Controllability of Fractional Neutral Stochastic Integro-Differential Systems with Infinite Delay

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This paper is concerned with the controllability of a class of fractional neutral stochastic integro-differential systems with infinite delay in an abstract space. By employing fractional calculus and Sadovskii’s fixed point principle without assuming severe compactness condition on the semigroup, a set of sufficient conditions are derived for achieving the controllability result.

1. Introduction

It is well known that the fractional calculus is a classical mathematical notion and is a generalization of ordinary differentiation and integration to arbitrary (noninteger) order. Nowadays, studying fractional-order calculus has become an active research field [1–7]. Much effort has been devoted to apply the fractional calculus to networks control. For example, Chen et al. [8], Delshad et al. [9], and Wang and Zhang [10] studied the synchronization for fractional-order complex dynamical networks; Zhang et al. [11] investigated a fractional order three-dimensional Hopfield neural network and pointed out that chaotic behaviors can emerge in a fractional network; Kaslik and Sivasundaram [12] discussed the local stability for fractional-order neural networks of Hopfield type by applying the linear stability theory of fractional-order system.

One of the emerging branches of this study is the theory of fractional evolution equations, say, evolution equations, where the integer derivative with respect to time is replaced by a derivative of fractional order. The increasing interest in this class of equations is motivated both by their application to problems from fluid dynamic traffic model, viscoelasticity, heat conduction in materials with memory, electrodynamics with memory, and also because they can be employed to approach nonlinear conservation laws (see [13] and references therein). In addition, neutral stochastic differential equations with infinite delay have become important in recent years as mathematical models of phenomena in both science and engineering, for instance, in the theory development in Gurtin and Pipkin [14] and Nunziato [15] for the description of heat conduction in materials with fading memory. It should be pointed out that the deterministic models often fluctuate due to noise, which is random or at least appears to be so. Therefore, we must move from deterministic problems to stochastic ones. We mention here the recent papers [16, 17] concerning the existence of mild solutions of fractional stochastic systems.

As one of the fundamental concepts in mathematical control theory, controllability plays an important role both in deterministic and stochastic control problems such as stabilization of unstable systems by feedback control. Roughly speaking, controllability generally means that it is possible to steer a dynamical control system from an arbitrary initial state to an arbitrary final state using the set of admissible controls. Controllability problems for different nonlinear stochastic systems in infinite dimensional spaces have been extensively studied in many papers; see [18–22] and references therein. We would also like to mention that the controllability for stochastic systems with infinite delay has been investigated by Balasubramaniam et al. [23, 24] and Ren et al. [25] using some abstract spaces. Nevertheless, to
the best of our knowledge, it seems that little is known about the controllability of fractional neutral stochastic differential equations with infinite delay, and the aim of this paper is to close this gap.

In this paper, we are interested in the controllability of a class of fractional neutral stochastic integro-differential systems with infinite delay of the follow form:

\[
\frac{d^\alpha x(t)}{dt^\alpha} + A x(t) + Bu(t) + \int_0^t \sigma(t-s,x(s))dW(s), \quad t \in J,
\]

such that \(\frac{d^\alpha x(t)}{dt^\alpha}\) takes value in a real separable Hilbert space \(H\) with inner product \((\cdot,\cdot)\) and norm \(\|\cdot\|\).

The fractional derivative \(D^\alpha\), \(\alpha \in (1/2,1]\), is understood in the Caputo sense. \(-A : \mathcal{D}(A) \subset H \to H\) is the infinitesimal generator of a analytic semigroup of a bounded linear operator \(S(t), t \geq 0, \in H\). Let \(K\) be another separable Hilbert space with inner product \((\cdot,\cdot)_{K}\) and norm \(\|\cdot\|_{K}\). \(W\) is a given \(K\)-valued Wiener process with a finite trace nuclear covariance operator \(Q \geq 0\) defined on a filtered complete probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\). The control function \(u(t)\) takes value in \(L^2(J,\mathbb{U})\) of admissible control functions for a separable Hilbert space \(\mathbb{U}\), and \(B\) is a bounded linear operator from \(\mathbb{U}\) into \(H\). The histories \(x_t : \Omega \to C_b=[x(t+\theta), \theta \in (-\infty,0]]\) belong to the phase space \(C_b\), which will be defined in Section 2. The initial data \(\phi = \{\phi(t), t \in (-\infty,0]\}\) is an \(\mathcal{F}_0\)-measurable, \(C_b\)-valued random variable independent of \(W\) with finite second moments, and \(G : J \times C_b \to H, \sigma : J \times J \times H \to \mathcal{L}_2(K,\mathbb{H})\) are appropriate mappings specified later (here, \(\mathcal{L}_2(K,\mathbb{H})\) denotes the space of all \(K\)-valued \(\mathbb{H}\)-valued \(\mathbb{L}_2\) operators from \(K\) into \(\mathbb{H}\), which is going to be defined later).

The structure of this paper is as follows. In Section 2, we briefly present some basic notations and preliminaries. The controllability result of system (1) is investigated by means of Sadovskii’s fixed point theorem and operator theory in Section 3. Conclusion is given in Section 4.

2. Preliminaries

For more details in this section, we refer the reader to Pazy [26], Da Prato and Zabczyk [27], and Samko et al. [28]. Throughout this paper, \((H,\|\cdot\|)\) and \((K,\|\cdot\|_K)\) denote two real separable Hilbert spaces. We denote by \(\mathcal{L}(K,\|\|)\) the set of all linear bounded operators from \(K\) into \(H\), equipped with the usual operator norm \(\|\cdot\|\). In this paper, we use the symbol \(\|\cdot\|\) to denote norms of operators regardless of the spaces potentially involved when no confusion possibly arises.

Let \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) be a filtered complete probability space satisfying the usual condition, which means that the filtration is a right continuous increasing family and \(\mathcal{F}_0\) contains all \(\mathbb{P}\)-null sets. \(W=(W_t)_{t \geq 0}\) is a \(Q\)-Wiener process defined on \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\) with covariance operator \(Q\) and norm \(\|\cdot\|\).

Let \(\frac{d^\alpha x(t)}{dt^\alpha} + A x(t) + Bu(t) + \int_0^t \sigma(t-s,x(s))dW(s), \quad t \in J,\)

such that \(\frac{d^\alpha x(t)}{dt^\alpha}\) takes value in \(H\) with inner product \((\cdot,\cdot)\) and norm \(\|\cdot\|\).

The integral that appear in the previous definitions are taken in Bochner’s sense.

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The integral that appear in the previous definitions are taken in Bochner’s sense.
Assume that \( h: (-\infty, 0] \rightarrow (0, +\infty) \) with \( l = \int_{-\infty}^{0} h(t) \, dt < +\infty \) is a continuous function. Recall that the abstract phase space \( \mathcal{E}_h \) is defined by

\[
\mathcal{E}_h = \left\{ \varphi: (-\infty, 0] \rightarrow H, \text{ for any } a > 0, \left( E[|\varphi(\theta)|^2]^2 \right)^{1/2} \text{ is bounded and measurable on } [-a, 0], \right. \]

\[
\int_{-\infty}^{0} h(s) \sup_{s \leq \theta \leq 0} \left( E[|\varphi(\theta)|^2]^2 \right)^{1/2} \, ds < +\infty \bigg\}.
\]

If \( \mathcal{E}_h \) is endowed with the norm

\[
\| \varphi \|_{\mathcal{E}_h} = \int_{-\infty}^{0} h(s) \sup_{s \leq \theta \leq 0} \left( E[|\varphi(\theta)|^2]^2 \right)^{1/2} \, ds, \quad \varphi \in \mathcal{E}_h,
\]

then \( (\mathcal{E}_h, \| \cdot \|_{\mathcal{E}_h}) \) is a Banach space (see Liu et al. [29]).

At the end of this section, we recall the fixed point theorem of Sadovskii [30].

**Lemma 4.** Let \( \Phi \) be a condensing operator on a Banach space \( H \); that is, \( \Phi \) is continuous, and take bounded sets into bounded sets, and \( \mu(\Phi(B)) \leq \mu(B) \) for any bounded set \( B \) of \( H \) with \( \mu(B) > 0 \). If \( \Phi(N) \subset N \) for a convex, closed, and bounded set \( N \) of \( H \), then \( \Phi \) has a fixed point in \( H \) (where \( \mu(\cdot) \) denotes Kuratowski’s measure of noncompactness.)

### 3. Main Results

In this section, we obtain controllability of system (1). We first present the definition of mild solutions.

**Definition 5.** An \( H \)-valued stochastic process \( \{x(t), t \in (-\infty, b]\} \) is said to be a mild solution of system (1) if

(i) \( x(t) \) is \( \mathcal{F}_T \)-adapted and measurable for each \( t \geq 0 \);

(ii) \( x(t) \) is continuous on \([0, b]\) almost surely and for each \( s \in [0, t] \), the function \( (t - s)^\alpha A_T(t - s)G(s, x_s) \) is integrable such that the following stochastic integral equation is verified:

\[
x(t) = S_\alpha(t) [x(0)] - G(t, x_t)
- \int_0^t (t - s)^\alpha A_T(t - s) G(s, x_s) \, ds
+ \int_0^t (t - s)^\alpha A_T(t - s) B u(s) \, ds
+ \int_0^t (t - s)^\alpha A_T(t - s)
\times \left[ \int_0^s \sigma(s, r, x_r) \, dW(r) \right] \, ds;
\]

(iii) \( x(t) = \phi(t) \) on \(( -\infty, 0] \) with \( \| \phi \|^2_{\mathcal{E}_h} < \infty \),

where

\[
S_\alpha(t) x = \int_0^\infty \eta_\alpha (\theta) S(\theta^2) x \, d\theta,
\]

\[
T_\alpha(t) x = \int_0^\alpha \theta \eta_\alpha (\theta) S(\theta^2) x \, d\theta
\]

with \( \eta \) a probability density function defined on \(( 0, +\infty ) \).

**Definition 6.** System (1) is said to be controllable on the interval \( J \), if for every initial stochastic process \( \phi \in \mathcal{E}_h \) defined on \((-\infty, 0]\), there exists a stochastic control \( u \in L^2(J, \mathcal{U}) \), which is adapted to the filtration \( \{ \mathcal{F}_t \}_{t \geq 0} \) such that the mild solution \( x(t) \) of (1) satisfies \( x(b) = x^* \), where \( x^* \) and \( b \) are preassigned terminal state and time, respectively.

The following properties of \( S_\alpha(t) \) and \( T_\alpha(t) \) that appeared in Zhou and Jiao [7] are useful.

**Lemma 7.** Under previous assumptions on \( S(t) \), \( t \geq 0 \), and \( A \), then

(i) for any fixed \( t \geq 0 \), \( S_\alpha(t) \) and \( T_\alpha(t) \) are linear and bounded operators such that for any \( x \in H \), \( |S_\alpha(t)x|_H \leq M|x|_H \) and \( |T_\alpha(t)x|_H \leq M|\alpha/(1 + \alpha)|/\Gamma(1 + \alpha) |x|_H \);

(ii) \( S_\alpha(t) \) and \( T_\alpha(t) \) are strongly continuous;

(iii) for any \( x \in H \), \( \beta \in (0, 1) \), and \( \theta \in (0, 1) \), one has

\[
\left\| A^\beta T_\alpha(t) \right\| \leq \frac{\alpha C_{\beta}}{\Gamma(2 - \beta)} \left( \frac{\Gamma(1 + \alpha) - \alpha}{\Gamma(1 + \alpha - \beta)} \right), \quad t \in [0, b].
\]

In this paper, we will work under the following assumptions.

(\( A_0 \)) -\( A \) is the infinitesimal generator of an analytic semigroup of bounded linear operators \( S(t) \) in \( H \); \( S(t), t > 0 \), is continuous in the uniform operator topology and \( S(t) \leq M \) for some \( M > 0 \).

(\( A_1 \)) The function \( G: J \times \mathcal{E}_h \rightarrow H \) is continuous, and there exist some constants \( M_G > 0 \), \( \beta \in (0, 1) \) with \( \alpha \beta > 1/2 \), such that \( G \) is \( \mathcal{H}_h \)-valued and

\[
E \left[ A^\beta G(t, x) - A^\beta G(t, y) \right]^2_{\mathcal{H}_h} \leq M_G \| x \|^2_{\mathcal{E}_h} \| y \|^2_{\mathcal{E}_h}, \quad x, y \in \mathcal{E}_h, \quad t \in J,
\]

\[
E \left[ A^\beta G(t, x) \right]^2_{\mathcal{H}_h} \leq M_G (\| x \|^2_{\mathcal{E}_h} + 1).
\]

(\( A_2 \)) \( \sigma: J \times \mathcal{E}_h \rightarrow L(K, H) \) satisfies the following:

(i) for each \( (t, s) \in D := J \times J \), \( \sigma(t, s, \cdot) : \mathcal{E}_h \rightarrow L(K, H) \) is continuous and for each \( x \in \mathcal{E}_h \), \( \sigma(t, \cdot, x) : D \rightarrow L(K, H) \) is strongly measurable;
(2b) there is a positive integrable function $m \in L^1([0, b])$ and a continuous nondecreasing function $\Xi: [0, \infty) \to (0, \infty)$ such that for every $(t, s, x) \in J \times J \times \mathcal{B}$, we have
\[
\lim_{r \to \infty} \frac{\Xi (r)}{r} ds = \Lambda < \infty;
\]
(2c) for any $x, y \in \mathcal{B}$, $t \geq 0$, there exists a positive constant $M_\xi$ such that
\[
\int_0^t E|\sigma (t, s, x) - \sigma (t, s, y)|^2 ds \leq M_\sigma \|x - y\|_{\mathcal{B}}^2.
\]
(A$_3$) The linear operator $W$ from $L^2(J, U)$ into $\mathcal{H}$ defined by
\[
Wu = \int_0^b (b-s)^{\alpha-1} A^\alpha (b-s) Bu (s) ds
\]
has an invertible operator $W^{-1}$ defined on $L^2(J, U)/\ker W$ (see [31]), and there exist a pair of constants $N_1, N_2 > 0$ such that
\[
\|W\|^2 \leq N_1, \quad \|W^{-1}\| \leq N_2.
\]
(A$_4$) There exists a compact set $U \subseteq \mathcal{H}$ such that $S(t) \sigma (t, r, \xi) \subseteq U$ for all $t \in J, 0 \leq r \leq s \leq b$ and $\xi \in \mathcal{B}$ with finite second moment.

Assume that the following relationship holds:
\[
20 \left(1 + 5N_1 N_2 N\right) I^2 \\
\times \left(M_0 + M_G P + N b^{-1} \text{Tr} QA \int_0^b m(s) ds\right) < 1,
\]
where
\[
N = \frac{b^{2\alpha}}{2\alpha - 1} \left(\frac{M \alpha}{\Gamma (1 + \alpha)} \right)^2, \quad M_0 = \|A^{-\beta}\|^2 M_G, \\
P = \frac{\alpha^2 C_{1-\beta}^2 (1 + \beta)}{\Gamma^2 (1 + \alpha \beta)} \frac{b^{2\alpha \beta}}{2\alpha \beta - 1}.
\]

Denote by $C((-\infty, b], \mathcal{H})$ the space of all continuous $\mathcal{H}$-valued stochastic processes $\xi(t), t \in (-\infty, b]$. Let
\[
\mathcal{B} = \{x : x \in C((-\infty, b], \mathcal{H}), x_0 = \phi \in \mathcal{B}_b\}.
\]
Set $\|x\|_{\mathcal{B}}$ to be a seminorm defined by
\[
\|x\|_{\mathcal{B}} = \|x_0\|_{\mathcal{B}} + \sup_{s \in [0, b]} \left(E|x(s)|^2\right)^{1/2}, \quad x \in \mathcal{B}.
\]
We have the following useful lemma that appeared in Liu et al. [29].

**Lemma 8.** Assume that $x \in \mathcal{B}_b$; then, for all $t \in J, x_t \in \mathcal{B}_b$. Moreover,
\[
\left(E|\sigma (t, s, x)|^2\right)^{1/2} \leq \|x_t\|_{\mathcal{B}} \leq l \sup_{s \in [0, b]} \left(E|x(s)|^2\right)^{1/2} + \|x_0\|_{\mathcal{B}},
\]
where $l = \int_0^b h(s) ds < \infty$.

The main object of this paper is to explain and prove the following theorem.

**Theorem 9.** Assume that assumptions (A$_0$)–(A$_3$) hold. Then, system (1) is controllable on $J$ provided that
\[
L := 3\left(1 + 3N_1 N_2 N\right) (M_0 + M_G P) + 9N_1 N_2 N^2 \text{Tr} Q M_\sigma \|^2 < 1.
\]

**Proof.** Using assumption (A$_1$), for an arbitrary process $x(\cdot)$, define the control process
\[
\begin{align*}
\mu_x (t) = W^{-1} \left\{ x^* - S_a (b) \left[ \phi (0) + G (0, \phi) \right] + G (b, x_b) \\
+ \int_0^b (b-s)^{\alpha-1} A^\alpha (b-s) G (s, x_s) ds \\
- \int_0^b (b-s)^{\alpha-1} A^\alpha (b-s) Bu_x (s) ds \\
\times \left[ \int_0^s \sigma (s, r, x_r) dW (r) \right] ds \right\} (t).
\end{align*}
\]

We transform (1) into a fixed point problem. Consider the operator $Q : \mathcal{B}_b \to \mathcal{B}_b$ defined by
\[
(Qx) (t) = \left\{ \begin{array}{ll}
\phi (t), & t \in (-\infty, 0] ; \\
S_a (t) \left[ \phi (0) + G (0, \phi) \right] - G (t, x_t) \\
- \int_0^t (t-s)^{\alpha-1} A^\alpha (t-s) G (s, x_s) ds \\
+ \int_0^t (t-s)^{\alpha-1} A^\alpha (t-s) Bu_x (s) ds \\
+ \int_0^t (t-s)^{\alpha-1} A^\alpha (t-s) \sigma (t, r, x_r) dW (r) ds, & t \in J.
\end{array} \right.
\]

It follows from Hölder inequality, Lemma 7, and assumption (A$_1$) that
\[
\begin{align*}
E \left| \int_0^t (t-s)^{\alpha-1} A^\alpha (t-s) G (s, x_s) ds \right|^2 \\
\leq E \left[ \int_0^t (t-s)^{\alpha-1} A^\alpha (t-s) G (s, x_s) \right|^2 ds.
\end{align*}
\]
\[
\left[ \int_{0}^{t} (t-s)^{\alpha(t-1)}(t-s)^{\alpha-1} \times A^{\alpha} G(s,x_s) \right] ds \right)^2 
\leq P \int_{0}^{t} (t-s)^{2(\alpha-1)} ds 
\times \int_{0}^{t} M_G (1 + |x_s|_{\mathcal{E}_b}) ds,
\]
which deduces that \((t-s)^{\alpha-1} A T_{\alpha}(t-s) G(s,x_s)\) is integrable on \(J\) by Bochner’s theorem (see [32]), where \(P\) is defined in (19).

In what follows, we will show that using the control \(u_\alpha(t)\), the operator \(Q\) has a fixed point, which is then a mild solution for system (1).

For \(\phi \in \mathcal{E}_b\), define
\[
\tilde{\phi}(t) = \begin{cases} 
\phi(t), & t \in (-\infty, 0] ; \\
S_{\alpha}(t) \phi(0), & t \in J.
\end{cases}
\]

Then, \(\tilde{\phi} \in \mathcal{E}_b\). Set \(x(t) = \tilde{\phi}(t) + z(t), t \in (-\infty, b]\). It is easy to check that \(x\) satisfies (1) if and only if \(z_0 = 0\) and
\[
\begin{align*}
z(t) &= S_{\alpha}(t) G(0,\phi) - G(t, \tilde{\phi}_t + z_t) \\
&\quad - \int_{0}^{t} (t-s)^{\alpha-1} A T_{\alpha}(t-s) G(s, \tilde{\phi}_s + z_s) ds \\
&\quad + \int_{0}^{t} (t-s)^{\alpha-1} T_{\alpha}(t-s) Bu_{\tilde{\phi}_s + z_s}(s) ds \\
&\quad + \int_{0}^{t} (t-s)^{\alpha-1} T_{\alpha}(t-s) \\
&\quad \times \left[ \int_{0}^{s} \sigma (s, \tau, \tilde{\phi}_\tau + z_\tau) dW(\tau) \right] ds,
\end{align*}
\]
where \(u_{\tilde{\phi}_s + z_s}\) is obtained by replacing \(x_s\) by \(\tilde{\phi}_s + z_s\) in (24). Let
\[
\mathcal{E}_{b_0} = \{ z \in \mathcal{E}_b, z_0 = 0 \in \mathcal{E}_h \}.
\]
For each \(z \in \mathcal{E}_{b_0}\), we have
\[
\|z\|_b = \|z_0\|_{\mathcal{E}_h} + \sup_{s \in [0,b]} \left( E|z(s)|^2 \right)^{1/2} = \sup_{s \in [0,b]} \left( E|z(s)|^2 \right)^{1/2}.
\]

Thus, \((\mathcal{E}_{b_0}, \| \cdot \|_b)\) is a Banach space. For \(q > 0\), set
\[
B_q = \{ y \in \mathcal{E}_{b_0}, \|y\|_b \leq q \};
\]
then, for each \(q\), \(B_q\) is clearly a bounded closed convex set in \(\mathcal{E}_{b_0}\). For \(z \in B_q\), from Lemma 8, we see that
\[
\|z_0 + \tilde{\phi}_0\|_{\mathcal{E}_h}^2 
\leq \left( \|z_0\|_{\mathcal{E}_h}^2 + \|\tilde{\phi}_0\|_{\mathcal{E}_h}^2 \right)
\leq 2 \left( \|z_0\|_{\mathcal{E}_h}^2 + \|\tilde{\phi}_0\|_{\mathcal{E}_h}^2 \right)
\leq 4 \left( \|z_0\|_{\mathcal{E}_h}^2 + \|\tilde{\phi}_0\|_{\mathcal{E}_h}^2 \right)
\leq 4t^2 \left( q + M^2 E|\phi(0)|^2 \right) + 4\|\phi\|_{\mathcal{E}_h}^2.
\]

Consider the map \(\Pi : \mathcal{E}_{b_0} \rightarrow \mathcal{E}_{b_0}\) defined by
\[
(\Pi z)(t) = \begin{cases} 
0, & t \in (-\infty, 0] ; \\
S_{\alpha}(t) G(0,\phi) - G(t, \tilde{\phi}_t + z_t) \\
- \int_{0}^{t} (t-s)^{\alpha-1} A T_{\alpha}(t-s) G(s, \tilde{\phi}_s + z_s) ds \\
- \int_{0}^{t} (t-s)^{\alpha-1} T_{\alpha}(t-s) Bu_{\tilde{\phi}_s + z_s}(s) ds \\
+ \int_{0}^{t} (t-s)^{\alpha-1} T_{\alpha}(t-s) Bu_{\tilde{\phi}_s + z_s}(s) ds \\
+ \int_{0}^{t} (t-s)^{\alpha-1} T_{\alpha}(t-s) \\
\times \left[ \int_{0}^{s} \sigma (s, \tau, \tilde{\phi}_\tau + z_\tau) dW(\tau) \right] ds, & t \in J.
\end{cases}
\]
A similar argument as (26), we can show that \(\Pi\) is well defined on \(B_q\) for each \(q > 0\). Note that the operator \(Q\) with a fixed point is equivalent to show that the operator \(\Pi\) has fixed point. To this end, we decompose \(\Pi = \Pi_1 + \Pi_2\), where the operators \(\Pi_1\) and \(\Pi_2\) are defined on \(B_q\), respectively, by
\[
(\Pi_1 z)(t) = S_{\alpha}(t) G(0,\phi) - G(t, \tilde{\phi}_t + z_t) \\
- \int_{0}^{t} (t-s)^{\alpha-1} A T_{\alpha}(t-s) G(s, \tilde{\phi}_s + z_s) ds \\
- \int_{0}^{t} (t-s)^{\alpha-1} T_{\alpha}(t-s) Bu_{\tilde{\phi}_s + z_s}(s) ds,
\]
\[
(\Pi_2 z)(t) = \int_{0}^{t} (t-s)^{\alpha-1} T_{\alpha}(t-s) \\
\times \left[ \int_{0}^{s} \sigma (s, \tau, \tilde{\phi}_\tau + z_\tau) dW(\tau) \right] ds.
\]
Thus, Theorem 9 follows from the next theorem. □

**Theorem 10.** Assume that the assumptions of Theorem 9 hold. Then, \(\Pi = \Pi_1 + \Pi_2\) satisfies all the conditions in Lemma 4.

**Proof.** The proof is followed by the several steps.

**Step 1.** There exists a positive number \(q\) such that \(\Pi(B_q) \subset B_q\). If it is not true, then for each positive number \(q\), there exists a
function $z^q(\cdot) \in B_q$, but $\Pi(z^q) \notin B_q$; that is, $E(\Pi z^q)(t)|^2_{H} > q$ for some $t = t(q) \in J$. An elementary calculation can show that

$$q \leq E|\Pi(z^q)(t)|^2_{H}$$

$$\leq 5E|\sigma(t)G(0,\phi)|^2_{H} + 5E|G(t,\phi_s + z^q_s)|^2_{H}$$

$$+ 5E\left|\int_{0}^{t}(t-s)^{\alpha - 1}AT_{\alpha}(t-s)\left(G(s,\phi_s + z^q_s)\right)ds\right|^2_{H}$$

$$+ 5E\left|\int_{0}^{t}(t-s)^{\alpha - 1}T_{\alpha}(t-s)\right.\left.\times \left[\int_{0}^{t}\sigma(s,\phi_s + z^q_s)dW(t)\right]ds\right|^2_{H}$$

$$+ 5E\left|\int_{0}^{t}(t-s)^{\alpha - 1}T_{\alpha}(t-s)Bu_{\phi_s + z^q_s}(s)ds\right|^2_{H}$$

$$= 5\sum_{i=1}^{I} I_i.$$ (35)

We estimate each term on the right hand side of (35). By virtue of Lemma 8, assumptions $(A_0)$-$(A_1)$, and (32), we obtain

$$I_1 + I_2 \leq M_0\left[2M^2(1 + \|\phi\|^2_{H}) + \left(2M^2 + 4E\|\phi(0)\|^2_{H}\right)\right]$$

$$\leq M_0\left[M^2\left(\beta + 1 + \|\phi\|^2_{H}\right)_{H} + \left(1 + \|\phi\|^2_{H}\right)_{H}\right]$$

$$(\text{36})$$

in view of Lemma 7 and H"{o}lder inequality, we have

$$I_3 \leq E\left[\left|\int_{0}^{t}(t-s)^{\alpha - 1}A^{1-\beta}T_{\alpha}(t-s)\right.\right.$$

$$\left.\times A^{\beta}G\left(s,\phi_s + z^q_s\right)ds\right|^2_{H}$$

$$\leq \frac{\alpha^2C^2_{1-\beta}}{\Gamma^2(1 + \alpha \beta)}E\int_{0}^{t}(t-s)^{2(\alpha \beta - 1)}ds$$

$$\times \int_{0}^{t}M_G\left(1 + \|\phi\|^2_{H}\right)_{H}ds$$

$$\leq \frac{M_G\alpha^2C^2_{1-\beta}}{\Gamma^2(1 + \alpha \beta)}E\int_{0}^{t}(t-s)^{2(\alpha \beta - 1)}ds$$

$$(\text{37})$$

where $M_0$ and $P$ are defined in (18) and (19), respectively. Applying Burkho"{o}lder-Davis-Gundy's inequality and assumptions $(A_2)$, we get

$$I_4 \leq E\left[\left|\int_{0}^{t}(t-s)^{\alpha - 1}T_{\alpha}(t-s)\right.\right.$$}

$$\times \left[\left[\int_{0}^{t}\sigma(s,\tau,\phi_s + z^q_s)dW(t)\right]ds\right|^2_{H}$$

$$\leq \left(\frac{M_0}{\Gamma(1 + \alpha)}\right)^2\int_{0}^{t}(t-s)^{2(\alpha \beta - 1)}ds$$

$$\times \int_{0}^{t}E\left[\left|\sigma(s,\tau,\phi_s + z^q_s)dW(t)\right|^2_{H}ds\right]$$

$$\leq \left(\frac{M_0}{\Gamma(1 + \alpha)}\right)^2\frac{b^{2\alpha - 1}}{2\alpha - 1}\int_{0}^{t}\text{Tr}Q\int_{0}^{t}\sigma(s,\tau,\phi_s + z^q_s)\right|^2_{H}d\tau ds$$

$$\times \int_{0}^{t}m(s)\Xi\left(\|\phi_s + z^q_s\|^2_{H}\right)ds$$

$$\leq \left(\frac{M_0}{\Gamma(1 + \alpha)}\right)^2\frac{\text{Tr}Qb^{2\alpha - 1}}{2\alpha - 1}\int_{0}^{t}m(s)ds.$$

On the other hand, in view of (24) and $(A_3)$, we have

$$E|Bu_{\phi_s + z^q_s}(t)|^2_{H}$$

$$\leq N_1N_2E\left[\|x^* - S_\alpha(b)\|\phi(0) + G(0,\phi)\right]$$

$$+ G\left(b,\phi_b + z^q_b\right)$$

$$+ \int_{0}^{b}(b-s)^{\alpha - 1}AT_{\alpha}(b-s)G\left(s,\phi_s + z^q_s\right)ds.$$
\[- \int_0^b (b-s)^{\alpha-1} T_a (b-s) \times \left[ \int_0^t \sigma (s, \tau, \tilde{\phi}_z + z_\tau') dW (\tau) \right] ds \right] (t) \right]_1^2 ; \]

thus, by the same procedure as (36)–(38), it follows that

\[
I_5 \leq \left( \frac{M_\alpha}{ \Gamma (1+\alpha) } \right)^2 \int_0^t (t-s)^{2\alpha-1} ds \int_0^t E \left[ Bu_{x+\tilde{\phi}} (s) \right] ds \]

\[
\leq N \left[ C_0 + 5N_1N_2 \times \left\{ 4M_G \| A^{-\beta} \|^2 \gamma + 4\| M_GPq \| + 2Nb^{-1} \text{Tr} Q \mathcal{E} \right. \]

\[
\times \left. (4t^2 (q + M^2 E |\bar{\phi} (0)|^2)_H + 4\| \bar{\phi} \|^2_{w_1}) \right\} \right] , \]

where

\[
C_0 = 5N_1N_2 \left[ E |x|^2 \right]_{l_1} + 2M^2 E \| \phi (0) \|^2_{l_1} + 2M^2 M_0 \left( 1 + \| \bar{\phi} \|^2_{w_1} \right) \]

\[
+ M_0 \left( 1 + 4t^2 M^2 E \| \phi (0) \|^2_{l_1} + 4\| \bar{\phi} \|^2_{w_1} \right) + M_GP \left( 1 + 4t^2 M^2 E \| \phi (0) \|^2_{l_1} + 4\| \bar{\phi} \|^2_{w_1} \right). \]

(40)

\[
M_0 \text{ and } P \text{ are defined in (18) and (19), respectively.}
\]

Combining these estimates (35) to (40) yields

\[
q \leq E\left[ \Pi (z^q) (t) \right]_{l_1}^2 \leq L_0 + 20M_0l^2 (1 + 5N_1N_2N) q \]

\[
+ 20N^2 M_GP (1 + 5N_1N_2N) q \]

\[
+ 5 \left( \frac{M_\alpha}{ \Gamma (1+\alpha) } \right)^2 \text{Tr} Q \mathcal{E} b^{2\alpha-1} \frac{1}{2\alpha-1} \]

\[
\times N \mathcal{E} \left( 4t^2 (q + M^2 E \| \phi (0) \|^2_{l_1}) + 4\| \phi \|^2_{w_1} \right) \int_0^b m (s) ds, \]

where

\[
L_0 = 5N + 5M^2 M_0 \left( 1 + \| \bar{\phi} \|^2_{w_1} \right) C_0 \]

\[
+ 5M_0 \left( 1 + 4t^2 M^2 E \| \phi (0) \|^2_{l_1} + 4\| \bar{\phi} \|^2_{w_1} \right) \]

\[
+ 5M_GP \left( 1 + 4t^2 M^2 E \| \phi (0) \|^2_{l_1} + 4\| \bar{\phi} \|^2_{w_1} \right). \]

(41)

Dividing both sides of (42) by \( q \) and taking \( q \rightarrow \infty \), we obtain that

\[
20 \left( 1 + 5N_1N_2N \right) l^2 \]

\[
\times \left( M_0 + M_GP + Nb^{-1} \text{Tr} Q \mathcal{E} \int_0^b m (s) ds \right) \geq 1, \]

(44)

which is a contradiction by assumption (A_5). Thus, for some positive number \( q, \Pi (B_q) \subset B_q \).

Step 2. \( \Pi_1 \) is a contractive mapping. Let \( x, v \in B_q \). From the assumptions on \( G \) and \( \sigma \), it is easy to verify that the following inequality holds:

\[
E \left[ u_{x+\tilde{\phi}} (s) - u_{y+\tilde{\phi}} (s) \right]_{l_1}^2 \leq 3N \left[ \left( A^{-\beta} \right) \| M_G \| x_b - v_{\tilde{\phi}} \|^2_{w_1} \]

\[
+ \frac{a^2 C_2^2 \Gamma^2 (1 + \beta)}{2\alpha-1} \int_0^b M_\alpha \| x_s - v_{\tilde{\phi}} \|^2_{w_1} ds \right) \]

\[
+ \left( \frac{M_\alpha}{ \Gamma (1+\alpha) } \right)^2 \frac{b^{2\alpha-1}}{2\alpha-1} \int_0^b M_\alpha \| x_s - v_{\tilde{\phi}} \|^2_{w_1} ds \right) . \]

(45)

Thus, by the assumptions, we have

\[
E\left[ (\Pi_1 x) (t) - (\Pi_1 v) (t) \right]_{l_1}^2 \leq 3 E \left[ G \left( t, \tilde{\phi}_z + x_t \right) - G \left( t, \tilde{\phi}_z + v_t \right) \right]_{l_1}^2 \]

\[
+ 3E \left[ \int_0^t (t-s)^{\alpha-1} T_a (t-s) \times \left[ G \left( s, \tilde{\phi}_z + x_s \right) - G \left( s, \tilde{\phi}_z + v_s \right) \right] ds \right]_{l_1}^2 \]

\[
+ 3E \left[ \int_0^t (t-s)^{\alpha-1} T_a (t-s) B \left( u_{\tilde{\phi}_z} - u_{\tilde{\phi}_z + v} \right) ds \right]_{l_1}^2 \]

\[
\leq 3M_\alpha l^2 \left( 1 + 3N_1N_2N \right) \sup_{0 \leq s \leq b} E \left[ (x (s) - v (s)) \right]_{l_1}^2 \]

\[
+ 3PM_G l^2 \left( 1 + 3N_1N_2N \right) \sup_{0 \leq s \leq b} E \left[ (x (s) - v (s)) \right]_{l_1}^2 \]

\[
+ 9N_1N_2 N^2 \text{Tr} Q M_\alpha l^2 \sup_{0 \leq s \leq b} E \left[ (x (s) - v (s)) \right]_{l_1}^2 \]

(46)

where we have used the fact that \( u_0 = v_0 = 0 \). Hence,

\[
\Pi_1 x - \Pi_1 v \leq L \| u - v \|^2_{l_1} ; \]

(47)

so, \( \Pi_1 \) is a contraction by (23).

Step 3. We show that the operator \( \Pi_2 \) is compact. Let \( q > 0 \) be such that \( \Pi_2 (B_q) \subset B_q \). The proof will be divided into the following claims.
Claim 1. \( \Pi_2 \) maps bounded sets into bounded sets in \( B_\eta \). Indeed, it is enough to show that there exists a positive constant \( \Delta \) such that for each \( t \in J, z \in B_\eta \), we have \( \| \Pi_2 z \|_\delta \leq \Delta \). If \( z \in B_\eta \), from (32), it follows that
\[
\| z + \Phi z \|_\delta \leq 4t^2 \left( q + \frac{M^2}{2} E \| \phi(0) \|_{H_\delta}^2 \right) + 4 \| \phi \|_{H_\delta}^2 := q'. \tag{48}
\]
By the similar argument as before, we get
\[
E \| \Pi_2 z (t) \|_{H_\delta}^2 \leq \left( \frac{M}{\Gamma(1 + \alpha)} \right)^2 \int_0^t (t-s)^{2(\alpha-1)} ds \times \left[ \int_0^t \sigma(s, \tau, \Phi z) dW(\tau) \right]_H^2 ds \\
+ \int_0^t \int_0^t \sigma(s, \tau, \Phi z + z) dW(\tau) dW(s) \\
\leq \left( \frac{M}{\Gamma(1 + \alpha)} \right)^2 \frac{\text{Tr} Q_b^{2(\alpha-1)}}{2\alpha - 1} \Xi(q') \int_0^t m(s) ds \\
\leq \left( \frac{M}{\Gamma(1 + \alpha)} \right)^2 \frac{\text{Tr} Q_b^{2(\alpha-1)}}{2\alpha - 1} \Xi(q') \int_0^b m(s) ds \\
:= \Delta. \tag{49}
\]
Therefore, for each \( z \in B_\eta \), it holds that \( \| \Pi_2 z \|_\delta \leq \Delta \).

Claim 2. The set \( \{ \Pi_2 z, z \in B_\eta \} \) is an equicontinuous family of functions on \( J \). Let \( 0 < \epsilon < t < b \) and \( \delta > 0 \) such that \( \| T_\alpha(s_1) - T_\alpha(s_2) \| < \epsilon \), for every \( s_1, s_2 \in J \) with \( |s_1 - s_2| < \delta \). For \( z \in B_\eta \), \( 0 < |h| < \delta, t + h \in J \), we have
\[
E \| \Pi_2 z (t + h) - \Pi_2 z (t) \|_{H_\delta}^2 \\
= E \int_0^{t+h} (t + h - s)^{\alpha-1} T_\alpha (t + h - s) \\
\times \left[ \int_0^t \sigma(s, \tau, \Phi z + z) dW(\tau) \right]_H^2 ds \\
- \int_0^t (t-s)^{\alpha-1} T_\alpha (t-s) \\
\times \left[ \int_0^t \sigma(s, \tau, \Phi z + z) dW(\tau) \right]_H^2 ds \\
\leq 3E \int_0^{t+h} (t + h - s)^{\alpha-1} T_\alpha (t + h - s) \\
\times \left[ \int_0^t \sigma(s, \tau, \Phi z + z) dW(\tau) \right]_H^2 ds \\
+ 3E \int_0^t [(t + h - s)^{\alpha-1} - (t - s)^{\alpha-1}] T_\alpha (t + h - s) \\
\times \left[ \int_0^t \sigma(s, \tau, \Phi z + z) dW(\tau) \right]_H^2 ds \\
\leq 3Z(\alpha) \int_0^{t+h} (t + h - s)^{\alpha-1} T_\alpha (t + h - s) \\
\times \left[ \int_0^t \sigma(s, \tau, \Phi z + z) dW(\tau) \right]_H^2 ds \\
+ 3Z(\alpha) \int_0^t \left[ (t + h - s)^{\alpha-1} - (t - s)^{\alpha-1} \right] T_\alpha (t + h - s) \\
\times \left[ \int_0^t \sigma(s, \tau, \Phi z + z) dW(\tau) \right]_H^2 ds,
\]
with \( Z(\alpha) = (M\alpha/\Gamma(1 + \alpha))^2 \). Therefore, for \( \epsilon \) sufficiently small, the right hand side of (51) tends to zero as \( h \to 0 \). Thus, the set \( \{ \Pi_2 z, z \in B_\eta \} \) is equicontinuous by assumption (A1).

Claim 3. \( \Pi_2 \) maps \( B_\eta \) into a precompact set in \( B_\eta \). Let \( 0 < t \leq b \) be fixed and \( \epsilon \) a real number satisfying \( 0 < \epsilon < t \). For \( \delta > 0 \), define an operator \( \Pi_2^\delta \) on \( B_\eta \) by
\[
(\Pi_2^\delta z)(t) = \alpha \int_0^{t-\epsilon} \int_0^\infty \theta(t-s)^{\alpha-1} \eta_\alpha(\theta) S((t-s)^\alpha \theta) \\
\times \left[ \int_0^t \sigma(s, \tau, \Phi z + z) dW(\tau) \right] d\theta ds.
\]
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\[ S(e^\alpha \delta) \alpha \int_{\delta}^{\infty} (t - s)^{\alpha - 1} \eta_s(\theta) S \times ((t - s)^{\alpha \theta}) \int_{0}^{\alpha} \sigma(s, \tau, \bar{\phi}_s + z_s) dW(\tau) d\theta ds. \]
\]
\[ \int_{0}^{t} \left[ \int_{\delta}^{\infty} \theta(t - s)^{\alpha - 1} \eta_s(\theta) S \times ((t - s)^{\alpha \theta}) \int_{0}^{\alpha} \sigma(s, \tau, \bar{\phi}_s + z_s) dW(\tau) d\theta ds \right] d\tau. \]
\[ \int_{0}^{t} \left[ \int_{\delta}^{\infty} \theta(t - s)^{\alpha - 1} \eta_s(\theta) S \times ((t - s)^{\alpha \theta}) \int_{0}^{\alpha} \sigma(s, \tau, \bar{\phi}_s + z_s) dW(\tau) d\theta ds \right] d\tau. \]
\[ \int_{0}^{t} \left[ \int_{\delta}^{\infty} \theta(t - s)^{\alpha - 1} \eta_s(\theta) S \times ((t - s)^{\alpha \theta}) \int_{0}^{\alpha} \sigma(s, \tau, \bar{\phi}_s + z_s) dW(\tau) d\theta ds \right] d\tau. \]
\[ \int_{0}^{t} \left[ \int_{\delta}^{\infty} \theta(t - s)^{\alpha - 1} \eta_s(\theta) S \times ((t - s)^{\alpha \theta}) \int_{0}^{\alpha} \sigma(s, \tau, \bar{\phi}_s + z_s) dW(\tau) d\theta ds \right] d\tau. \]
\[ \int_{0}^{t} \left[ \int_{\delta}^{\infty} \theta(t - s)^{\alpha - 1} \eta_s(\theta) S \times ((t - s)^{\alpha \theta}) \int_{0}^{\alpha} \sigma(s, \tau, \bar{\phi}_s + z_s) dW(\tau) d\theta ds \right] d\tau. \]
\[ \int_{0}^{t} \left[ \int_{\delta}^{\infty} \theta(t - s)^{\alpha - 1} \eta_s(\theta) S \times ((t - s)^{\alpha \theta}) \int_{0}^{\alpha} \sigma(s, \tau, \bar{\phi}_s + z_s) dW(\tau) d\theta ds \right] d\tau. \]


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