Research Article

An Alternating Direction Implicit Method for Solving Projected Generalized Continuous-Time Sylvester Equations

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We present the generalized low-rank alternating direction implicit method and the low-rank cyclic Smith method to solve projected generalized continuous-time Sylvester equations with low-rank right-hand sides. Such equations arise in control theory including the computation of inner products and $\mathbb{H}_2$ norms and the model reduction based on balanced truncation for descriptor systems. The requirements of these methods are moderate with respect to both computational cost and memory. Numerical experiments presented in this paper show the effectiveness of the proposed methods.

1. Introduction

In this paper we consider the projected generalized continuous-time Sylvester equation of the form

$$\begin{aligned} AXC + DXB + P_{l,1}EP_{r,2} &= 0, \\ X &= P_{r,1}XP_{l,2}, \end{aligned} \quad (1)$$

where $A, D \in \mathbb{R}^{n \times n}$, $B, C \in \mathbb{R}^{m \times m}$, $E \in \mathbb{R}^{n \times m}$, and $X \in \mathbb{R}^{n \times m}$ are the sought-after solution. Here, $P_{r,1}$ and $P_{r,2}$ are the spectral projectors onto the right deflating subspaces corresponding to the finite eigenvalues of the pencils $\lambda D - A$ and $\lambda C - B$, respectively; $Q_{r,1} = I - P_{r,1}$ and $Q_{r,2} = I - P_{r,2}$ are the spectral projectors onto the right deflating subspaces corresponding to the eigenvalue at infinity. The spectral projectors onto the left deflating subspaces corresponding to the finite eigenvalues of $\lambda D - A$ and $\lambda C - B$ are denoted by $P_{l,1}$ and $P_{l,2}$, respectively, while $Q_{l,1} = I - P_{l,1}$ and $Q_{l,2} = I - P_{l,2}$ are the spectral projectors onto the left deflating subspaces corresponding to the eigenvalue at infinity. We assume that the matrices C and D are singular, but the pencils $\lambda D - A$ and $\lambda C - B$ are regular; that is, $\det(\lambda D - A)$ and $\det(\lambda C - B)$ are not identically zero. Under the assumption, the pencils $\lambda D - A$ and $\lambda C - B$ have the Weierstrass canonical forms: there exist

nonsingular $n \times n$ matrices W_1, T_1 and $m \times m$ matrices W_2, T_2 such that

$$\begin{aligned} D &= W_1 \begin{bmatrix} I & 0 \\ 0 & N^{(A)} \end{bmatrix} T_1, & A &= W_1 \begin{bmatrix} J^{(A)} & 0 \\ 0 & I \end{bmatrix} T_1, \\ C &= W_2 \begin{bmatrix} I & 0 \\ 0 & N^{(B)} \end{bmatrix} T_2, & B &= W_2 \begin{bmatrix} J^{(B)} & 0 \\ 0 & I \end{bmatrix} T_2, \end{aligned} \quad (2)$$

where $J^{(A)}$, $J^{(B)}$, $N^{(A)}$, and $N^{(B)}$ are block diagonal matrices with each diagonal block being a Jordan block. The eigenvalues of $J^{(A)}$ and $J^{(B)}$ are the finite eigenvalues of the pencils $\lambda D - A$ and $\lambda C - B$, respectively. $N^{(A)}$ and $N^{(B)}$ correspond to the eigenvalue at infinity. The index γ of nilpotency of $N^{(A)}$ is called the index of the pencil $\lambda D - A$. Using (2), $P_{l,1}$, $P_{l,2}$, $P_{r,1}$, and $P_{r,2}$ can be expressed as

$$\begin{aligned} P_{l,1} &= W_1 \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} W_1^{-1}, & P_{l,2} &= W_2 \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} W_2^{-1}, \\ P_{r,1} &= T_1^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_1, & P_{r,2} &= T_2^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} T_2. \end{aligned} \quad (3)$$

A number of numerical solution methods have been proposed for the standard/generalized Lyapunov and Sylvester

equations. Two classical direct methods are the Bartels-Stewart method [1] and the Hammarling method [2]. These methods need to compute the real Schur forms/generalized real Schur forms of the underlying matrices/matrix pencils by means of the QR/QZ algorithm [3]. Besides direct methods, we mention, among several iterative methods, the Smith method [4], the alternating direction implicit iteration (ADI) method [5, 6], the Smith(l) method [7], the low-rank Smith methods [7–9], the Cholesky factor-alternating direction implicit method [10–13], and the (generalized) matrix sign function method [14–25].

As $m = n$, $B = A^T$, and $C = D^T$, (1) is referred to as the projected generalized continuous-time Lyapunov equation, which arises in stability analysis and control design problems for descriptor systems including the characterization of controllability and observability properties, balanced truncation model order reduction, determining the minimal and balanced realizations, and computing H_2 and Hankel norms; see [26–31] and the references therein. If the pencil $\lambda D - A$ is c-stable, that is, all its finite eigenvalues have negative real part, then the projected generalized Lyapunov equation has a unique solution for each E , and if, additionally, E is symmetric and positive semidefinite, then the solution X is symmetric and positive semidefinite; see, for example, [32] for details. Recently, several numerical methods have been proposed in the literature for solving the projected generalized Lyapunov equation. In [33], two direct methods, the generalized Bartels-Stewart method and the generalized Hammarling method, were proposed for the projected generalized Lyapunov equation. The generalized Hammarling method is designed to obtain the Cholesky factor of the solution. These two methods are based on the generalized real Schur form of the pencil $\lambda D - A$ and require $\mathcal{O}(n^3)$ flops and $\mathcal{O}(n^2)$ memory. Iterative methods to solve the projected generalized Lyapunov equation have also been proposed. Stykel [31] extended the ADI method and the Smith method to the projected equation. Moreover, their low-rank versions were also presented, which could be used to compute low-rank approximations to the solution. These methods are especially suitable for large sparse equations with low-rank E . Another iterative method for the projected generalized Lyapunov equation is the modified generalized matrix sign function method [25]. Unlike the classical generalized matrix sign function method, the variant converges quadratically independent of the index of the underlying matrix pencil; see [25] for more details.

The projected generalized Sylvester equation (1) also plays an important role in control theory for descriptor systems.

Let $H(s) = G(sD - A)^{-1}F$ and $\tilde{H}(s) = \tilde{G}(sC - B)^{-1}\tilde{F}$ be two c-stable systems. As shown in [34], $H(s)$ and $\tilde{H}(s)$ can be decomposed into

$$H(s) = H_{\text{sp}}(s) + P(s), \quad \tilde{H}(s) = \tilde{H}_{\text{sp}}(s) + \tilde{P}(s), \quad (4)$$

where

$$\begin{aligned} H_{\text{sp}}(s) &= G(sD - A)^{-1}P_{l,1}F, \\ P(s) &= G(I - P_{r,1})(sD - A)^{-1}(I - P_{l,1})F, \end{aligned}$$

$$\tilde{H}_{\text{sp}}(s) = \tilde{G}(sC - B)^{-1}P_{l,2}F, \quad (5)$$

$$\tilde{P}(s) = \tilde{G}(I - P_{r,2})(sC - B)^{-1}(I - P_{l,2})\tilde{F}.$$

Here, $H_{\text{sp}}(s), \tilde{H}_{\text{sp}}(s)$ are called the strictly proper parts of $H(s), \tilde{H}(s)$, while $P(s), \tilde{P}(s)$ are called the polynomial parts of $H(s), \tilde{H}(s)$, respectively. Define the $\mathbb{H}_2$ inner product of $H(s)$ and $\tilde{H}(s)$ by

$$\begin{aligned} \langle H(s), \tilde{H}(s) \rangle_{\mathbb{H}_2} &= \langle H_{\text{sp}}(s), \tilde{H}_{\text{sp}}(s) \rangle_{\mathbb{H}_2} + \langle P(s), \tilde{P}(s) \rangle_{\mathbb{L}_2} \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{trace}(\overline{H_{\text{sp}}(i\omega)} \tilde{H}_{\text{sp}}^T(i\omega)) d\omega \\ &\quad + \frac{1}{2\pi} \int_0^{2\pi} \text{trace}(\overline{P(e^{i\omega})} \tilde{P}^T(e^{i\omega})) d\omega, \end{aligned} \quad (6)$$

where the notation $\text{trace}(\cdot)$ denotes the trace of a matrix. Then, the $\mathbb{H}_2$ norm of $H(s)$ induced by the $\mathbb{H}_2$ inner product is

$$\begin{aligned} \|H(s)\|_{\mathbb{H}_2} &= \sqrt{\|H_{\text{sp}}(s)\|_{\mathbb{H}_2}^2 + \|P(s)\|_{\mathbb{L}_2}^2} \\ &= \sqrt{\langle H_{\text{sp}}(s), H_{\text{sp}}(s) \rangle_{\mathbb{H}_2} + \langle P(s), P(s) \rangle_{\mathbb{L}_2}}. \end{aligned} \quad (7)$$

For $F, G^T \in \mathbb{R}^n, \tilde{F}, \tilde{G}^T \in \mathbb{R}^m$, it can be shown that $\langle H_{\text{sp}}(s), \tilde{H}_{\text{sp}}(s) \rangle_{\mathbb{H}_2}$ could be expressed as

$$\langle H_{\text{sp}}(s), \tilde{H}_{\text{sp}}(s) \rangle_{\mathbb{H}_2} = \text{trace}(GX\tilde{F}), \quad (8)$$

where X is the solution of the projected generalized continuous-time Sylvester equation (1) with $E = F\tilde{G}$. Therefore, the norm $\|H_{\text{sp}}(s)\|_{\mathbb{H}_2}$ can be computed by

$$\|H_{\text{sp}}(s)\|_{\mathbb{H}_2} = \sqrt{\text{trace}(GX\tilde{F})}, \quad (9)$$

where X is the solution of the following projected generalized continuous-time Sylvester equation:

$$\begin{aligned} AXD + DXA + P_{l,1}FGP_{r,1} &= 0, \\ X &= P_{r,1}XP_{l,1}. \end{aligned} \quad (10)$$

We now present the application of the projected generalized continuous-time Sylvester equation in model reduction of c-stable descriptor systems. Let $\mathcal{E}_{\text{pc}}$ and $\mathcal{E}_{\text{po}}$ denote the proper controllability and observability Gramians of the continuous-time descriptor system $H(s) = G(sD - A)^{-1}F$, respectively. For the definitions of these Gramians and their applications in model reduction of $H(s)$, the reader is referred to [30]. The proper Gramians $\mathcal{E}_{\text{pc}}$ and $\mathcal{E}_{\text{po}}$ are the unique symmetric, positive semidefinite solutions of the projected generalized continuous-time controllability and observability Lyapunov equations associated with the system $H(s)$, respectively.

There exists another kind of Gramians, called cross-Gramians; for a standard square system, see [35]. We now extend the definition to square descriptor systems. The proper cross-Gramian for a c -stable square descriptor system $H(s) = G(sD - A)^{-1}F$ is the solution X of the projected generalized continuous-time Sylvester equation (10). It can be shown that the proper cross-Gramian X satisfies

$$(XD)^2 = \mathcal{E}_{pc} D^T \mathcal{E}_{po} D. \quad (11)$$

By making use of the relation above and a similar approach as proposed in [35], we can design a cross-Gramian-based model reduction method for the square descriptor system $H(s)$.

In this paper, we propose two iterative methods for solving the projected generalized Sylvester equation (1). This work presented here is an extension of [31]. Numerical experiments show the effectiveness of the proposed methods.

We note that the perturbation analysis for (1) has been presented in [36]. The following theorem, for example, [37], gives sufficient conditions for the existence, uniqueness, and analytic formula of the solution of the projected generalized continuous-time Sylvester equation (1).

Theorem 1. *Let $\lambda D - A$ and $\lambda C - B$ be regular pencils with finite eigenvalues $\{\mu_1, \mu_2, \dots, \mu_{n_1}\}$ and $\{\nu_1, \nu_2, \dots, \nu_{m_1}\}$ counted according to their multiplicities, respectively. Then, the projected generalized continuous-time Sylvester equation (1) has a unique solution for every E if $\mu_i + \nu_j \neq 0$ for any $i = 1, 2, \dots, n_1$ and $j = 1, 2, \dots, m_1$.*

Moreover, if $\lambda D - A$ and $\lambda C - B$ are c -stable, that is, all their finite eigenvalues have negative real part, then X can be expressed as

$$X = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-i\omega D - A)^{-1} P_{l,1} E P_{r,2} (i\omega C - B)^{-1} d\omega. \quad (12)$$

Throughout this paper, we adopt the following notations. The square identity and zero matrices are denoted by I and 0 , respectively. The spaces of $m \times n$ real matrices are denoted by $\mathbb{R}^{m \times n}$. The 2-norm and the Frobenius matrix norm are denoted by $\|\cdot\|_2$ and $\|\cdot\|_F$, respectively. The superscript “ T ” denotes the transposition of a vector or a matrix. $\rho(A)$ is the spectral radius of square matrix A , and $\kappa(A) = \|A\|_2 \|A^{-1}\|_2$ is the spectral condition number of A . The open left and right half-plane are denoted by $\mathbb{C}^-$ and $\mathbb{C}^+$, respectively. We will also adopt MATLAB-like convention to access the entries of vectors and matrices. For a matrix X , $X_{(i,j)}$ is X 's (i, j) th entry; X 's submatrices $X_{(k:l,i:j)}$, $X_{(k:l,:)}$, and $X_{(:,i:j)}$ consist of intersections of row k to row l and column i to column j , row k to row l , and column i to column j , respectively.

The remainder of the paper is organized as follows. In Section 2, we propose the generalized low-rank alternating direction implicit method for the solution of (1). In Section 3, we discuss the choice of shift parameters. In Section 4, we show how to solve the projected Sylvester equations by the low-rank cyclic Smith method. Section 5 is devoted to some numerical tests. Some concluding remarks are given in the last section.

2. Alternating Direction Implicit Method

The alternating direction implicit (ADI) method was first introduced by Peaceman and Rachford [38] to solve linear systems arising from the discretization of elliptic boundary value problems and then used in [5–7, 12, 13, 31] to solve Lyapunov or Sylvester matrix equations. In this section, we consider how to generalize the ADI method for iteratively solving (1).

We always assume that the pencils $\lambda D - A$ and $\lambda C - B$ are c -stable; that is, all their finite eigenvalues have negative real part. Hence, the matrices A and B are nonsingular. Multiplying the first equation in (1) on the left by A^{-1} and on the right by B^{-1} , we get the following projected standard Sylvester equation:

$$\begin{aligned} (A^{-1}D)X + X(CB^{-1}) + A^{-1}P_{l,1}EP_{r,2}B^{-1} &= 0, \\ X &= P_{r,1}XP_{l,2}. \end{aligned} \quad (13)$$

The iterates X_k of the ADI iteration for (13) are usually generated by the alternating solution of two linear systems with multiple right-hand sides:

$$\begin{aligned} (A^{-1}D - \beta_k I)X_{k-(1/2)} &= X_{k-1}(CB^{-1} + \beta_k I) + A^{-1}P_{l,1}EP_{r,2}B^{-1}, \\ X_k(CB^{-1} + \alpha_k I) &= (A^{-1}D - \alpha_k I)X_{k-(1/2)} - A^{-1}P_{l,1}EP_{r,2}B^{-1}, \end{aligned} \quad (14)$$

where $X_0 = 0$ and the shift parameters $\{\alpha_i\}_{i=1}^k$ and $\{\beta_i\}_{i=1}^k$ are elements of $\mathbb{C}^-$ and $\mathbb{C}^+$, respectively. These two equations are equivalent to the following single iteration step:

$$\begin{aligned} X_k &= (A^{-1}D - \beta_k I)^{-1} (A^{-1}D - \alpha_k I) X_{k-1} \\ &\quad \times (CB^{-1} + \beta_k I) (CB^{-1} + \alpha_k I)^{-1} \\ &\quad + (\beta_k - \alpha_k) (A^{-1}D - \beta_k I)^{-1} A^{-1}P_{l,1}EP_{r,2}B^{-1} \\ &\quad \times (CB^{-1} + \alpha_k I)^{-1}. \end{aligned} \quad (15)$$

We have from

$$\begin{aligned} P_{r,1}(A^{-1}D - \beta_k I)^{-1} &= (A^{-1}D - \beta_k I)^{-1} P_{r,1}, \\ P_{r,1}(A^{-1}D - \alpha_k I) &= (A^{-1}D - \alpha_k I) P_{r,1}, \\ (CB^{-1} + \alpha_k I)^{-1} P_{l,2} &= P_{l,2}(CB^{-1} + \alpha_k I)^{-1}, \\ (CB^{-1} + \beta_k I) P_{l,2} &= P_{l,2}(CB^{-1} + \beta_k I) \end{aligned} \quad (16)$$

that $X_k = P_{r,1} X_k P_{l,2}$, that is, the second equation in (1) and (13) is satisfied exactly. We can rewrite iteration (15) as

$$\begin{aligned} X_k &= (D - \beta_k A)^{-1} (D - \alpha_k A) X_{k-1} \\ &\quad \times (C + \beta_k B) (C + \alpha_k B)^{-1} \\ &\quad + (\beta_k - \alpha_k) (D - \beta_k A)^{-1} P_{l,1} E P_{r,2} (C + \alpha_k B)^{-1}. \end{aligned} \quad (17)$$

Let X denote the exact solution of (1). Then it is easily verified that the error matrix $X - X_k$ obeys the recursion:

$$\begin{aligned} X - X_k &= (D - \beta_k A)^{-1} (D - \alpha_k A) (X - X_{k-1}) \\ &\quad \times (C + \beta_k B) (C + \alpha_k B)^{-1} \\ &= \dots = \mathcal{A}_k X \mathcal{B}_k, \end{aligned} \quad (18)$$

where

$$\begin{aligned} \mathcal{A}_k &= P_{r,1} (D - \beta_k A)^{-1} (D - \alpha_k A) \dots (D - \beta_1 A)^{-1} (D - \alpha_1 A), \\ \mathcal{B}_k &= (C + \beta_1 B) (C + \alpha_1 B)^{-1} \dots (C + \beta_k B) (C + \alpha_k B)^{-1} P_{l,2}. \end{aligned} \quad (19)$$

By using the Weierstrass canonical form (2)–(3), we have

$$\begin{aligned} \mathcal{A}_k &= T_1^{-1} \begin{bmatrix} J_k^{(A)} & 0 \\ 0 & 0 \end{bmatrix} T_1, \\ \mathcal{B}_k &= W_2 \begin{bmatrix} J_k^{(B)} & 0 \\ 0 & 0 \end{bmatrix} W_2^{-1} \end{aligned} \quad (20)$$

with

$$\begin{aligned} J_k^{(A)} &= (I - \beta_k J^{(A)})^{-1} (I - \alpha_k J^{(A)}) \\ &\quad \dots (I - \beta_1 J^{(A)})^{-1} (I - \alpha_1 J^{(A)}), \\ J_k^{(B)} &= (I + \beta_1 J^{(B)}) (I + \alpha_1 J^{(B)})^{-1} \\ &\quad \dots (I + \beta_k J^{(B)}) (I + \alpha_k J^{(B)})^{-1}. \end{aligned} \quad (21)$$

This implies that if $\{\alpha_j\}_{j=1}^k$ contains all finite eigenvalues (multiple eigenvalues counted by their algebraic multiplicities) of $A^{-1}D$ or if $\{\beta_j\}_{j=1}^k$ contains all finite eigenvalues of $-B^{-1}C$, then $X - X_k \equiv 0$. This is due to the Cayley-Hamilton theorem [39], which states that $p(A) \equiv 0$ for A 's characteristic polynomial $p(\lambda) = \det(\lambda I - A)$ and $q(B) \equiv 0$ for $q(\lambda) = \det(\lambda I - B)$. From (18), we can see that the parameters $\{\alpha_i\}_{i=1}^k$ and $\{\beta_i\}_{i=1}^k$ should be chosen to achieve $\min_{\alpha_i, \beta_i} \|\mathcal{A}_k\|_2 \cdot \|\mathcal{B}_k\|_2$; we will discuss the details of the choice of parameters in the next section.

About the k th approximate solution X_k of the ADI method, by using (18)–(20), we have the following estimate.

Theorem 2. Assume that the pencils $\lambda D - A$ and $\lambda C - B$ are in Weierstrass form (2) with $J^{(A)}$ and $J^{(B)}$ being diagonal. Then

$$\|X - X_k\|_2 \leq \kappa(T_1) \kappa(W_2) \rho(\mathcal{A}_k) \rho(\mathcal{B}_k) \|X\|_2. \quad (22)$$

Note that the solution X is explicitly computed by the ADI iteration (17), so the storage requirement is $\mathcal{O}(mn)$. One should notice that in many cases the storage requirement is the limiting factor rather than the amount of computation. We note that low-rank schemes are the only existing methods that can effectively solve large-scale Lyapunov/Sylvester equations. Assume that the low-rank right-hand side E has the factored form $E = FG$ with $F \in \mathbb{R}^{n \times r}$ and $G \in \mathbb{R}^{r \times m}$. Instead of explicitly forming the solution X , the low-rank method computes and stores approximate solutions in low-rank factored form. If the numerical rank l of X is much smaller than $\min\{m, n\}$, that is, $l \ll \min\{m, n\}$, then the storage is reduced from $\mathcal{O}(mn)$ to $\mathcal{O}(ml)$ or $\mathcal{O}(nl)$.

The key idea in the low-rank version of the ADI iteration is to rewrite the iteration X_k in (17) as an outer product:

$$X_k = Y_k H_k Z_k. \quad (23)$$

This is always possible since starting with the initial guess $X_0 = 0_{n \times m}$. The low-rank alternating direction implicit (LR-ADI) method is based on (17). Replacing X_{k-1} with $Y_{k-1} H_{k-1} Z_{k-1}$, (17) can be reformulated in terms of the low-rank factors as

$$\begin{aligned} X_k &= [(D - \beta_k A)^{-1} P_{l,1} F \quad (D - \beta_k A)^{-1} (D - \alpha_k A) Y_{k-1}] \\ &\quad \times \begin{bmatrix} (\beta_k - \alpha_k) I \\ H_{k-1} \end{bmatrix} \\ &\quad \times \begin{bmatrix} G P_{r,2} (C + \alpha_k B)^{-1} \\ Z_{k-1} (C + \beta_k B) (C + \alpha_k B)^{-1} \end{bmatrix} \\ &\equiv Y_k H_k Z_k, \end{aligned} \quad (24)$$

where

$$\begin{aligned} Y_k &= [(D - \beta_k A)^{-1} P_{l,1} F \quad (D - \beta_k A)^{-1} (D - \alpha_k A) Y_{k-1}], \\ H_k &= \begin{bmatrix} (\beta_k - \alpha_k) I \\ H_{k-1} \end{bmatrix}, \\ Z_k &= \begin{bmatrix} G P_{r,2} (C + \alpha_k B)^{-1} \\ Z_{k-1} (C + \beta_k B) (C + \alpha_k B)^{-1} \end{bmatrix}. \end{aligned} \quad (25)$$

From the fact that Y_0, H_0 , and Z_0 are all zero matrices, it can be seen that Y_k is $n \times kr$, H_k is $kr \times kr$, and Z_k is $kr \times m$. Thus the rank of X_k is no more than kr . Since the order of the ADI parameters $\{\alpha_i\}_{i=1}^k$ and $\{\beta_i\}_{i=1}^k$ is not important, the ordering of $\{\alpha_i\}_{i=1}^k$ and $\{\beta_i\}_{i=1}^k$ can be reversed. As shown in [12], we have the following iterative scheme:

$$Y_k = [Y^{(1)} \quad Y^{(2)} \quad \dots \quad Y^{(k)}], \quad \text{with}$$

$$Y^{(1)} = (D - \beta_1 A)^{-1} P_{l,1} F,$$

$$Y^{(i+1)} = (D - \beta_{i+1} A)^{-1} (D - \alpha_i A) Y^{(i)} \quad (i = 1, 2, \dots, k-1),$$

Input: $A, D \in \mathbb{R}^{n \times n}$, $B, C \in \mathbb{R}^{m \times m}$, $F \in \mathbb{R}^{n \times r}$ and $G \in \mathbb{R}^{r \times m}$ with $\lambda D - A$ and $\lambda C - B$ are c -stable. The number of ADI steps k and the ADI shifts $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ and $\{\beta_1, \beta_2, \dots, \beta_k\}$ computed by Algorithm 2.

Output: Y, H and Z such that $X = YHZ$ is an approximate solution of the projected generalized Sylvester equation (1) with $E = FG$.

(1) Let $Y_{(:,1:r)} = (D - \beta_1 A)^{-1} P_{1,1} F$, $Z_{(1:r,:)} = GP_{r,2}(C + \alpha_1 B)^{-1}$ and $H_{(1:r,1:r)} = (\beta_1 - \alpha_1) I_r$.

(2) For $i = 1, 2, \dots$

$$Y_{(:,ir+1:(i+1)r)} = (D - \beta_{i+1} A)^{-1} (D - \alpha_i A) Y_{(:,(i-1)r+1:ir)},$$

$$Z_{(ir+1:(i+1)r,:)} = Z_{((i-1)r+1:ir,:)} (C + \beta_i B) (C + \alpha_{i+1} B)^{-1},$$

$$H_{(ir+1:(i+1)r,ir+1:(i+1)r)} = (\beta_{i+1} - \alpha_{i+1}) I_r.$$

End For

ALGORITHM 1: The generalized LR-ADI method for the projected generalized Sylvester equation (1) with $E = FG$.

$$Z_k = [Z^{(1)} \ Z^{(2)} \ \dots \ Z^{(k)}], \text{ with}$$

$$Z^{(1)} = GP_{r,2}(C + \alpha_1 B)^{-1},$$

$$Z^{(i+1)} = Z^{(i)} (C + \beta_i B) (C + \alpha_{i+1} B)^{-1} \quad (i = 1, 2, \dots, k-1), \quad (26)$$

and so we have

$$X_k = Y_k H_k Z_k \quad (27)$$

with

$$H_k = \text{diag}((\beta_1 - \alpha_1) I_r, \dots, (\beta_k - \alpha_k) I_r). \quad (28)$$

The algorithm is summarized as in Algorithm 1.

We make a few comments on Algorithm 1.

- (1) Since the computation of the Weierstrass canonical form is sensitive under small perturbations, we should make use of the generalized real Schur factorization to compute the spectral projectors; see, for example, [33]. For large-scale problems, the computation of the spectral projectors by the generalized real Schur factorization may be very expensive. However, in some applications the spectral projectors can be expressed in explicit form by using the special block structure of the matrices A, D, B , and C ; see numerical examples in this paper or [31].
- (2) If the number of shift parameters k is smaller than the number of iterations required to obtain a prescribed tolerance, then we reuse these parameters in a cyclic manner.

3. Choice of Shift Parameters

The selection of good parameters is vitally important to the successful application of the generalized LR-ADI iteration. The rate of convergence is dominated by spectral radii of matrices $\mathcal{A}_k$ and $\mathcal{B}_k$ given by (20). The minimization of

these spectral radii with respect to the parameters $\{\alpha_j\}_{j=1}^k$ and $\{\beta_j\}_{j=1}^k$ leads to the generalized ADI minimax problem

$$\min_{\substack{\{\alpha_1, \dots, \alpha_k\} \in \mathbb{C}^- \\ \{\beta_1, \dots, \beta_k\} \in \mathbb{C}^+}} \max_{\substack{x \in \mathbb{E}_f \\ y \in \mathbb{F}_f}} \prod_{j=1}^k \left| \frac{(1 - \alpha_j x)(1 - \beta_j y)}{(1 - \beta_j x)(1 - \alpha_j y)} \right|, \quad (29)$$

where $\mathbb{E}_f$ and $\mathbb{F}_f$ denote two sets of finite eigenvalues of the pencils $\lambda D - A$ and $\lambda C + B$, respectively. In practice since the eigenvalues of the pencils $\lambda D - A$ and $\lambda C + B$ are unknown and expensive to compute, $\mathbb{E}_f$ and $\mathbb{F}_f$ are often replaced by domains that contain the eigenvalues of $\lambda D - A$ and $\lambda C + B$, respectively. Noting that A and B are nonsingular, this problem is equivalent to find $\{\alpha_j\}_{j=1}^k$ and $\{\beta_j\}_{j=1}^k$ such that

$$\min_{\substack{\{\alpha_1, \dots, \alpha_k\} \in \mathbb{C}^- \\ \{\beta_1, \dots, \beta_k\} \in \mathbb{C}^+}} \max_{\substack{x \in \mathbb{E} \setminus \{0\} \\ y \in \mathbb{F} \setminus \{0\}}} \prod_{j=1}^k \left| \frac{(x - \alpha_j)(y - \beta_j)}{(x - \beta_j)(y - \alpha_j)} \right|, \quad (30)$$

where $\mathbb{E}$ and $\mathbb{F}$ denote spectrums of the matrices $A^{-1}D$ and $-B^{-1}C$, respectively.

To compute the suboptimal ADI shift parameters for the standard Lyapunov equation with $B = A^T$ and $C = D = I$, a heuristic algorithm has been proposed in [7]. It chooses suboptimal ADI parameters from a set R , which is taken to be the union of the Ritz values of A and the reciprocals of the Ritz values of A^{-1} , obtained by two Arnoldi processes, with A and A^{-1} . As shown in [7], the Ritz values obtained by the Arnoldi process with A tend to be located near the “outer” eigenvalues, that is, the eigenvalues near the convex hull of the spectrum. In particular, the eigenvalues of large magnitude are usually approximated well. In contrast, they are generally poor approximations to the eigenvalues near the origin. Therefore, one computes the reciprocals of the Ritz values obtained by the Arnoldi process with A^{-1} to approximate the eigenvalues near the origin.

In [12], this idea has been extended to the case for Sylvester equations. A heuristic procedure which is easy to implement has been proposed in [12]. It can also be naturally extended to the generalized problem (30). Here, we need to compute the largest and smallest (in modulus) nonzero approximate eigenvalues of $A^{-1}D$ and $-B^{-1}C$, respectively.

Input: $A, D \in \mathbb{R}^{n \times n}$, $B, C \in \mathbb{R}^{m \times m}$ with $\lambda D - A$ and $\lambda C - B$ c-stable, $F \in \mathbb{R}^{n \times r}$ and $G \in \mathbb{R}^{r \times m}$.

Output: ADI parameters $\{\alpha_1, \alpha_2, \dots, \alpha_k\}$ and $\{\beta_1, \beta_2, \dots, \beta_k\}$.

- (1) Run Arnoldi process with $A^{-1}D$ on F to give the set $\mathbb{E}_A^+$ of Ritz values,
- (2) Run Arnoldi process with $P_1 A$ on F to give the set $\mathbb{E}_A^-$ of Ritz values,
- (3) $\mathbb{E} = \mathbb{E}_A^+ \cup (1/\mathbb{E}_A^-)$,
- (4) Run Arnoldi process with $-B^{-1}C$ on G^T to give the set $\mathbb{F}_B^+$ of Ritz values,
- (5) Run Arnoldi process with $-P_2 B$ on G^T to give the set $\mathbb{F}_B^-$ of Ritz values,
- (6) $\mathbb{F} = (\mathbb{F}_B^+ \cup (1/\mathbb{F}_B^-))$,

$$(7) \text{ Set } \{\alpha_1, \beta_1\} = \arg \min_{\alpha \in \mathbb{E}} \max_{\substack{x \in \mathbb{E} \\ \beta \in \mathbb{F}}} \left| \frac{(x - \alpha)(y - \beta)}{(x - \beta)(y - \alpha)} \right|,$$

- (8) For $i = 2, 3, \dots, k$

$$\text{Set } \{\alpha_i, \beta_i\} = \arg \min_{\alpha \in \mathbb{E}'} \max_{\substack{x \in \mathbb{E} \\ \beta \in \mathbb{F}'}} \left| \frac{(x - \alpha)(y - \beta)}{(x - \beta)(y - \alpha)} \right| \prod_{j=1}^{i-1} \left| \frac{(x - \alpha_j)(y - \beta_j)}{(x - \beta_j)(y - \alpha_j)} \right|,$$

where $\mathbb{E}'$ is $\mathbb{E}$ with $\alpha_1, \dots, \alpha_{i-1}$ deleted, and similarly for $\mathbb{F}'$,
End For

ALGORITHM 2: ADI parameters by Ritz values.

Noting that C and D are assumed to be singular, inverses of $A^{-1}D$ and $-B^{-1}C$ do not exist. In [31], Stykel proposed a strategy to overcome this difficulty. Define

$$\begin{aligned} P_1 &= P_{r,1}(DP_{r,1} - AQ_{r,1})^{-1} \\ &= (P_{l,1}D - Q_{l,1}A)^{-1}P_{l,1} = T_1^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} W_1^{-1}, \\ P_2 &= P_{r,2}(CP_{r,2} - BQ_{r,2})^{-1} \\ &= (P_{l,2}C - Q_{l,2}B)^{-1}P_{l,2} = T_2^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} W_2^{-1}, \end{aligned} \quad (31)$$

where T_1, T_2, W_1 , and W_2 are the transformation matrices as in (2). A simple calculation gives that

$$\begin{aligned} P_1 A &= T_1^{-1} \begin{bmatrix} J^{(A)} & 0 \\ 0 & 0 \end{bmatrix} T_1, \\ -P_2 B &= T_2^{-1} \begin{bmatrix} -J^{(B)} & 0 \\ 0 & 0 \end{bmatrix} T_2. \end{aligned} \quad (32)$$

Then it is clear that the reciprocals of the smallest nonzero eigenvalues of $A^{-1}D$ and $-B^{-1}C$ are the largest eigenvalues of $P_1 A$ and $-P_2 B$, respectively. Thus, we can run two Arnoldi processes with the matrices $P_1 A$ and $-P_2 B$ to compute the smallest nonzero eigenvalues of $A^{-1}D$ and $-B^{-1}C$, respectively. In some applications, similar to the projectors $P_{r,1}, P_{l,1}, P_{r,2}$ and $P_{l,2}$, the matrices P_1 , and P_2 can be also obtained in explicit form by making use of the special block structure of the matrices A, D, B , and C . The algorithm for choosing $\{\alpha_j\}_{j=1}^k$ and $\{\beta_j\}_{j=1}^k$ is summarized as in Algorithm 2.

For more details about Algorithm 2, the interesting reader is referred to [12].

4. Smith Method

The Smith method [4] is derived from the projected discrete-time Sylvester equation

$$\begin{aligned} \mathcal{A}X\mathcal{B} - X + (\beta - \alpha)P_{r,1}\mathcal{F}\mathcal{G}P_{l,2} &= 0, \\ X &= P_{r,1}XP_{l,2}, \end{aligned} \quad (33)$$

with

$$\begin{aligned} \mathcal{A} &= (D - \beta A)^{-1}(D - \alpha A), \\ \mathcal{B} &= (C + \beta B)(C + \alpha B)^{-1}, \end{aligned} \quad (34)$$

$$\mathcal{F} = (D - \beta A)^{-1}F, \quad \mathcal{G} = G(C + \alpha B)^{-1},$$

which is equivalent to (1) for any two parameters $\alpha \in \mathbb{C}^-$ and $\beta \in \mathbb{C}^+$. Under this assumption, the sequence $\{X_i\}_{i=0}^\infty$ generated by the iteration

$$\begin{aligned} X_0 &= (\beta - \alpha)P_{r,1}\mathcal{F}\mathcal{G}P_{l,2}, \\ X_k &= \mathcal{A}X_{k-1}\mathcal{B} + (\beta - \alpha)P_{r,1}\mathcal{F}\mathcal{G}P_{l,2} \end{aligned} \quad (35)$$

converges to the solution X .

If all of the shifts in the generalized ADI iteration (17) are constant, that is, $\alpha_k = \alpha, \beta_k = \beta$ ($k = 1, 2, \dots$), then the generalized ADI iteration reduces to the Smith method. In [7], Penzl illustrated that the ADI iteration with a single shift (Smith method) converges very slowly, while a moderate increase in the number of shifts s accelerates the convergence nicely. However, it is also observed that the speed of convergence is hardly improved by a further increase of s ; see Table 2.1 in [7]. These observations lead to the idea of the cyclic Smith(s) iteration, a special case of ADI where s different shifts are used in a cyclic manner.

Input: $A, D \in \mathbb{R}^{n \times n}$, $B, C \in \mathbb{R}^{m \times m}$, $F \in \mathbb{R}^{n \times r}$ and $G \in \mathbb{R}^{r \times m}$. $\lambda D - A$ and $\lambda C - B$ are c-stable.

Output: Y, H and Z such that $X = YHZ$ is an approximate solution of the projected generalized Sylvester equation (1) with $E = FG$.

(1) Compute Y_s, H_s and Z_s by using Algorithm 1 with $k = s$.

(2) Set $Y = \bar{Y} = Y_s, H = \bar{H} = H_s$ and $Z = \bar{Z} = Z_s$.

(3) For $i = 1, 2, \dots$

For $j = 1, 2, \dots, s$

$\bar{Y} = (D - \beta_j A)^{-1}(D - \alpha_j A)\bar{Y}$,

$\bar{Z} = \bar{Z}(C + \beta_j B)(C + \alpha_j B)^{-1}$,

End For

$Y = \begin{bmatrix} Y & P_{r,1}\bar{Y} \end{bmatrix}, \quad Z = \begin{bmatrix} Z \\ \bar{Z}P_{l,2} \end{bmatrix}$,

$H = \begin{bmatrix} H & 0 \\ 0 & \bar{H} \end{bmatrix}$,

End For

ALGORITHM 3: The generalized LR-Smith(s) method for the projected generalized Sylvester equation (1) with $E = FG$.

The low-rank scheme based on the Smith(s) iteration was also introduced in [7]. The method is called the cyclic low-rank Smith method (LR-Smith(s)) and is a special case of LR-ADI where s number of shifts are reused in a cyclic manner. This idea can be generalized for (1). The generalized LR-Smith method consists of two steps. First the matrices Y_s, H_s , and Z_s are obtained by an s step generalized LR-ADI iteration. Then one solves the discrete-time Sylvester equation

$$\mathcal{A}_s X \mathcal{B}_s - X + Y_s H_s Z_s = 0, \quad (36)$$

where $\mathcal{A}_s$ and $\mathcal{B}_s$ are as in (19) with $k = s$, respectively.

The generalized LR-Smith method for solving the projected generalized Sylvester (1) is described as in Algorithm 3.

5. Numerical Experiments

In this section, we present some numerical examples to illustrate the performance of the generalized LR-ADI method for the projected generalized Sylvester equation (1). Let LR-ADI and LR-Smith(s) denote Algorithms 1 and 3, respectively. In the following examples, we compare the numerical behavior of LR-ADI with LR-Smith(s) with respect to the number of iterations (Iter for short), CPU time (CPU for short), and the relative residuals $\|R_k\|_F / \|R_0\|_F$ (Res for short). All numerical experiments are performed on a PC with the usual double precision, where the floating point relative accuracy is 2.22×10^{-16} .

The stopping criterion for both methods is

$$\frac{\|R_k\|_F}{\|R_0\|_F} = \frac{\|AX_k C + DX_k B + P_{l,1} E P_{r,2}\|_F}{\|P_{l,1} E P_{r,2}\|_F} \leq 10^{-12}. \quad (37)$$

Let mvp denote the number of matrix vector products and let lss denote the number of linear system solvers at every iteration for the two methods. Table 1 has a rough count of the expenses of LR-ADI and LR-Smith(s) at every iteration. Only the major expenses are considered. From Table 1, we can

TABLE 1: mvp and lss at every iteration step.

	LR-ADI	LR-Smith(s)
mvp for $D - \alpha A$	r	rs^2
mvp for $(C + \beta B)^T$	r	rs^2
lss for $D - \beta A$	r	rs^2
lss for $(C + \alpha B)^T$	r	rs^2

see that LR-Smith(s) needs more mvp and lss than LR-ADI at every iteration, so LR-ADI is more efficient than LR-Smith(s) for the computation cost at every iteration.

Example 1. For the first experiment, we consider the 2D instationary Stokes equation that describes the flow of an incompressible fluid in a domain. The spatial discretization of this equation by the finite difference method on a uniform staggered grid leads to the descriptor systems

$$D\dot{x}(t) = Ax(t) + Fu(t), \quad (38)$$

$$y(t) = \tilde{G}x(t),$$

$$C\dot{x}(t) = Bx(t) + \tilde{F}u(t), \quad (39)$$

$$y(t) = Gx(t).$$

The matrix coefficients in (38) are given by

$$D = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}, \quad (40)$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{bmatrix} \in \mathbb{R}^{n \times n}.$$

These matrices are sparse and have a special block structure. Using this structure, the projectors $P_{l,1}$ and $P_{r,1}$ onto the left

FIGURE 1: Comparison for Example 1.

FIGURE 2: Comparison for Example 2.

and right deflating subspaces of the pencil $\lambda D - A$ can be computed as

$$\begin{aligned} P_{l,1} &= \begin{bmatrix} \Pi & -\Pi A_{11} A_{12} (A_{21} A_{12})^{-1} \\ 0 & 0 \end{bmatrix}, \\ P_{r,1} &= \begin{bmatrix} \Pi & 0 \\ -(A_{21} A_{12})^{-1} A_{21} A_{11} \Pi & 0 \end{bmatrix}, \end{aligned} \quad (41)$$

where $\Pi = I - A_{12} (A_{21} A_{12})^{-1} A_{21}$ is the orthogonal projector onto the kernel of A_{21} along the image of A_{12} ; see [40]. The matrices A_{12} and A_{21} have full rank, and the pencil $\lambda D - A$ is of index 2. Analogously, we can get the projectors $P_{l,2}$ and $P_{r,2}$ of the pencil $\lambda B - C$. In our experiment the state space dimensions of the problems are $n = 2295$ and $m = 1975$, respectively. The matrix E in (1) is $E = FG$ with $F \in \mathbb{R}^{2295 \times 1}$ and $G \in \mathbb{R}^{1 \times 1975}$.

See Figure 1 for a graph of the convergence. The results in Table 2 clearly indicate that the LR-ADI method is more efficient than LR-Smith(3) for this example.

Example 2. For the second experiment, we consider a holonomically constrained damped mass-spring system with g masses as in [25]. The i th mass is connected to the $(i + 1)$ th mass by a spring and a damper and also to the ground by another spring and damper. Moreover, the first mass is connected to the last one by a rigid bar, and it can be influenced by a control. The vibration of this system is described by the descriptor system (38) with the matrices

$$\begin{aligned} D &= \begin{bmatrix} I & 0 & 0 \\ 0 & M_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ A &= \begin{bmatrix} 0 & I & 0 \\ K_1 & H_1 & -N_1^T \\ N_1 & 0 & 0 \end{bmatrix}, \end{aligned} \quad (42)$$

where $M_1 \in \mathbb{R}^{g \times g}$ is the symmetric positive definite mass matrix, $K_1 \in \mathbb{R}^{g \times g}$ is the stiffness matrix, $H_1 \in \mathbb{R}^{g \times g}$ is the damping matrix, and N_1 is the matrix of constraints. If $N_1 \in \mathbb{R}^{1 \times g}$ has full row rank, the pencil $\lambda D - A$ is of index 3, and the spectral projectors $P_{l,1}$ and $P_{r,1}$ can be computed as

$$\begin{aligned} P_{l,1} &= \begin{bmatrix} \Pi & 0 & -\Pi M_1^{-1} H_1 Q \\ -\Pi^T H_1 (I - \Pi) & \Pi^T & -\Pi^T (K_1 + H_1 \Pi M_1^{-1} H_1) Q \\ 0 & 0 & 0 \end{bmatrix}, \\ P_{r,1} &= \begin{bmatrix} \Pi & 0 & 0 \\ -\Pi M_1^{-1} H_1 (I - \Pi) & \Pi & 0 \\ -Q^T (K_1 \Pi - H_1 \Pi M_1^{-1} H_1 (I - \Pi)) & Q^T H_1 \Pi & 0 \end{bmatrix}. \end{aligned} \quad (43)$$

Here $Q = M_1^{-1} N_1^T (N_1 M_1^{-1} N_1^T)^{-1}$ and $\Pi = I - M_1^{-1} N_1^T (N_1 M_1^{-1} N_1^T)^{-1} N_1 = I - Q N_1$ are a projector onto the kernel of N_1 along the image of $M_1^{-1} N_1^T$; see [41] for details. The matrix F has the form $F = [e_1, e_2, e_{g-1}] \in \mathbb{R}^{n \times 3}$, where e_j denotes the j th column of the identity matrix I_{2g+1} . Analogously, we can get the projectors $P_{l,2}$ and $P_{r,2}$ of the pencil $\lambda B - C$. In this experiment the state space dimensions of the problems are $n = 1261$ and $m = 1161$, respectively. The matrix E in (1) is $E = FG$ with $F \in \mathbb{R}^{1261 \times 3}$ and $G \in \mathbb{R}^{3 \times 1161}$.

The computational results were reported in Figure 2 and Table 3. We note that the iteration steps of the LR-Smith(6) method is 21 while the steps of LR-ADI is 42, but the CPU time of LR-Smith(6) method is much more than that of LR-ADI method. This is because the computation cost at every iteration of two methods is very different.

6. Conclusions

In this paper, we have proposed the generalized low-rank alternating direction implicit method and the generalized

TABLE 2: LR-ADI, $k = 12$ versus LR-Smith(3).

	LR-ADI	LR-Smith(3)
Iter	30	12
mvp for $D - \alpha A$	30	108
mvp for $(C + \beta B)^T$	30	108
lss for $D - \beta A$	30	108
lss for $(C + \alpha B)^T$	30	108
CPU	10.68	12.21
Res	8.93×10^{-13}	3.72×10^{-13}

TABLE 3: LR-ADI, $k = 12$ versus LR-Smith(6).

	LR-ADI	LR-Smith(6)
Iter	42	21
mvp for $D - \alpha A$	126	2268
mvp for $(C + \beta B)^T$	126	2268
lss for $D - \beta A$	126	2268
lss for $(C + \alpha B)^T$	126	2268
CPU	30.80	44.13
Res	7.16×10^{-13}	7.69×10^{-13}

low-rank cyclic Smith method to solve the projected generalized Sylvester equations. Numerical experiments presented in this paper show the effectiveness of the proposed methods.

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