Research Article

Bifurcation and Hybrid Control for A Simple Hopfield Neural Networks with Delays

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Received 30 March 2013; Accepted 6 May 2013

Academic Editor: Guanghui Wen

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A detailed analysis on the Hopf bifurcation of a delayed Hopfield neural network is given. Moreover, a new hybrid control strategy is proposed, in which time-delayed state feedback and parameter perturbation are used to control the Hopf bifurcation of the model. Numerical simulation results confirm that the new hybrid controller using time delay is efficient in controlling Hopf bifurcation.

1. Introduction

It is well known that neural networks are complex and large-scale nonlinear dynamical system. In the last decade, the dynamical characteristics (including stable, unstable, oscillatory, and chaotic behavior) of Hopfield neural networks (HNNs) with time delays have become a subject of intense research activities. Many stability criteria are obtained. We refer the reader to [1–8] and the references cited therein. However, the periodic nature of neural impulses is of fundamental significance in the control of regular dynamical functions such as breathing and heart beating. Neural networks involving persistent oscillations such as limit cycle may be applied to pattern recognition and associative memory. In differential equations with delays, periodic oscillatory behavior can arise through the Hopf bifurcation. Therefore, it is also very significant to study the class of problem. Olien and Bélair [9] investigated the bifurcation of the following HNNs system:

$$\begin{aligned}\dot{x}_1 &= -a_1 x_1(t) + b_{11} f_1(x_1(t - \tau_1)) \\ &\quad + b_{12} f_2(x_2(t - \tau_2)), \\ \dot{x}_2 &= -a_2 x_2(t) + b_{21} f_1(x_1(t - \tau_3)) \\ &\quad + b_{22} f_2(x_2(t - \tau_1)),\end{aligned}\tag{1}$$

in which $\tau_1 = \tau_2 = \tau_3 = \tau$, $a_i = 1$ and $f_i(0) = 0$, $i = 1, 2$, and Huang et al. [10] study further the bifurcation and periodic nature of system (1) with $2\tau_1 = \tau_2 + \tau_3$, $a_i = 1$ and $f_i(0) = 0$, $i = 1, 2$. Moreover, many authors also consider discrete form of system (1); we can see [11–13].

In recent years, bifurcation control has attracted many researchers from various disciplines. The aim of bifurcation control is to design a controller to modify the bifurcation properties of a given nonlinear system, thereby to achieve some desirable dynamical behaviors. After the pioneering work initiated by Ott et al. [14], there have been many ideas and methods of bifurcation control [15–20]. However, from the control theory point of view, we may classify the current methods into two main categories: the first one is feedback control where state feedback is applied to control bifurcation or chaos, and the other is nonfeedback methods. Recently, Luo et al. [21] proposed a new control strategy for period-doubling bifurcations in a discrete nonlinear dynamical system. Moreover, Liu and Chung [22] investigated the same control strategy in a continuous dynamical system without time delays. Now, we extend this strategy to deal with bifurcation control in HNNs system (1).

In the paper, we will propose a new hybrid control strategy in which the parameter perturbation and time-delayed state feedback are combined and used to control Hopf bifurcation in system (1). Simulation results demonstrate the correctness of our theoretical analysis. The comparison shows

that the control strategy is effective as it meets the purpose of retarding the occurrence of bifurcation.

2. Stability and Hopf Bifurcation of System (1) without Control

In this section, we will consider system (1) with $2\tau_1 = \tau_2 + \tau_3 \triangleq 2\tau$ and $f_i(0) = 0, i = 1, 2$. It is obviously that $(0, 0)$ is an equilibrium point of system (1).

To simplify, here we denote $c_{11} = b_{11}f'_1(0)$, $c_{12} = b_{12}f'_2(0)$, $c_{21} = b_{21}f'_1(0)$, $c_{22} = b_{22}f'_2(0)$, $a_1 = a_2 = b$. Consider the linearized system of system (1) at $(0, 0)$

$$\begin{aligned}\dot{x}_1 &= -bx_1(t) + c_{11}x_1(t - \tau) + c_{12}x_2(t - \tau_2), \\ \dot{x}_2 &= -bx_2(t) + c_{21}x_1(t - \tau_3) + c_{22}x_2(t - \tau).\end{aligned}\quad (2)$$

The characteristic equation of the linearized system (2) is

$$(\lambda + b)^2 - (c_{11} + c_{22})(\lambda + b)e^{-\lambda\tau} + (c_{11}c_{22} - c_{12}c_{21})e^{-2\lambda\tau} = 0, \quad (3)$$

which determines the local stability of the equilibrium solution. Thus, we will find some conditions which ensure that all roots of (3) have negative real parts. To facilitate the calculation in this paper, we rewrite the characteristic equation (3) as follows:

$$(\lambda + b)^2 e^{2\lambda\tau} - 2T(\lambda + b)e^{\lambda\tau} + D = 0, \quad (4)$$

where $D = c_{11}c_{22} - c_{12}c_{21}$, $T = (1/2)(c_{11} + c_{22})$. Obviously, (4) is a quadratic polynomial in the variable $(\lambda + b)e^{\lambda\tau}$ and has roots given by

$$(\lambda + b)e^{\lambda\tau} = T \pm \sqrt{T^2 - D}. \quad (5)$$

In the following, we distinguish two cases to discuss (5).

2.1. As $T^2 \geq D$. In this part, we state a result due to [23] as a lemma to analyze (5), which is, for the convenience of the reader, stated as follows.

Lemma 1. *For the transcendental equation*

$$\begin{aligned}\lambda^n + p_1^{(0)}\lambda^{n-1} + \dots + p_{n-1}^{(0)}\lambda + p_n^{(0)} \\ + [p_1^{(1)}\lambda^{n-1} + \dots + p_{n-1}^{(1)}\lambda + p_n^{(1)}]e^{-\lambda\sigma_1} \\ + \dots + [p_1^{(m)}\lambda^{n-1} + \dots + p_{n-1}^{(m)}\lambda + p_n^{(m)}]e^{-\lambda\sigma_m} = 0,\end{aligned}\quad (6)$$

where $\sigma_i \geq 0$ ($i = 1, \dots, m$) and $p_j^{(i)}$ ($i = 1, \dots, m; j = 1, \dots, n$) are constants. As $(\sigma_1, \dots, \sigma_m)$ varies, the sum of the orders of the zeros of (6) in the open right half-plane can change only if a zero appears on or crosses the imaginary axis.

For convenience, we make the following assumptions:

- (H1) $b > T \pm \sqrt{T^2 - D}$.
- (H2) $b^2 > (T \pm \sqrt{T^2 - D})^2$.
- (H3) $(T + \sqrt{T^2 - D})^2 < b^2 < (T - \sqrt{T^2 - D})^2$.
- (H4) $b^2 < (T \pm \sqrt{T^2 - D})^2$.

Lemma 2. *If (H1) and (H2) hold, then all roots of (3) have negative real parts for every $\tau \in [0, +\infty)$.*

Proof. For (3), when $\tau = 0$, its roots can be expressed as $\lambda_{1,2} = -b + T \pm \sqrt{T^2 - D}$. Clearly, all roots of (3) are negative if (H1) holds. We want to determine if the real part of some root increases to reach zero and eventually becomes positive as $\tau \neq 0$. We can see that λ is a root of (3) if and only if λ is a root of (5).

We write $\lambda = \rho + i\omega$ for a root of the characteristic equations (5), separate the real and imaginary parts of the ensuing equations (5), and obtain

$$\begin{aligned}e^{\rho\tau} [(\rho + b) \cos(\omega\tau) - \omega \sin(\omega\tau)] &= T \pm \sqrt{T^2 - D}, \\ e^{\rho\tau} [(\rho + b) \sin(\omega\tau) + \omega \cos(\omega\tau)] &= 0.\end{aligned}\quad (7)$$

A change in the stability of the stationary solution can only occur when $\rho = 0$, that is,

$$\begin{aligned}b \cos(\omega\tau) - \omega \sin(\omega\tau) &= T \pm \sqrt{T^2 - D}, \\ b \sin(\omega\tau) + \omega \cos(\omega\tau) &= 0.\end{aligned}\quad (8)$$

By (8), we have

$$\omega^2 = (T \pm \sqrt{T^2 - D})^2 - b^2. \quad (9)$$

By (9), if (H2) holds, we know that (5) has no purely imaginary roots, and then applying Lemma 1 one obtains that all roots of (3) have negative real parts. This completes the proof of lemma. $\square$

Lemma 3. *For (5), one obtains the following results.*

- (1) *If (H1) and (H3) hold, then (5) have a pair of purely imaginary roots $\pm i\omega_-$ at $\tau = \tau_{-,j}$.*
- (2) *If (H1) and (H4) hold, then (5) have a pair of purely imaginary roots $\pm i\omega_+$ at $\tau = \tau_{+,j}$ and have another pair of purely imaginary roots $\pm i\omega_-$ at $\tau = \tau_{-,j}$.*

where

$$\omega_{\pm}^2 = (T \pm \sqrt{T^2 - D})^2 - b^2, \quad (10)$$

$$\begin{aligned}\tau_{\pm,j} &= \frac{1}{\omega_{\pm}} \arctan\left(-\frac{\omega_{\pm}}{b}\right) + \frac{j\pi}{\omega_{\pm}}, \quad j = 0, 1, \dots; \\ \tau_0 &= \min\{\tau_{+,0}, \tau_{-,0}\}.\end{aligned}\quad (11)$$

Here one denotes $\pm\omega_0$ especially as a pair of purely imaginary roots of (5) at $\tau = \tau_0$. To see if $\tau_{-,0}$ and τ_0 are bifurcation values, one needs to verify if the transversality conditions hold. In fact, one has the following.

Lemma 4. *The following transversality conditions:*

$$\operatorname{Re} \left(\frac{d\lambda}{d\tau} \right) \Big|_{\tau_{\pm,j}} > 0 \quad (12)$$

are satisfied.

Proof. By (5), we have

$$\frac{d\lambda}{d\tau} e^{\lambda\tau} + (\lambda + b) \left(\tau e^{\lambda\tau} \frac{d\lambda}{d\tau} + \lambda e^{\lambda\tau} \right) = 0. \quad (13)$$

Hence,

$$\frac{d\lambda}{d\tau} (1 + (\lambda + b) \tau) = -\lambda (\lambda + b). \quad (14)$$

Obviously, we have

$$\left(\frac{d\lambda}{d\tau} \right)^{-1} = \frac{(1 + (\lambda + b) \tau)}{-\lambda (\lambda + b)} = -\frac{1}{\lambda (\lambda + b)} - \frac{\tau}{\lambda}, \quad (15)$$

then

$$\begin{aligned} \operatorname{sign} \left(\operatorname{Re} \left(\frac{d\lambda}{d\tau} \right) \Big|_{\tau_{\pm,j}} \right) &= \operatorname{sign} \left(\operatorname{Re} \left(\frac{d\lambda}{d\tau} \right)^{-1} \Big|_{\tau_{\pm,j}} \right) \\ &= \operatorname{sign} \left(\operatorname{Re} \left(\frac{-1}{i\omega (i\omega + b)} \right) \right) \\ &= \operatorname{sign} \left(\operatorname{Re} \left(\frac{-1}{-\omega^2 + ib\omega} \right) \right) \\ &= \operatorname{sign} \left(\frac{-1 \cdot -\omega^2}{\omega^4 + b^2 \omega^2} \right) \\ &= \operatorname{sign} \left(\frac{1}{\omega^2 + b^2} \right) > 0. \end{aligned} \quad (16)$$

We complete the proof of Lemma 4. $\square$

From Lemmas 2–4, we can obtain the following theorem about the distribution of the characteristic roots of (3).

Theorem 5. *Let τ_{-0}, τ_0 be defined by (11).*

- (i) *If (H1) and (H2) hold, then all roots of (3) have negative real parts for all $\tau \geq 0$.*
- (ii) *If (H1) and (H3)((H4)) hold, then when $\tau < \tau_{-0}$ ($\tau < \tau_0$), all roots of (3) have negative real parts, when $\tau = \tau_{-0}$ ($\tau = \tau_0$), (3) has a pair of purely imaginary roots $\pm i\omega_0$ ($\pm i\omega_0$), and when $\tau > \tau_{-0}$ ($\tau > \tau_0$), (3) has at least one root with positive real part.*

By using Theorem 5, the stability and bifurcation of system (1) can be summarized as the following theorem.

Theorem 6. *For system (1), let (H1) hold and let τ_{-0}, τ_0 be defined by (11).*

- (i) *If (H2) holds, then the equilibrium point $(0, 0)$ is asymptotically stable for discrete delays $\tau \geq 0$.*

- (ii) *If (H3)((H4)) holds, there is a critical value $\tau = \tau_{-0}$ ($\tau = \tau_0$) of the discrete delay so that if $\tau < \tau_{-0}$ ($\tau < \tau_0$), then the equilibrium point $(0, 0)$ is asymptotically stable; if $\tau > \tau_{-0}$ ($\tau > \tau_0$), then $(0, 0)$ is unstable; Hopf bifurcation occurs when $\tau = \tau_{-0}$ ($\tau = \tau_0$).*

2.2. As $T^2 < D$. For convenience, we have the following assumptions:

(H5) $b > T$.

(H6) $b^2 > D$.

(H7) $b^2 < D$.

Similar to the deduction of Lemma 2, we have the following result.

Lemma 7. *If (H5) and (H6) hold, then all roots of (3) have negative real parts for every $\tau \in [0, +\infty)$.*

Proof. For (3), when $\tau = 0$, its roots can be expressed as $\lambda_{1,2} = -b \pm \sqrt{T^2 - D}$. Clearly, all roots of (3) have negative real parts if (H5) holds. We want to determine if the real part of some root increases to reach zero and eventually becomes positive as $\tau \neq 0$. We can see that λ is a root of (3) if and only if λ is a root of (5).

We write $\lambda = \rho + i\omega$ for a root of the characteristic equation (3), separate the real and imaginary parts of the ensuing equations (5), and obtain

$$\begin{aligned} e^{\rho\tau} [(\rho + b) \cos(\omega\tau) - \omega \sin(\omega\tau)] &= T, \\ e^{\rho\tau} [(\rho + b) \sin(\omega\tau) + \omega \cos(\omega\tau)] &= \pm \sqrt{D - T^2}. \end{aligned} \quad (17)$$

A change in the stability of the equilibrium point can only occur when $\rho = 0$, that is,

$$\begin{aligned} b \cos(\omega\tau) - \omega \sin(\omega\tau) &= T, \\ b \sin(\omega\tau) + \omega \cos(\omega\tau) &= \pm \sqrt{D - T^2}. \end{aligned} \quad (18)$$

Hence, we have

$$\omega^2 = D - b^2. \quad (19)$$

By (19), if (H6) holds, we know that (5) has no purely imaginary roots, and then applying Lemma 1 one obtains that all roots of (3) have negative real parts. This completes the proof of lemma. $\square$

Lemma 8. *For (3), one obtains the following results.*

If (H5) and (H7) hold, then (3) have a pair of purely imaginary roots $\pm i\omega$ at $\tau = \tau_{\pm,j}$, where

$$\omega^2 = D - b^2, \quad (20)$$

$$\tau_{\pm,j} = \frac{1}{\omega} \arccos \left(\frac{bT \pm \omega \sqrt{D - T^2}}{b^2 + \omega^2} \right) + \frac{2j\pi}{\omega}, \quad j = 0, 1, \dots;$$

$$\tau_0 = \min \{ \tau_{+,0}, \tau_{-,0} \}. \quad (21)$$

According to Lemma 4, one knows that τ_0 is bifurcation values.

From Lemmas 1 and 2, one can obtain the following theorem about the distribution of the characteristic roots of (3).

Theorem 9. Let τ_0 be defined by (21).

- (i) If (H5) and (H6) hold, then all roots of (3) have negative real parts for all $\tau \geq 0$.
- (ii) If (H5) and (H7) hold, then when $\tau < \tau_0$, all roots of (3) have negative real parts, when $\tau = \tau_0$, (3) has a pair of purely imaginary roots $\pm i\omega$, and when $\tau > \tau_0$, (3) has at least one root with positive real part.

By using Theorem 9, the stability and bifurcation of system (1) can be summarized as the following theorem.

Theorem 10. For system (1), let (H5) hold and let τ_0 be defined by the following: (21).

- (i) If (H6) holds, then the equilibrium point $(0, 0)$ is asymptotically stable for discrete delays $\tau \geq 0$.
- (ii) If (H7) holds, there is a critical value $\tau = \tau_0$ of the discrete delay so that if $\tau < \tau_0$, then the equilibrium point $(0, 0)$ is asymptotically stable; if $\tau > \tau_0$, then $(0, 0)$ is unstable; Hopf bifurcation occurs when $\tau = \tau_0$.

3. Stability Analysis and Bifurcation with Hybrid Control

In this section, we will consider system (1) with hybrid control described by the following differential equation:

$$\begin{aligned}\dot{x}_1 &= \alpha(-a_1x_1(t) + b_{11}f_1(x_1(t - \tau_1)) + b_{12}f_2(x_2(t - \tau_2))) \\ &\quad + \beta x_1(t - \tau_1), \\ \dot{x}_2 &= \alpha(-a_2x_2(t) + b_{21}f_1(x_1(t - \tau_3)) + b_{22}f_2(x_2(t - \tau_1))) \\ &\quad + \beta x_2(t - \tau_1),\end{aligned}\quad (22)$$

where $\alpha > 0$ and $\beta \in \mathbb{R}$. Obviously, $(0, 0)$ is also an equilibrium point of system (22).

Linearizing the system (22) at the equilibrium point $(0, 0)$, we obtain

$$\begin{aligned}\dot{x}_1 &= -\alpha b x_1(t) + (c_{11}\alpha + \beta)x_1(t - \tau_1) + c_{12}\alpha x_2(t - \tau_2), \\ \dot{x}_2 &= -\alpha b x_2(t) + c_{21}\alpha x_1(t - \tau_3) + (c_{22}\alpha + \beta)x_2(t - \tau_1).\end{aligned}\quad (23)$$

Then the characteristic equation for the linearized system around $(0, 0)^T$ is given by

$$\begin{aligned}(\lambda + \alpha b)^2 e^{2\lambda\tau} - (\alpha(c_{11} + c_{22}) + 2\beta)(\lambda + \alpha b)e^{\lambda\tau} \\ + ((\alpha c_{11} + \beta)(\alpha c_{22} + \beta) - \alpha^2 c_{12}c_{21}) = 0,\end{aligned}\quad (24)$$

which is a quadratic polynomial in the variable $(\lambda + \alpha b)e^{\lambda\tau}$ and has roots given by

$$(\lambda + \alpha b)e^{\lambda\tau} = T' \pm \sqrt{(T')^2 - D'}, \quad (25)$$

where

$$D' = \alpha^2 D + 2\alpha\beta T + \beta^2, \quad T' = \alpha T + \beta. \quad (26)$$

By (26), we know that $(T')^2 - D' = \alpha^2(T^2 - D)$, and thus, $(T')^2 \geq D'$ ($(T')^2 < D'$) holds if and only if $T^2 \geq D$ ($T^2 < D$) holds.

In the following, we also distinguish two cases to discuss (25).

3.1. As $T^2 \geq D$. Corresponding to Part I of Section 2, we make the following assumptions for convenience:

- (H1)' $b > T \pm \sqrt{T^2 - D} + (\beta/\alpha)$.
- (H2)' $b^2 > (T \pm \sqrt{T^2 - D} + (\beta/\alpha))^2$.
- (H3)' $(T + \sqrt{T^2 - D} + (\beta/\alpha))^2 < b^2 < (T - \sqrt{T^2 - D} + (\beta/\alpha))^2$.
- (H4)' $b^2 < (T \pm \sqrt{T^2 - D} + (\beta/\alpha))^2$.

Denote

$$(\omega'_\pm)^2 = \left(T' \pm \sqrt{(T')^2 - D'}\right)^2 - \alpha^2 b^2, \quad (27)$$

$$\tau'_{\pm,j} = \frac{1}{\omega'_\pm} \arctan\left(-\frac{\alpha b}{\omega'_\pm}\right) + \frac{j\pi}{\omega'_\pm}, \quad j = 0, 1, \dots, \quad (28)$$

$$\tau'_0 = \min\{\tau'_{+,0}, \tau'_{-,0}\}.$$

Similarly, we can obtain the following theorem.

Theorem 11. For system (22), let (H1)' hold and let $\tau'_{\pm,0}$, τ'_0 be defined by (28).

- (i) If (H2)' holds, then the equilibrium point $(0, 0)$ is asymptotically stable for discrete delays $\tau \geq 0$.
- (ii) If (H3)'((H4)') holds, there is a critical value $\tau = \tau'_{-,0}$ ($\tau = \tau'_0$) of the discrete delay so that if $\tau < \tau'_{-,0}$ ($\tau < \tau'_0$), then the equilibrium point $(0, 0)$ is asymptotically stable; if $\tau > \tau'_{-,0}$ ($\tau > \tau'_0$), then $(0, 0)$ is unstable; Hopf bifurcation occurs when $\tau = \tau'_{-,0}$ ($\tau = \tau'_0$).

3.2. As $T^2 < D$. Similar to deduction of Section 2.2, we have the following assumptions:

- (H5)' $b > T + (\beta/\alpha)$,
- (H6)' $b^2 > D + (2\beta/\alpha)T + (\beta/\alpha)^2$,
- (H7)' $b^2 < D + (2\beta/\alpha)T + (\beta/\alpha)^2$.

In this part, we denote

$$(\omega')^2 = D' - \alpha^2 b^2, \quad (29)$$

$$\tau'_{\pm,j} = \frac{1}{\omega'} \arccos\left(\frac{\alpha b T' \pm \omega' \sqrt{D' - (T')^2}}{\alpha^2 b^2 + (\omega')^2}\right) \quad (30)$$

$$+ \frac{2j\pi}{\omega'}, \quad j = 0, 1, \dots; \quad \tau'_0 = \min\{\tau'_{+,j}, \tau'_{-,j}\}.$$

Hence, we can obtain the following theorem.

Theorem 12. For system (22), let $(H5)'$ hold and let τ'_0 be defined by (30).

- (i) If $(H6)'$ holds, then the equilibrium point $(0, 0)$ is asymptotically stable for discrete delays $\tau \geq 0$.
- (ii) If $(H7)'$ holds, there is a critical value $\tau = \tau'_0$ of the discrete delay so that if $\tau < \tau'_0$ then the equilibrium point $(0, 0)$ is asymptotically stable; If $\tau > \tau'_0$, then $(0, 0)$ is unstable; Hopf bifurcation occurs when $\tau = \tau'_0$.

Remark 13. When $\alpha = 1 - \gamma$ and $\beta = \gamma$ in the system (22), then we obtain the same hybrid control with [22]; however, a control model based on delayed feedback is proposed in this paper; it is well know that control theory should contain delay since any control action takes effect only after a certain delay. Hence, our hybrid control is more helpful than [22].

Remark 14. When $\alpha = 1$ in the system (23), then we obtain a control model only based on delayed feedback in this paper, it is clear that our hybrid control is more general than control strategy proposed by [17].

Remark 15. In [10], authors investigated the Hopf bifurcation of following HNNs with $\alpha = 1$ and $\beta = 0$:

$$\begin{aligned}\dot{x}_1 &= \alpha \left(-x_1(t) - (\sqrt{2} - 1) f_1(x_1(t - \tau_1)) \right. \\ &\quad \left. + b_{12} f_2(x_2(t - \tau_2)) \right) + \beta x_1(t - \tau_1), \\ \dot{x}_2 &= \alpha \left(-x_2(t) + b_{21} f_1(x_1(t - \tau_3)) \right. \\ &\quad \left. - (\sqrt{2} - 1) f_2(x_2(t - \tau_1)) \right) + \beta x_2(t - \tau_1),\end{aligned}\quad (31)$$

By choosing $\tau_1 = 3\pi/4$, $\tau_2 = \pi/8$, $\tau_3 = 11\pi/8$, the authors obtained that Hopf bifurcation occurs when $b_{12}b_{21} = 1$. However, if choosing the parameters $\alpha = 0.8$ and $\beta = 0$, by the hybrid control strategy of this paper, the Hopf bifurcation in [10] will be eliminated. We can see Figures 1 and 2.

Remark 16. It is known to all that neural networks are a special case of complex networks. Thus, it is interesting and important to further study how to expand the application of theoretical results in [24–27] and any other complex networks.

4. Examples

In this section, we give two examples to illustrate our results.

Example 1. Consider the following HNNs system with hybrid control:

$$\begin{aligned}\dot{x}_1 &= \alpha (-a_1 x_1(t) + b_{11} f_1(x_1(t - \tau_1)) + b_{12} f_2(x_2(t - \tau_2))) \\ &\quad + \beta x_1(t - \tau_1), \\ \dot{x}_2 &= \alpha (-a_2 x_2(t) + b_{21} f_1(x_1(t - \tau_3)) + b_{22} f_2(x_2(t - \tau_1))) \\ &\quad + \beta x_2(t - \tau_1),\end{aligned}\quad (32)$$

FIGURE 1: The trajectory of $x_1(t)$ versus time in the system (31) with $\tau = 3\pi/4$.

FIGURE 2: The trajectory of $x_2(t)$ versus time in the system (31) with $\tau = 3\pi/4$.

where $\tau_1 = \tau_2 = \tau_3 = \tau$, $a_i = 0.5$, $b_{ij} = -0.3$, and $f_i(x) = \tanh(x)$, $i, j = 1, 2$. It is obvious that $(0, 0)$ is an equilibrium point of system (32). Choosing $\alpha = 1$, $\beta = 0$, by calculation, the periodic oscillatory behavior can arise through the Hopf bifurcation as $\tau_0 = 7.7063$; we can see Figures 3 and 5 ($\tau = 7.7063$, $\tau = 17.7063$). However, when $\alpha/5 < 2\beta < \alpha$, with complicated calculation, $(H2)'$ holds; by Theorem 11(i), the equilibrium point $(0, 0)$ is asymptotically stable for any discrete delays $\tau \geq \tau_0$. For the convenience of numerical

FIGURE 3: The trajectory of $x_1(t)$ and $x_2(t)$ versus time in the system (32) without control ($\tau = 7.7063$).

FIGURE 4: The trajectory of $x_1(t)$ and $x_2(t)$ versus time in the system (32) with control ($\tau = 7.7063$).

FIGURE 5: The trajectory of $x_1(t)$ and $x_2(t)$ versus time in the system (32) without control ($\tau > 7.7063$).

FIGURE 6: The trajectory of $x_1(t)$ and $x_2(t)$ versus time in the system (32) with control ($\tau > 7.7063$).

FIGURE 7: The trajectory of $x_1(t)$ and $x_2(t)$ versus time in the system (33) without control ($\tau = 1.2092$).

FIGURE 8: The trajectory of $x_1(t)$ and $x_2(t)$ versus time in the system (33) with control ($\tau = 1.2092$).

simulation, here we choose $\alpha = 0.75$, $\beta = 0.25$, and $\tau = 17.7063$ as an example. it can be seen in Figure 4 ($\tau = 7.7063$). Fix all coefficients of system (32) and let τ vary, and the waveforms $x_1(t)$, $x_2(t)$ without and with control are shown, respectively. Obviously, we obtain that the Hopf bifurcation in (32) without hybrid control could be eliminated by hybrid control; we can see Figure 6 ($\tau = 17.7063 > \tau_0$).

Example 2. Consider the following HNNs system with hybrid control:

$$\begin{aligned}\dot{x}_1 &= \alpha (-a_1 x_1(t) + b_{11} f_1(x_1(t - \tau_1)) + b_{12} f_2(x_2(t - \tau_2))) \\ &\quad + \beta x_1(t - \tau_1), \\ \dot{x}_2 &= \alpha (-a_2 x_2(t) + b_{21} f_1(x_1(t - \tau_3)) + b_{22} f_2(x_2(t - \tau_1))) \\ &\quad + \beta x_2(t - \tau_1),\end{aligned}\tag{33}$$

where $\tau_1 = \tau_2 = \tau_3 = \tau$, $a_i = 1$, $b_{ij} = -1$, and $f_i(x) = \tanh(x)$, $i, j = 1, 2$. It is obvious that $(0, 0)$ is an equilibrium point of system (33). Choosing $\alpha = 1$, $\beta = 0$, by calculation, we know $\tau_0 = 1.2092$. However, when $-\alpha < \beta < \alpha$, a family of periodic solutions bifurcates from $(0, 0)$ at τ'_0 (see Figure 7). Choosing $\alpha = 0.75$ and $\beta = 0.25$, with complicated calculation, we know $\tau'_0 = 2.2143$ (see Figure 8). Fix all coefficients of system (33) and let $\tau = \tau_0$; the waveforms $x_1(t)$, $x_2(t)$ without and with control are shown, respectively. However, the Hopf

FIGURE 9: The trajectory of $x_1(t)$ and $x_2(t)$ versus time in the system (33) with control ($\tau = 2.2143$).

bifurcation in (33) could be delayed by hybrid control which could be seen by Figure 9.

5. Conclusions

In this paper, the bifurcation and the bifurcation control problems have further been investigated for a HNNs model with delays. For the model, hybrid control strategy in which the parameter perturbation and time-delayed state feedback are combined and used to control various bifurcations in a continuous nonlinear dynamical system. It should be pointed out that, although Liu also have dealt with hybrid control, the time delayed feedback control used in our paper is more helpful than the controller in [22]. On the other hand, using parameter perturbation in this paper, our control strategy is more general than the other feedback control. Numerical simulations are given to justify the validity of hybrid controller in bifurcation control.

Acknowledgments

This research was supported by the Pre-research Foundation of PLA University of Science and Technology. Youth Research Foundation of College of Science of PLA University of Science and Technology.

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