{\mathrm{T}}(t) \dot{e}_1(t) + e_2^{\mathrm{T}}(t) \dot{e}_2(t) + e_3^{\mathrm{T}}(t) \dot{e}_3(t) \\ & + e_4^{\mathrm{T}}(t) \dot{e}_4(t) + e_{a_1}^{\mathrm{T}}(t) \dot{e}_{a_1}(t) + e_{b_1}^{\mathrm{T}}(t) \dot{e}_{b_1}(t) \\ & + e_{c_1}^{\mathrm{T}}(t) \dot{e}_{c_1}(t) + e_{d_1}^{\mathrm{T}}(t) \dot{e}_{d_1}(t) + e_{\alpha}^{\mathrm{T}}(t) \dot{e}_{\alpha}(t) \\ & + e_{\beta}^{\mathrm{T}}(t) \dot{e}_{\beta}(t) + e_{\gamma}^{\mathrm{T}}(t) \dot{e}_{\gamma}(t) + e_{\theta}^{\mathrm{T}}(t) \dot{e}_{\theta}(t) \\ & + \dot{k}_i \sum_{i=1}^{4} \frac{1}{\varepsilon_i} (k_i - k)^2. \end{aligned} \quad (44)$$

Constituting (39), (40), (41), and (42) into (44), we get the following:

$$\dot{V}(t) = -k \left(e_1^{\mathrm{T}}(t) e_1 + e_2^{\mathrm{T}}(t) e_2 + e_3^{\mathrm{T}}(t) e_3 + e_4^{\mathrm{T}}(t) e_4 \right) < 0. \quad (45)$$

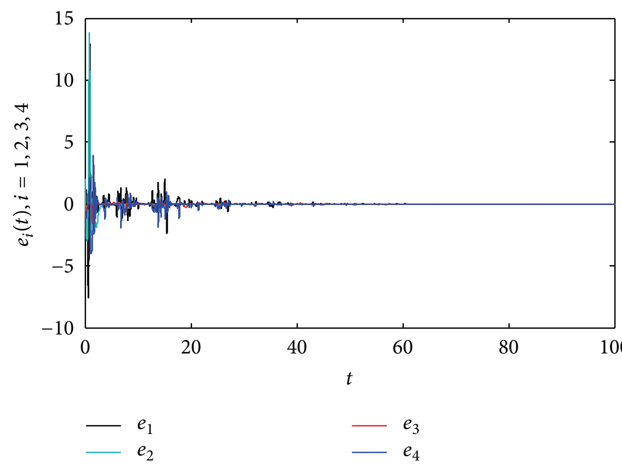

FIGURE 4: Time evolution of MFPLS errors between systems (35) and (36).

$V(t)$ is positive definite and $\dot{V}(t) < 0$; thus, according to Barbalat's Lemma, MFPLS between the drive system (35) and the response system (36) is achieved. The uncertain parameters and control gain strengths can be identified and defined as well. □

In the numerical simulation, the system parameters are selected as $a_1 = 36$, $b_1 = 3$, $c_1 = 20$, $d_1 = 1$, $\alpha = 1$, $\beta = 1.5$, $\gamma = 26$, and $\theta = 0.7$, such that the drive system (35) and the response system (36) are hyperchaotic with no control applied. The initial conditions of the drive system (35) and the response system (36) are taken as $\mathbf{x}(0) = (-2, -1, -4, 2)^{\mathrm{T}}$, $\mathbf{y}(0) = (-2, -1, 3, 5)^{\mathrm{T}}$, respectively. The initial values of the unknown parameters are arbitrarily set as $\hat{a}_1(0) = \hat{b}_1(0) = \hat{c}_1(0) = \hat{d}_1(0) = \hat{\alpha}(0) = \hat{\beta}(0) = \hat{\gamma}(0) = \hat{\theta}(0) = 0.001$. The constants are $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 5$ and the initial values of the control gain strength are chosen as $k_1(0) = k_2(0) = k_3(0) = k_4(0) = 10$. The scaling constants are set as $(m_1, m_2, m_3, m_4) = (-3, 2, -1, 2)$, and nonzero continuous differentiable function is selected as $h(t) = \sin(t)$. The time delays are chosen as $(\tau_1, \tau_2, \tau_3, \tau_4) = (0.1, 0.2, 0.3, 0.4)$.

Numerical results are displayed in Figures 4–7. Figure 4 shows the time evolution of the MFPLS errors, which displays that the errors tend to zero as $t \to \infty$. Figures 5 and 6 depict the evolution of the estimated parameters of Lü hyperchaotic system and Lorenz-Stenflo (LS) hyperchaotic system, which displays that the estimates of the uncertain parameters converge to $a_1 = 36$, $b_1 = 3$, $c_1 = 20$, $d_1 = 1$, $\alpha = 1$, $\beta = 1.5$, $\gamma = 26$, and $\theta = 0.7$ as $t \to \infty$, respectively. Figure 7 illustrates that the control gain strengths approach to some certain values, that is, $k_1 = 73.95$, $k_2 = 74.16$, $k_3 = 12.34$, and $k_4 = 35.46$ as $t \to \infty$. In Figure 7, the middle small figure is the magnified drawing of the circling part. These results show that MFPLS between four-dimensional Lü hyperchaotic system and four-dimensional Lorenz-Stenflo (LS) hyperchaotic system is realized by using the adaptive controllers (40) and the parameter update rule (41), and the control gain strength can be gotten adaptively.

4.3. Case 3: MFPLS between a Four-Dimensional Hyperchaotic System and a Three-Dimensional Chaotic System. In this

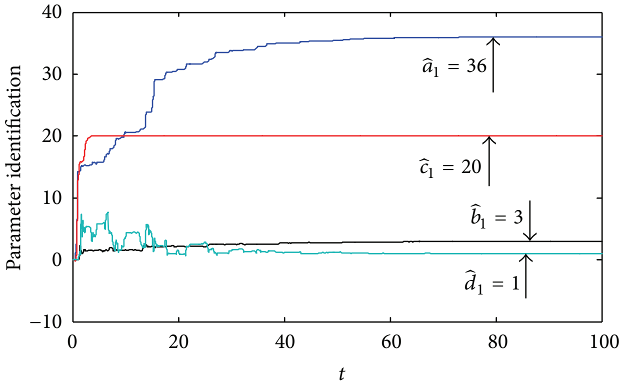

Figure 5: Time evolution of parameter estimation for Lü hyperchaotic system (35).

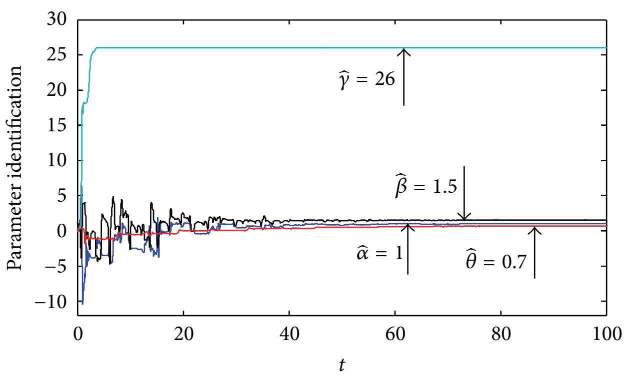

Figure 6: Time evolution of parameter estimation for Lorenz-Stenflo (LS) hyperchaotic system (36).

subsection, we consider the case that $m > n$ and the dimension of drive system is larger than that of the response system.

The four-dimensional Lorenz-Stenflo (LS) hyperchaotic system [33], as the drive system, is given by

$$\begin{aligned} \dot{x}_1 &= \alpha(x_2 - x_1) + \beta x_4, \\ \dot{x}_2 &= \gamma x_1 - x_1 x_3 - x_2, \\ \dot{x}_3 &= x_1 x_2 - \theta x_3, \\ \dot{x}_4 &= -x_1 - \alpha x_4, \end{aligned} \tag{46}$$

where x_1, x_2, x_3, and x_4 are state variables and α, β, γ, and θ are the uncertain system parameters to be estimated. When $\alpha = 1$, $\beta = 1.5$, $\gamma = 26$, and $\theta = 0.7$, system (46) exhibits hyperchaotic behavior.

In order to achieve the full-state MFPLS, an auxiliary state should be added to the response system. Since adding subcontroller to the response system to compensate it as extra-dimensions is a practicable way [21], the auxiliary state is $\dot{y}_4 = u_4$.

Then, the three-dimensional Lorenz system, as the response system, is described by

$$\begin{aligned} \dot{y}_1 &= l(y_2 - y_1) + u_1, \\ \dot{y}_2 &= m y_1 - y_2 - y_1 y_3 + u_2, \\ \dot{y}_3 &= -n y_3 + y_1 y_2 + u_3, \\ \dot{y}_4 &= u_4, \end{aligned} \tag{47}$$

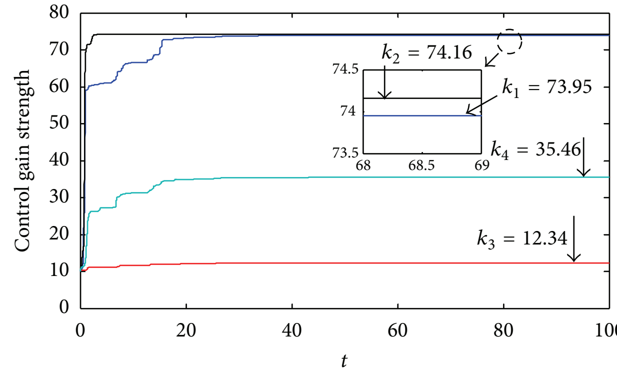

Figure 7: Time evolution of the control gain strength.

where y_1, y_2, y_3, and y_4 are state variables l, m, and n are the unknown system parameters. When $l = 10$, $m = 28$, $n = 8/3$, system (47) is chaotic, u_1, u_2, u_3, and u_4 are the controllers to be designed.

The MFPLS error is defined as

$$\begin{aligned} e_1(t) &= y_1(t) - m_1 h(t) x_1(t - \tau_1), \\ e_2(t) &= y_2(t) - m_2 h(t) x_2(t - \tau_2), \\ e_3(t) &= y_3(t) - m_3 h(t) x_3(t - \tau_3), \\ e_4(t) &= y_4(t) - m_4 h(t) x_4(t - \tau_4), \end{aligned} \tag{48}$$

where m_i $(i = 1, 2, 3, 4)$ is a scaling constant, $h(t)$ is a nonzero continuous differentiable function, and $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3, \tau_4)^{\mathrm{T}}$ are the time delays.

Based on Theorem 3, the controller, parameter update rules, and control gain strength adapt laws are as follows:

$$\begin{aligned} u_1(t) &= -\hat{l}(y_2(t) - y_1(t)) + m_1 \dot{h}(t) x_1(t - \tau_1) - m_1 h(t) \\ &\quad \times \left(-\hat{\alpha}(x_2(t - \tau_1) - x_1(t - \tau_1)) - \hat{\beta} x_4(t - \tau_1)\right) \\ &\quad - k_1 e_1, \\ u_2(t) &= -\hat{m} y_1 + y_2 + y_1 y_3 + m_2 \dot{h}(t) x_2(t - \tau_2) - m_2 h(t) \\ &\quad \times (-\hat{\gamma} x_1(t - \tau_2) + x_1(t - \tau_2) x_3(t - \tau_2) + x_2(t - \tau_2)) \\ &\quad - k_2 e_2, \\ u_3(t) &= +\hat{n} y_3 - y_1 y_2 + m_3 \dot{h}(t) x_3(t - \tau_3) - m_3 h(t) \\ &\quad \times \left(-x_1(t - \tau_3) x_2(t - \tau_3) + \hat{\theta} x_3(t - \tau_3)\right) - k_3 e_3, \\ u_4(t) &= +m_4 \dot{h}(t) x_4(t - \tau_4) \\ &\quad - m_4 h(t)(x_1(t - \tau_4) + \hat{\alpha} x_4(t - \tau_4)) - k_4 e_4, \end{aligned} \tag{49}$$

$$\begin{aligned} \dot{e}_\alpha &= \dot{\hat{\alpha}} = -m_1 h(t) e_1 (x_2(t - \tau_1) - x_1(t - \tau_1)) \\ &\quad + m_4 h(t) e_4 x_4(t - \tau_4), \\ \dot{e}_\beta &= \dot{\hat{\beta}} = -m_1 h(t) e_1 x_4(t - \tau_1), \end{aligned}$$

$$\begin{aligned}\dot{e}_\gamma &= \dot{\hat{\gamma}} = -m_2 h(t) e_2 x_1 (t-\tau_2),\\ \dot{e}_\theta &= \dot{\hat{\theta}} = m_3 h(t) e_3 x_3 (t-\tau_3),\\ \dot{e}_l &= \dot{\hat{l}} = (y_2(t) - y_1(t)) e_1,\\ \dot{e}_m &= \dot{\hat{m}} = e_2 y_1(t),\\ \dot{e}_n &= \dot{\hat{n}} = -e_3 y_3(t),\end{aligned} \tag{50}$$

$$\dot{k}_i = \varepsilon_i e_i^2, \quad \varepsilon_i > 0, \; i = 1,2,\ldots,4, \tag{51}$$

where $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, $\hat{\theta}$, $\hat{l}$, $\hat{m}$, and $\hat{n}$ are estimated values of the unknown parameters α, β, γ, θ, l, m, and n, respectively and $e_\alpha = \hat{\alpha} - \alpha$, $e_\beta = \hat{\beta} - \beta$, $e_\gamma = \hat{\gamma} - \gamma$, $e_\theta = \hat{\theta} - \theta$, $e_l = \hat{l} - l$, $e_m = \hat{m} - m$, and $e_n = \hat{n} - n$ are the parameter errors.

Proof. The Lyapunov function is chosen as

$$\begin{aligned}V(t) = \frac{1}{2}\Big(&e_1^{\mathrm{T}}(t) e_1(t) + e_2^{\mathrm{T}}(t) e_2(t) + e_3^{\mathrm{T}}(t) e_3(t)\\ &+ e_4^{\mathrm{T}}(t) e_4(t) + e_\alpha^{\mathrm{T}}(t) e_\alpha(t) + e_\beta^{\mathrm{T}}(t) e_\beta(t)\\ &+ e_\gamma^{\mathrm{T}}(t) e_\gamma(t) + e_\theta^{\mathrm{T}}(t) e_\theta(t) + e_l^{\mathrm{T}}(t) e_l(t)\\ &+ e_m^{\mathrm{T}}(t) e_m(t) + e_n^{\mathrm{T}}(t) e_n(t)\Big)\\ &+ \frac{1}{2}\sum_{i=1}^{4}\frac{1}{\varepsilon_i}(k_i - k)^2,\end{aligned} \tag{52}$$

where $k > 0$ is a positive constant.

Then the time derivation of the Lyapunov function along the trajectory of error systems is

$$\begin{aligned}\dot{V}(t) = {}& e_1^{\mathrm{T}}(t)\dot{e}_1(t) + e_2^{\mathrm{T}}(t)\dot{e}_2(t) + e_3^{\mathrm{T}}(t)\dot{e}_3(t)\\ &+ e_4^{\mathrm{T}}(t)\dot{e}_4(t) + e_\alpha^{\mathrm{T}}(t)\dot{e}_\alpha(t) + e_\beta^{\mathrm{T}}(t)\dot{e}_\beta(t)\\ &+ e_\gamma^{\mathrm{T}}(t)\dot{e}_\gamma(t) + e_\theta^{\mathrm{T}}(t)\dot{e}_\theta(t) + e_l^{\mathrm{T}}(t)\dot{e}_l(t)\\ &+ e_m^{\mathrm{T}}(t)\dot{e}_m(t) + e_n^{\mathrm{T}}(t)\dot{e}_n(t)\\ &+ \dot{k}_i\sum_{i=1}^{4}\frac{1}{\varepsilon_i}(k_i - k)^2.\end{aligned} \tag{53}$$

Constituting (48), (49), (50), and (51) into (53), we get the following:

$$\dot{V}(t) = -k\left(e_1^{\mathrm{T}}(t) e_1 + e_2^{\mathrm{T}}(t) e_2 + e_3^{\mathrm{T}}(t) e_3 + e_4^{\mathrm{T}}(t) e_4\right) < 0. \tag{54}$$

$V(t)$ is positive definite and $\dot{V}(t) < 0$; thus, according to Barbalat's Lemma, MFPLS between the drive system (46) and the response system (47) is achieved. The uncertain parameters and control gain strengths can be identified and defined as well. □

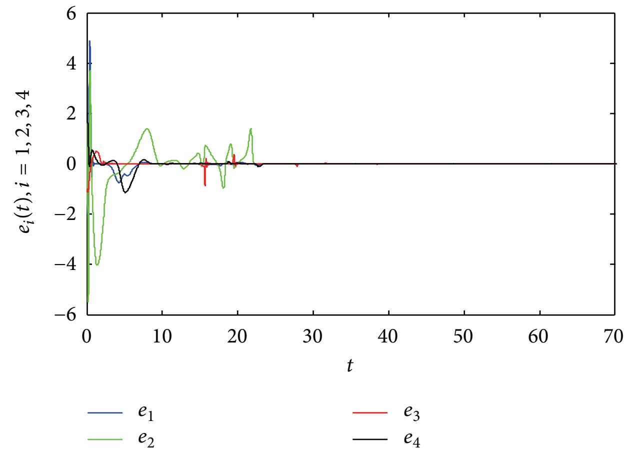

Figure 8: Time evolution of MFPLS errors between systems (46) and (47).

In the numerical simulation, the system parameters are selected as $\alpha = 1$, $\beta = 1.5$, $\gamma = 26$, $\theta = 0.7$, $l = 10$, $m = 28$, $n = 8/3$, such that the drive system (46) and the response system (47) are chaotic without control. We take that the initial conditions of the drive system (46) and the response system (47) are $\mathbf{x}(0) = (-2, 1, 4, 2)^{\mathrm{T}}$, $\mathbf{y}(0) = (2, -2, 3, 4)^{\mathrm{T}}$, respectively. The constants are $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 10$, and the initial values of the control gain strengths are chosen as $k_1(0) = k_2(0) = k_3(0) = k_4(0) = 8$. The initial values of the unknown parameters are set as $\hat{\alpha}(0) = \hat{\beta}(0) = \hat{\gamma}(0) = \hat{\theta}(0) = \hat{l}_1(0) = \hat{m}_1(0) = \hat{n}_1(0) = 0.001$. The time delays are randomly selected as $(\tau_1, \tau_2, \tau_3, \tau_4) = (0.2, 0.1, 0.3, 0.1)$. The scaling constants are taken as $(m_1, m_2, m_3, m_4) = (2, -3, -2, 3)$ and nonzero continuous differentiable function is chosen as $h(t) = 2\sin(0.5t) + 1$.

Corresponding simulation results are displayed in Figures 8–11. Figure 8 shows that the error variables e_1, e_2, e_3, e_4 tend to zero, respectively. Figures 9 and 10 illustrate that the estimated values of the unknown parameters approach to $\alpha = 1$, $\beta = 1.5$, $\gamma = 26$, $\theta = 0.7$, $l = 10$, $m = 28$, $n = 8/3$ as $t \to \infty$, respectively. Figure 11 depicts that the control gain strength k_1, k_2, k_3, and k_4 tend to some certain value, $k_1 = 32.42$, $k_2 = 121.3$, $k_3 = 12.24$, $k_4 = 12.62$ as $t \to \infty$. As shown in Figures 8–11, MFPLS between Lorenz-Stenflo (LS) hyperchaotic system (46) and the three-dimensional Lorenz system (47) is obtained and all the uncertain parameters are identified successfully by using the controller (49) and the parameter update rule (50). At the meantime, the control gain strengths can be estimated.

4.4. Case 4: MFPLS between a Novel Three-Dimensional Chaotic System and a Five-Dimensional Hyperchaotic System. In this subsection, MFPLS between a novel three-dimensional chaotic system and a five-dimensional hyperchaotic system is analyzed.

Recently, Wu and Li [34] introduced a novel three-dimensional autonomous chaotic system by adding a quadratic cross-product term to the first equation and modifying the state variable in the third equation of a chaotic system

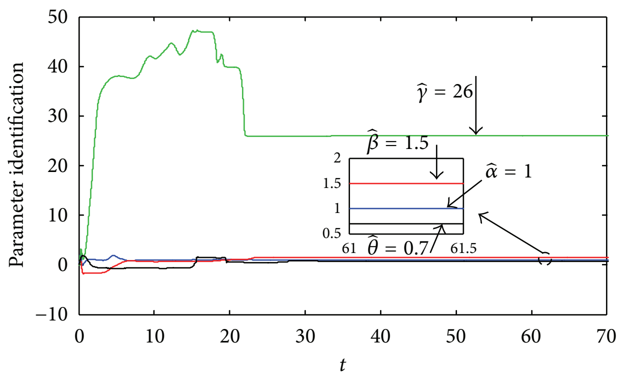

FIGURE 9: Time evolution of parameter estimation for the drive system (46).

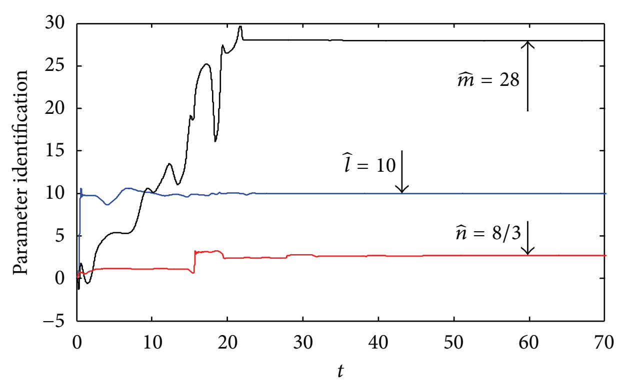

FIGURE 10: Time evolution of parameter estimation for the response system (47).

proposed by Cai et al. [35], investigated some basic dynamical properties, such as Lyapunov exponent spectrum, bifurcations, equilibria, and chaotic dynamical behaviors of the new chaotic system, and studied hybrid function projective synchronization (HFPS) of the new chaotic system. The system is described by

$$\begin{aligned} \dot{x}_1 &= a\left(x_2 - x_1\right) + x_2 x_3, \\ \dot{x}_2 &= b x_1 + c x_2 - x_1 x_3, \\ \dot{x}_3 &= x_2^2 - d x_3, \end{aligned} \tag{55}$$

where x_1, x_2, and x_3 are state variables and a, b, c, and d are the uncertain system parameters to be estimated. When $a = 20$, $b = 5$, $c = 10$, $d \in [0, +\infty)$, the system behaves as hyperchaos.

The response system is a five-dimensional hyperchaotic Lorenz system, which is given by

$$\begin{aligned} \dot{y}_1 &= -\sigma y_1 + \sigma y_2 + y_4 + u_1, \\ \dot{y}_2 &= r y_1 - y_2 - y_1 y_3 - y_5 + u_2, \\ \dot{y}_3 &= -\beta y_3 + y_1 y_2 + u_3, \\ \dot{y}_4 &= -y_1 y_3 + d_1 y_4 + u_4, \\ \dot{y}_5 &= d_2 y_2 + u_5, \end{aligned} \tag{56}$$

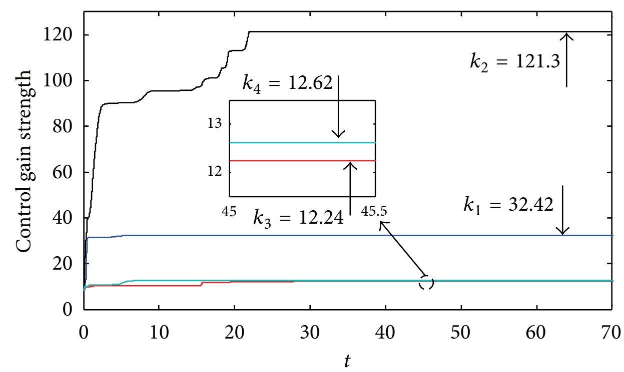

FIGURE 11: Time evolution of the control gain strength.

where y_1, y_2, y_3, y_4, and y_5 are state variables and σ, β, r, d_1, and d_2 are the unknown system parameters to be identified. When $\sigma = 10$, $\beta = 8/3$, $r = 28$, $d_1 = 2$, and $d_2 \in (2, 12)$, the system is hyperchaotic with three positive LEs. u_1, u_2, u_3, u_4, and u_5 are the controllers to be designed.

Based on the above method, we construct two auxiliary state variable $x_4 = x_1$, $x_5 = x_2 + x_3$. Then the drive system can be written as follows:

$$\begin{aligned} \dot{x}_1 &= a\left(x_2 - x_1\right) + x_2 x_3, \\ \dot{x}_2 &= b x_1 + c x_2 - x_1 x_3, \\ \dot{x}_3 &= x_2^2 - d x_3, \\ \dot{x}_4 &= a\left(x_2 - x_1\right) + x_2 x_3, \\ \dot{x}_5 &= b x_1 + c x_2 - x_1 x_3 + x_2^2 - d x_3. \end{aligned} \tag{57}$$

The MFPLS error is

$$\begin{aligned} e_1(t) &= y_1(t) - m_1 h(t) x_1\left(t - \tau_1\right), \\ e_2(t) &= y_2(t) - m_2 h(t) x_2\left(t - \tau_2\right), \\ e_3(t) &= y_3(t) - m_3 h(t) x_3\left(t - \tau_3\right), \\ e_4(t) &= y_4(t) - m_4 h(t) x_4\left(t - \tau_4\right), \\ e_5(t) &= y_5(t) - m_5 h(t) x_5\left(t - \tau_5\right), \end{aligned} \tag{58}$$

where m_i $(i = 1, 2, 3, 4, 5)$ is a scaling constant, $h(t)$ is a nonzero continuous differentiable function, and the time delay vector is $\boldsymbol{\tau} = (\tau_1, \tau_2, \tau_3, \tau_4, \tau_5)^{\mathrm{T}}$.

Based on Theorem 3, the controller, parameter update rules, and control gain strength adapt laws are chosen as follows:

$$\begin{aligned} u_1(t) = {} & \hat{\sigma}\left(y_1(t) - y_2(t)\right) - y_4(t) + m_1 \dot{h}(t) x_1\left(t - \tau_1\right) \\ & + m_1 h(t) \hat{a}\left(x_2\left(t - \tau_1\right) - x_1\left(t - \tau_1\right)\right) \\ & + m_1 h(t) x_2\left(t - \tau_1\right) x_3\left(t - \tau_1\right) - k_1 e_1, \end{aligned}$$

$$
\begin{aligned}
u_2(t) &= -\hat{r}y_1(t) + y_2(t) + y_1(t)y_3(t) + y_5(t) \\
&\quad + m_2\dot{h}(t)x_2(t-\tau_2) - m_2h(t) \\
&\quad \times \left(-\hat{b}x_1(t-\tau_2) - \hat{c}x_2(t-\tau_2) + x_1(t-\tau_2)x_3(t-\tau_2)\right) \\
&\quad - k_2e_2, \\
u_3(t) &= \hat{\beta}y_3(t) - y_1(t)y_2(t) + m_3\dot{h}(t)x_3(t-\tau_3) \\
&\quad - m_3h(t)\left(-x_2^2(t-\tau_3) + \hat{d}x_3(t-\tau_3)\right) - k_3e_3, \\
u_4(t) &= y_1(t)y_3(t) - \hat{d}_1y_4(t) + m_4\dot{h}(t)x_4(t-\tau_4) \\
&\quad + m_4h(t)\hat{a}(x_2(t-\tau_4) - x_1(t-\tau_4)) \\
&\quad + m_4h(t)x_2(t-\tau_4)x_3(t-\tau_4) - k_4e_4, \\
u_5(t) &= -\hat{d}_2y_2(t) + m_5\dot{h}(t)x_5(t-\tau_5) - m_5h(t) \\
&\quad \times \left(-\hat{b}x_1(t-\tau_5) - \hat{c}x_2(t-\tau_5) \right.\\
&\quad + x_1(t-\tau_5)x_3(t-\tau_5) \\
&\quad \left. - x_2^2(t-\tau_5) + \hat{d}x_3(t-\tau_5)\right) - k_5e_5, \\
\dot{e}_a &= \dot{\hat{a}} = -e_1m_1h(t)(x_2(t-\tau_1) - x_1(t-\tau_1)) \\
&\quad - m_4h(t)e_4(x_2(t-\tau_4) - x_1(t-\tau_4)), \\
\dot{e}_b &= \dot{\hat{b}} = -e_2m_2h(t)x_1(t-\tau_2) - m_5h(t)e_5x_1(t-\tau_5), \\
\dot{e}_c &= \dot{\hat{c}} = -m_2h(t)e_2x_2(t-\tau_2) - m_5h(t)e_5x_2(t-\tau_5), \\
\dot{e}_d &= \dot{\hat{d}} = m_3h(t)e_3x_3(t-\tau_3) + m_5h(t)e_5x_3(t-\tau_5), \\
\dot{e}_\sigma &= \dot{\hat{\sigma}} = -e_1(y_1(t) - y_2(t)), \\
\dot{e}_\beta &= \dot{\hat{\beta}} = -e_3y_3(t), \\
\dot{e}_r &= \dot{\hat{r}} = e_2y_1(t), \\
\dot{e}_{d_1} &= \dot{\hat{d}}_1 = e_4y_4(t), \\
\dot{e}_{d_2} &= \dot{\hat{d}}_2 = e_5y_2(t), \\
\dot{k}_i &= \varepsilon_ie_i^2, \quad \varepsilon_i > 0, \ i = 1, 2, \ldots, 5,
\end{aligned}
\tag{59}
$$

where $\hat{a}$, $\hat{b}$, $\hat{c}$, $\hat{d}$, $\hat{\sigma}$, $\hat{\beta}$, $\hat{r}$, $\hat{d}_1$, and $\hat{d}_2$ are the estimated values of a, b, c, d, σ, β, r, d_1, and d_2, respectively and $e_a = \hat{a} - a$, $e_b = \hat{b} - b$, $e_c = \hat{c} - c$, $e_d = \hat{d} - d$, $e_\sigma = \hat{\sigma} - \sigma$, $e_\beta = \hat{\beta} - \beta$, $e_r = \hat{r} - r$, $e_{d_1} = \hat{d}_1 - d_1$, $e_{d_2} = \hat{d}_2 - d_2$ are parameter errors.

Proof. The Lyapunov function is chosen as

$$
\begin{aligned}
V(t) = \frac{1}{2}\Big(&e_1^{\mathrm{T}}(t)e_1(t) + e_2^{\mathrm{T}}(t)e_2(t) + e_3^{\mathrm{T}}(t)e_3(t) \\
&+ e_4^{\mathrm{T}}(t)e_4(t) + e_5^{\mathrm{T}}(t)e_5(t) + e_a^{\mathrm{T}}(t)e_a(t) \\
&+ e_b^{\mathrm{T}}(t)e_b(t) + e_c^{\mathrm{T}}(t)e_c(t) + e_d^{\mathrm{T}}(t)e_d(t) \\
&+ e_\sigma^{\mathrm{T}}(t)e_\sigma(t) + e_\beta^{\mathrm{T}}(t)e_\beta(t) + e_r^{\mathrm{T}}(t)e_r(t) \\
&+ e_{d_1}^{\mathrm{T}}(t)e_{d_1}(t) + e_{d_2}^{\mathrm{T}}(t)e_{d_2}(t)\Big) \\
&+ \frac{1}{2}\sum_{i=1}^{5}\frac{1}{\varepsilon_i}(k_i - k)^2,
\end{aligned}
\tag{60}
$$

where $k > 0$ is a positive constant.

Then the time derivation of the Lyapunov function along the trajectory of error systems is

$$
\begin{aligned}
\dot{V}(t) &= e_1^{\mathrm{T}}(t)\dot{e}_1(t) + e_2^{\mathrm{T}}(t)\dot{e}_2(t) + e_3^{\mathrm{T}}(t)\dot{e}_3(t) \\
&\quad + e_4^{\mathrm{T}}(t)\dot{e}_4(t) + e_5^{\mathrm{T}}(t)\dot{e}_5(t) + e_a^{\mathrm{T}}(t)\dot{e}_a(t) \\
&\quad + e_b^{\mathrm{T}}(t)\dot{e}_b(t) + e_c^{\mathrm{T}}(t)\dot{e}_c(t) + e_d^{\mathrm{T}}(t)\dot{e}_d(t) \\
&\quad + e_\sigma^{\mathrm{T}}(t)\dot{e}_\sigma(t) + e_\beta^{\mathrm{T}}(t)\dot{e}_\beta(t) + e_r^{\mathrm{T}}(t)\dot{e}_r(t) \\
&\quad + e_{d_1}^{\mathrm{T}}(t)\dot{e}_{d_1}(t) + e_{d_2}^{\mathrm{T}}(t)\dot{e}_{d_2}(t) \\
&\quad + \dot{k}_i\sum_{i=1}^{5}\frac{1}{\varepsilon_i}(k_i - k)^2.
\end{aligned}
\tag{61}
$$

Constituting (58) and (59) into (61), we get the following:

$$
\begin{aligned}
\dot{V}(t) &= -k\left(e_1^{\mathrm{T}}(t)e_1 + e_2^{\mathrm{T}}(t)e_2 + e_3^{\mathrm{T}}(t)e_3 + e_4^{\mathrm{T}}(t)e_4 + e_5^{\mathrm{T}}(t)e_5\right) \\
&< 0.
\end{aligned}
\tag{62}
$$

$V(t)$ is positively definite and $\dot{V}(t) < 0$; thus, according to Barbalat's Lemma, MFPLS between the drive system (57) and the response system (56) is achieved. The uncertain parameters and control gain strengths can be identified and defined as well. □

In the simulation, the system parameters are selected as $a = 20$, $b = 5$, $c = 10$, $d = 2$, $\sigma = 10$, $\beta = 8/3$, $r = 28$, $d_1 = 2$, and $d_2 = 8$, such that the drive system (57) and the response system (56) can exhibit chaotic behaviors without control. We assume that the initial conditions of the drive system (57) and the response system (56) as $\mathbf{x}(0) = (-2, 1, 4, 2, -3)^{\mathrm{T}}$, $\mathbf{y}(0) = (2, -2, 3, 6, 1)^{\mathrm{T}}$, respectively. We choose the initial values of the unknown parameters as $\hat{a}(0) = \hat{b}(0) = \hat{c}(0) = \hat{d}(0) = \hat{\sigma}(0) = \hat{\beta}(0) = \hat{r}(0) = \hat{d}_1(0) = \hat{d}_2(0) = 0.001$. We select the initial values of the control gain strength as $k_1(0) = k_2(0) = k_3(0) = k_4(0) = k_5(0) = 4$, and the constants as $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 6$. The time delay is arbitrarily chosen as $(\tau_1, \tau_2, \tau_3, \tau_4, \tau_5) = (0.3, 0.4, 0.2, 0.1, 0.2)$. We take the scaling constants as $(m_1, m_2, m_3, m_4, m_5) = (-2, 3, -1, 2, -5)$ and nonzero continuous differentiable function as $h(t) = \sin(t)$. The corresponding simulation results are illustrated in Figures 12–15.

Figure 12 shows that the synchronization errors converge to zero after a transient time, which indicates that MFPLS between the drive system (57) and the response system (56)

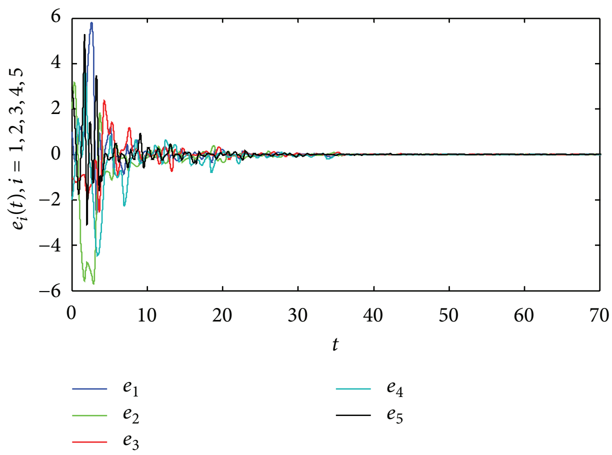

Figure 12: Time evolution of MFPLS errors between the system (57) and the system (56).

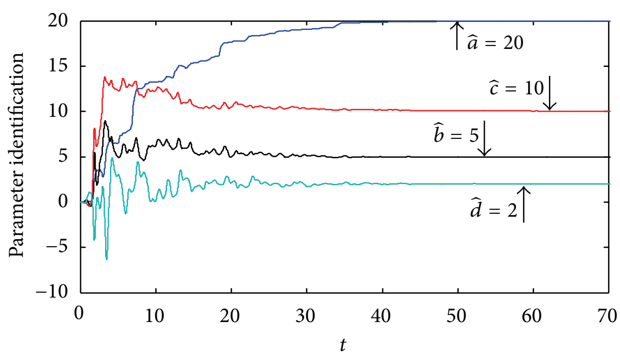

Figure 13: Time evolution of parameter estimation for the drive system (57).

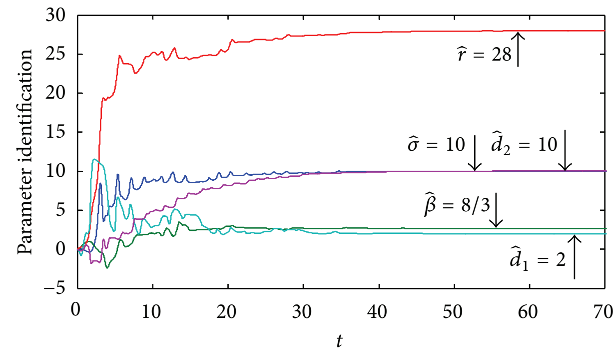

Figure 14: Time evolution of parameter estimation for the response system (56).

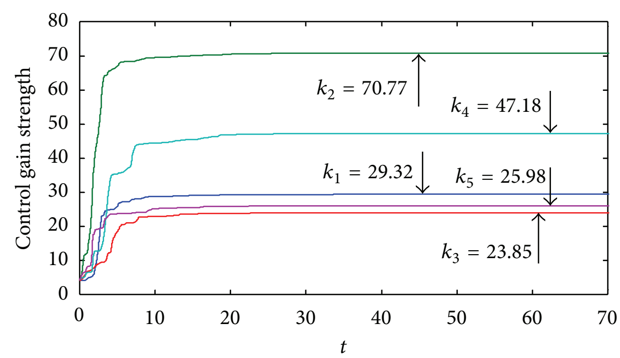

Figure 15: Time evolution of the control gain strength.

is achieved. From Figures 13 and 14, it is easy to see that the nine unknown parameters are also identified. Time evolution of the adaptive control strength is illustrated in Figure 15, and the control strengths approach to $k_1 = 29.32$, $k_2 = 70.77$, $k_3 = 23.85$, $k_4 = 47.18$ and $k_5 = 25.98$ as $t \to \infty$. Hence, the MFPLS between the 3-dimensional hyperchaotic system and the 5-dimensional hyperchaotic system are attained, all the uncertain parameters can be estimated and the control gain strengths can be given, too.

5. Conclusions

The paper investigated MFPLS of hyperchaotic or chaotic system, when the system parameters are all uncertain and the dimension and structure of the drive system and the response system are the same or different. A general theorem for controller designing, parameter update rule designing, and control gain strength adapt law designing is introduced by using adaptive control method, and it has been proven effective theoretically based on Lyapunov stability theory. It is worth mentioning that the systems can have the identical or different dimensions and structures, and the control gain strengths can be identified adaptively.

Furthermore, the proposed method is applied to four typical examples, which include MFPLS between two five-dimensional hyperchaotic systems with identical structures, MFPLS between two four-dimensional hyperchaotic systems with different structures, MFPLS between a four-dimensional hyperchaotic system and a three-dimensional chaotic system, and MFPLS between a novel three-dimensional chaotic system and a five-dimensional hyperchaotic system. In every case, controller, parameter update rule, and control gain strength adapt law are constructed in detail. The corresponding numerical simulations are performed to show the effectiveness of our results.

By now, many fractional-order differential systems such as fractional-order Chua's circuit, the fractional-order van der pol system, and the fractional-order Lorenz system, are chaotic. And study on the synchronization of chaotic fractional-order differential systems has greatly attracted interest of many researchers due to its potential applications in secure communication and control processing [36–38]. Wang et al. [36, 37] introduced projective synchronization of fractional-order chaotic systems based on linear separation and synchronization of fractional-order chaotic systems with activation feedback control. In [38], modified projective synchronization of fractional-order chaotic systems via active sliding mode control is analyzed, and active sliding mode controller is proposed to synchronize two different fractional-order differential systems based on the stability theorems of fractional-order linear system. Research on MFPLS of fractional-order chaotic systems is interesting and useful. The adding of time delay makes the solving of the equation more difficult. To the best of our knowledge, modified function projective lag synchronization of fractional-order chaotic systems are not studied. So, we will investigate the MFPLS of fractional-order chaotic systems and derive some stability criteria in a near future.

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