Research Article

Fractional Describing Function Analysis of PWPF Modulator

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Received 9 November 2012; Accepted 8 January 2013

Academic Editor: Clara Ionescu

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Pulse-width pulse-frequency (PWPF) modulators are widely used in spacecraft thruster control. Their dynamic characteristic is still lack of effective analysis tools. This paper presents a fractional describing function method to describe the frequency characteristics of PWPF. A frequency-dependent gain and phase shift are clearly described by fractional-order expression, and the fractional-order behaviors depict the nonlinear properties of PWPF modulators. This fractional describing function method can also be applied to other kinds of modulators.

1. Introduction

Spacecrafts commonly deploy thrusters as actuators for attitude control [1]. As thrusters work on on/off switching, there is a conversion from continuous desired torque control command to an on/off signal for spacecraft thrusters. There are two common approaches for thruster activation. The simplest converter is a bang-bang controller which is, however, vulnerable to noise. Several improved controllers have been proposed, such as bang-bang controller with dead zone or time-optimal bang-bang controller [2]. Another kind of the converter is pulse modulators, which produce a pulse command sequence to the thruster valves by adjusting pulse width and/or pulse frequency according to the level of input [3]. Among the known pulse modulators, the pulse-width pulse-frequency (PWPF) modulators are the most common and enjoy advantages over bang-bang control systems [4]. But the inherent nonlinearity of PWPF has inhibited the dynamic analysis of attitude control system.

Describing function method is a well-known analysis tool for a kind of nonlinear system with certain structure. If the output signal of a nonlinear device can be approximated by the fundamental harmonics, the fundamental harmonics can be used to define the frequency characteristics of the nonlinear element and is called a describing function [5].

The describing function method is primarily used to analyze the stability and limit cycle of nonlinear control systems and is inherently approximate. In the last decades, many new criteria were established to improve the accuracy. These new types of describing function methods include area-matching method, root-mean-square method, and corrected RMS method [3].

Recently, fractional calculus has been increasingly applied to mechanical systems, electricity, and bioengineering [6, 7]. Fractional calculus studies derivatives and integrals of fractional order. It is shown that fractional-order system works more accurately than integer-order system [8, 9]. With the development of computational tools, more and more fractional-order models and controllers are studied [10, 11]. Noticeably, the describing function method is also considered in the framework of nonlinear fractional-order systems [12, 13]. In this paper, fractional describing function will be introduced to describe the dynamics of PWPF actuators.

2. Pulse-Width Pulse-Frequency Modulator

Pulse modulators are commonly used in thruster control of fuel valves. There are various kinds of pulse modulators, such as pulse-width modulator, pulse-frequency modulator,

FIGURE 1: Structure of PWPF modulator.

TABLE 1: Static characteristic variables of PWPF modulator.

On time	$T_{on} = -T_m \ln \left(1 - \frac{h}{U_{on} - K_m(C - U)} \right)$
Off time	$T_{off} = -T_m \ln \left(1 - \frac{h}{K_m C - U_{off}} \right)$
Modulator frequency	$f = \frac{1}{T_{on} + T_{off}}$
Duty cycle	$DC = \frac{T_{on}}{T_{on} + T_{off}}$
Equivalent internal deadband	$C_d = \frac{U_{on}}{K_m}$
Equivalent internal saturation level	$C_s = 1 + \frac{U_{off}}{K_m}$

pseudo-rate modulator, and pulse-width pulse-frequency (PWPF) modulator [14, 15]. PWPF modulator is preferred for its operation has almost linear input/output relationship [16, 17]. A PWPF modulator mainly comprises two components: a first-order lag filter and a Schmitt trigger inside a feedback loop, as shown in Figure 1. A Schmitt trigger is an on-off relay with a dead zone and hysteresis. It differs from bang-bang controller in that there are two thresholds: one on-value U_{on} and one off-value U_{off} . These values define a hysteresis as $h = U_{on} - U_{off}$. The output of the Schmitt trigger is compared with the reference signal, and the error is fed to the first-order filter whose output is the input of the Schmitt trigger. PWPF modulator operates in a quasilinear mode by modulating the width of the output pulses and the distance between them simultaneously. And it can produce pulses in two directions: positive and negative pulses.

With a constant input C , the PWPF modulator drives the thruster valve with an on-off pulse sequence having a nearly linear duty cycle. The time interval in which the modulator has a nonzero output is denoted T_{on} , and the time interval with a zero output is denoted T_{off} . Static characteristics variables of PWPF are collected in Table 1. These variables are considered in the modulator design. Paper [18] shows the relationship between the static characteristics of PWPF modulator and selection of its parameters.

The static analysis of PWPF modulator shows that it operates near linear to the constant input over a large range between the deadband C_d and saturation level C_s . The modulator's operations are independent from the spacecraft's parameters and allow easy parameter tuning, especially when there are different requirements through different phases of operation. In addition, this modulator has the superiority in fuel consumption and pointing accuracy in the presence of

vibrations. However, attitude control systems usually operate on dynamic mode, and PWPF modulators will introduce phase lag to the attitude control systems, which can cause instability. Dynamic analysis is necessary for the attitude control system design. But as to this nonlinear device, effective tools are limited. This work will propose the use of fractional-order describing function method and develops some useful techniques.

3. Derivation of Fractional Describing Function

Describing function method is an approximation method for analyzing nonlinear dynamics, because only the first harmonic of the output of a nonlinear element is considered. But here describing function method is beneficial for PWPF modulator analysis for the following reasons. Firstly, the first-order filter which is in series with the nonlinear element serves as a low-pass filter in PWPF modulator. Secondly, the attitude control system is a high-order system, where high harmonics are attenuated substantially. In this section, the PWPF modulator is considered as a single device and its fractional describing function is developed.

3.1. Describing Function of Hysteresis. A nonlinear element to a sinusoidal input $x(t) = X \sin(\omega t)$ in general does not generate a sinusoidal output, but the output $y(t)$ is periodic. If the nonlinearity is symmetrical with respect to the variation around zero, the output signal can be decomposed to the Fourier series:

$$y(t) = \sum_{n=1}^{\infty} (A_n \cos n\omega t + B_n \sin n\omega t) \quad (1)$$

$$= \sum_{n=1}^{\infty} Y_n \sin(n\omega t + \varphi_n),$$

where

$$A_n = \frac{1}{\pi} \int_0^{2\pi} y(t) \cos n\omega t d(\omega t),$$

$$B_n = \frac{1}{\pi} \int_0^{2\pi} y(t) \sin n\omega t d(\omega t), \quad (2)$$

$$Y_n = \sqrt{A_n^2 + B_n^2},$$

$$\varphi_n = \tan^{-1} \frac{A_n}{B_n},$$

and the frequency of the fundamental harmonic is the same frequency as the input with amplitude Y_1 and phase shift φ_1 :

$$y_1(t) = A_1 \cos \omega t + B_1 \sin \omega t \quad (3)$$

$$= Y_1 \sin(\omega t + \varphi_1).$$

With the assumption that the amplitude of fundamental harmonic is much larger than the amplitude of other harmonics, the describing function is defined as the complex ratio of

the fundamental harmonic component of the output and the input, that is,

$$N(X, \omega) = \frac{Y_1}{X} e^{j\varphi_1}, \quad (4)$$

where $N(X, \omega)$ is the describing function of the nonlinear element, X is the sinusoidal input amplitude, ω is the frequency of input sinusoid, Y_1 is the amplitude of the fundamental harmonic, Y_1/X is the describing function gain, and φ_1 is the describing function phase.

Imposing this harmonic balance principle, $N(X, \omega)$ can be used as frequency characteristic of nonlinear element to analyze the dynamics of a linear closed-loop system. The Schmitt trigger in PWPF modulator exhibits hysteresis nonlinearity, whose describing function of hysteresis can be deduced as shown above directly:

$$N(X) = \frac{4}{\pi X} \sqrt{1 - \left(\frac{h}{X}\right)^2} - j \frac{4h}{\pi X^2}. \quad (5)$$

Remark 1. The describing function defined above is based on fundamental harmonic equivalence. The influence of high order harmonics can be considered as system uncertainty.

3.2. Frequency Characteristics of PWPF. The PWPF modulator is a unit in the attitude control system. Considering the frequency characteristics of PWPF as a whole is convenient for analysis and design of the attitude control system. In the structure of Figure 1, the frequency characteristic of the first-order filter is

$$L(j\omega) = \frac{K_m}{1 + j\omega T_m}. \quad (6)$$

Then the frequency characteristic of the PWPF closed-loop system is

$$NN(X, \omega) = \left(\frac{4K_m}{\pi X} \sqrt{1 - \left(\frac{h}{X}\right)^2} - j \frac{4hK_m}{\pi X^2} \right) \times \left(\left(1 + \frac{4K_m}{\pi X} \sqrt{1 - \left(\frac{h}{X}\right)^2} \right) + j \left(\omega T_m - \frac{4hK_m}{\pi X^2} \right) \right)^{-1}. \quad (7)$$

If the describing function is independent of frequency ω , it is plotted with varying nonlinear input X . But when it is a function of both amplitude X and frequency ω , certain ω values are selected to view the plot of frequency characteristics. Consequently, this kind of plot lacks frequency dependent information. For control system dynamic analysis, information about how the amplitude and phase of a nonlinear element change with frequency is important because the nonlinear part usually adds phase lag to control system, which would destroy close-loop stability of the control system.

Now, a fractional describing function is introduced to give a direct relationship between the nonlinear characteristic

FIGURE 2: Nichols plot of $NN(X, \omega)$.

and frequency. For illustration purpose, set the parameters of PWPF as follows: $K_m = 7.46$, $T_m = 1.33$, $U_{on} = 0.45$, and $U_{off} = 0.25$. Figure 2 shows the Nichols plot of the closed-loop describing function $NN(X, \omega)$. It can be seen that with the amplitude X changing from 0.5 to 10 the magnitude of $NN(X, \omega)$ decreases, and the phase lag increases. To show the characteristic with frequency, several curves corresponding with different frequencies are plotted. Though it is clear that higher frequency induces larger phase lag, but the curves are separated and cannot show continuous information about frequency.

To reveal the relationship between the describing function and the frequency of interest, the real part and imaginary part of $NN(X, \omega)$ are studied, respectively. For attitude control, the interested frequency is relatively low. Figures 3 and 4 show the log-log plots of the real part and imaginary part of $NN(X, \omega)$ versus the exciting frequency ω , respectively, where the nonlinear input X is from 0.5 to 1. Approximately, the curves can be considered as straight lines. Then a fractional-order behavior is investigated.

The fractional-order behavior is described by power functions, so it is called fractional describing function. The new function can be written as

$$FNN(X, \omega) = -a\omega^{-b} - jc\omega^{-d}, \quad \{a, b, c, d\} \in \mathbb{R}^+. \quad (8)$$

Figure 3 shows that the real part of $NN(X, \omega)$ can be considered as a constant, that is, $b = 0$ and $-0.9 < a < -0.88$. Figure 4 shows that the imaginary part of $NN(X, \omega)$ is fractional with $-0.75 < d < -0.7$ and $0.11 < c < 0.13$. These values will vary with system parameters, but the fractional property remains the same. It can be seen that the range of parameters $\{a, b, c, d\}$ does not vary much. We can choose one set of values to simulate the frequency characteristic of the PWPF modulator. Here, we set the values $\{a, b, c, d\} = \{-0.9, 0, 0.12, -0.74\}$. With the help of available fractional calculator software [19, 20], frequency characteristic diagrams can be plotted. Figure 5 shows the Bode diagram of $FNN(X, \omega)$. It predicts a phase lag of

FIGURE 3: Log-log plots of $|\operatorname{Re}(NN)|$. FIGURE 4: Log-log plots of $|\operatorname{Im}(NN)|$.

about 10 degrees to 25 degrees between frequency 1 rad/s to 5 rad/s, which relates well with the analysis result in [3]. This information is crucial to the stability analysis of attitude control system.

Remark 2. The deduced fractional describing function is useful for system stability analysis for it is considered around the crossover frequency.

4. Conclusions

PWPF modulator is a nonlinear actuator in spacecraft attitude control system. The nonlinear dynamic behavior of PWPF modulator is investigated by the fractional describing function in this paper. The nonlinear element of PWPF is a Schmitt trigger, and its frequency characteristic can be described by describing function, and the fractional behaviors are caused by nonlinear element in PWPF. The frequency

FIGURE 5: Bode diagram of $FNN(X, \omega)$.

characteristic of the actuator is frequency dependent. The log-log plots of the real part and imaginary part of a modulator clearly reveal the fractional-order behavior. The imaginary component is described by fractional-order power function over a certain frequency range. With fractional calculus, these frequency-dependent gain and phase information can be plotted in Bode diagram and used for control system design. Furthermore, the fractional describing function method should be an effective tool for other kinds of modulators.

Acknowledgments

This work is funded in part by the China Scholarship Council (CSC) and in part under Project Agreement no. DSOCL10004 with DSO National Laboratories, Singapore.

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