Research Article

Breathers and Soliton Solutions for a Generalization of the Nonlinear Schrödinger Equation

Hai-Feng Zhang, Hui-Qin Hao, and Jian-Wen Zhang

College of Mathematics, Taiyuan University of Technology, Taiyuan 030024, China

Correspondence should be addressed to Hui-Qin Hao; gr81@sina.com

Received 27 December 2012; Accepted 21 February 2013

Academic Editor: Farzad Khani

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A generalized nonlinear Schrödinger equation, which describes the propagation of the femtosecond pulse in single mode optical silica fiber, is analytically investigated. By virtue of the Darboux transformation method, some new soliton solutions are generated: the bright one-soliton solution on the zero background, the dark one-soliton solution on the continuous wave background, the Akhmediev breather which delineates the modulation instability process, and the breather evolving periodically along the straight line with a certain angle of x -axis and t -axis. Those results might be useful in the study of the femtosecond pulse in single mode optical silica fiber.

1. Introduction

Investigations on the dynamic features of solitons have attracted certain interest in nonlinear optics [1–4]. Optical solitons have been regarded as a candidate for the optical communication networks [5–8]. On the basis of the balance between the group velocity dispersion and self-phase modulation [9, 10], the propagation of optical soliton is usually governed by the nonlinear Schrödinger (NLS) equation [11–14]:

$$iE_{\zeta} + \frac{1}{2}E_{\tau\tau} + |E|^2E = 0. \quad (1)$$

However, when optical pulses are shorter, the NLS equation becomes inadequate, and it is necessary to include additional terms [6, 7]. For example, in single mode optical silica fiber, in order to describe the propagation of femtosecond pulse, the higher order asymptotic terms should be retained [15]; to understand such phenomena, we consider the following generalization of the NLS equation [16]:

$$iu_t - u_{xt} + u_{xx} + |u|^2u + i|u|^2u_x = 0. \quad (2)$$

Analogous to the circumstance that the Camassa-Holm equation provides a better approximation of the KdV

equation [15], (2) is related to the NLS equation, provided that one retains terms of the next asymptotic order. Under the transformation $q = e^{ix}u$, (2) can be converted into the following equation [16, 17]:

$$iq_{xt} - iq_{xx} + 2q_x - |q|^2q_x + iq = 0, \quad (3)$$

where q denotes the complex field envelope and the subscripts x and t are the longitudinal distance and retarded time, respectively. In recent years, some results have been obtained for (1): (1) Reference [15] has analyzed the dynamic features of the rogue wave solutions; (2) Reference [16] has analyzed the conservation laws, bi-Hamiltonian structure, Lax pair, and initial-value problem; (3) Reference [17] has derived some soliton solutions by using the bilinear method. The aim of this paper is mainly to derive some new soliton solutions for (3) using the Darboux transformation (DT) method and analyze the dynamic features of soliton solutions.

This paper will be organized as follows. In Section 2, we will give the Lax pair and construct the DT for (3). In Section 3, we will obtain bright one-soliton, dark one-soliton, and breather solutions and analyze the dynamic features of soliton solutions by using some figures. Finally, our conclusions will be addressed in Section 4.

2. Lax Pair and Darboux Transformation

Employing the Ablowitz-Kaup-Newell-Segur formalism [18], [15, 16] has given the Lax pair associated with (3) as

$$\Psi_x = U\Psi, \quad (4a)$$

$$\Psi_t = V\Psi, \quad (4b)$$

where $\Psi = (\psi_1, \psi_2)^T$ (T denotes the transpose of a matrix), and the matrices U and V have the form

$$U = \lambda^2 U_2 + \lambda U_1, \quad (5a)$$

$$V = \lambda^2 V_2 + \lambda V_1 + V_0 + \frac{1}{\lambda} V_{-1} + \frac{1}{\lambda^2} V_{-2}, \quad (5b)$$

where λ is a spectral parameter and

$$U_2 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad U_1 = \begin{pmatrix} 0 & q_x \\ q_x^* & 0 \end{pmatrix},$$

$$V_2 = U_2, \quad V_1 = U_1, \quad V_{-2} = \frac{1}{4} U_2,$$

$$V_0 = \begin{pmatrix} i - \frac{i}{2}|q|^2 & 0 \\ 0 & -i + \frac{i}{2}|q|^2 \end{pmatrix}, \quad V_{-1} = \begin{pmatrix} 0 & \frac{i}{2}q \\ -\frac{i}{2}q^* & 0 \end{pmatrix}. \quad (6)$$

Through direct computations, it can be verified that the zero curvature equation $U_t - V_x + UV - VU = 0$ exactly gives rise to (3).

Next, based on Lax pair (4a) and (4b), we will give the DT [19–22] formalism for (3). Define

$$\Psi' = \left(\lambda D_1 + D_0 + \frac{1}{\lambda} I \right) \Psi, \quad (7)$$

where I denotes the identity matrix and

$$D_1 = \begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}, \quad D_0 = \begin{pmatrix} 0 & s_3 \\ s_4 & 0 \end{pmatrix}, \quad (8)$$

with

$$\begin{aligned} s_1 &= \frac{-(1/\lambda)|\phi_1|^2 + (1/\lambda^*)|\phi_2|^2}{\lambda|\phi_1|^2 - \lambda^*|\phi_2|^2}, \\ s_2 &= \frac{-(1/\lambda)|\phi_2|^2 + (1/\lambda^*)|\phi_1|^2}{\lambda|\phi_2|^2 - \lambda^*|\phi_1|^2}, \\ s_3 &= \frac{(\lambda^*/\lambda - \lambda/\lambda^*)\phi_1\phi_2^*}{\lambda|\phi_1|^2 - \lambda^*|\phi_2|^2}, \\ s_4 &= \frac{(\lambda^*/\lambda - \lambda/\lambda^*)\phi_2\phi_1^*}{\lambda|\phi_2|^2 - \lambda^*|\phi_1|^2}. \end{aligned} \quad (9)$$

One can verify that if $(\phi_1, \phi_2)^T$ is the solution of Lax pair (4a) and (4b) with λ , then $(\phi_2^*, \phi_1^*)^T$ is also the solution of Lax pair (4a) and (4b) corresponding to λ^* .

Through direct computation, we can obtain

$$q' = q + \frac{(\lambda^{*2} - \lambda^2)\phi_1\phi_2^*}{\lambda\lambda^*(\lambda|\phi_1|^2 - \lambda^*|\phi_2|^2)}. \quad (10)$$

So, if q is a seed solution of (3), q' is also a solution of (3).

3. One and Breather Solutions (3)

In this section, we will apply the DT constructed to obtain one and breather solutions for (3). Now we take the nonzero continuous wave (cw) solution $q = a \exp i(At + Bx)$ as the initial seed for (3), where a , A , and B are all real parameters. Equation (3) requires that the frequency A satisfies the nonlinear dispersion relation:

$$A = 2 - a^2 + B + \frac{1}{B}. \quad (11)$$

Solving (4a) and (4b) and setting $\phi_1 = f_1 \exp i(At + Bx)$, $\phi_2 = f_2$, one can obtain

$$\begin{aligned} f_{1x} + iBf_1 &= -i\lambda^2 f_1 + iaB\lambda f_2, \\ f_{2x} &= -iaB\lambda f_1 + i\lambda^2 f_2, \\ f_{1t} + iAf_1 &= \alpha f_1 + \beta f_2, \\ f_{2t} &= -\beta f_1 - \alpha f_2, \end{aligned} \quad (12)$$

where

$$\alpha = -i\lambda^2 + i - \frac{i}{2}a^2 - \frac{i}{4\lambda^2}, \quad \beta = iaB\lambda + \frac{i}{2\lambda}a. \quad (13)$$

Through tedious computations, one can arrive at

$$\begin{aligned} \phi_1 &= (c_1 \exp \theta_1 + c_2 \exp \theta_2) \exp i(At + Bx), \\ \phi_2 &= c_3 \exp \theta_1 + c_4 \exp \theta_2, \end{aligned} \quad (14)$$

where

$$\begin{aligned} \theta_1 &= \frac{x}{2} [-iB + \zeta_1 + i\eta_1] + \frac{t}{2} [-iA + \zeta_2 + i\eta_2], \\ \theta_2 &= \frac{x}{2} [-iB - \zeta_1 - i\eta_1] + \frac{t}{2} [-iA - \zeta_2 - i\eta_2], \end{aligned} \quad (15)$$

and c_1 , c_2 , c_3 , and c_4 are complex constants satisfying:

$$c_2 = (L + iM)c_4, \quad c_3 = (L + iM)c_1, \quad (16)$$

with

$$\begin{aligned} \lambda &= \frac{1}{2}(\lambda_1 + i\lambda_2), \\ \zeta_1 + i\eta_1 &= \sqrt{-B^2 - 4(\lambda^2 B + \lambda^4 - a^2 B^2 \lambda^2)}, \\ \zeta_2 + i\eta_2 &= \sqrt{-A^2 - 4(\beta^2 - \alpha^2 + iA\alpha)}, \end{aligned} \quad (17)$$

FIGURE 1: Evolution of the one-soliton solution of (22). The parameters are (a) $\lambda_1 = 2$, $\lambda_2 = 4$; (b) $\lambda_1 = 1$, $\lambda_2 = 4$.

FIGURE 2: Evolution of the soliton solutions of (26). The parameters are (a) $a = 1$, $\lambda_2 = 0.5$; (b) $a = -2$, $\lambda_2 = 1.1$; (c) $a = 1$, $\lambda_2 = 3$.

with λ_1 , λ_2 , ζ_1 , η_1 , ζ_2 , and η_2 as real numbers, and we can derive

$$\begin{aligned} L &= \frac{-2\zeta_1\lambda_2 + \lambda_1(\lambda_1^2 + \lambda_2^2 + 2B + 2\eta_1)}{2aB(\lambda_1^2 + \lambda_2^2)}, \\ M &= \frac{2\zeta_1\lambda_1 + \lambda_2(\lambda_1^2 + \lambda_2^2 - 2B - 2\eta_1)}{2aB(\lambda_1^2 + \lambda_2^2)}. \end{aligned} \quad (18)$$

Now, substituting (14) into (10), we can obtain the solution for (1) as

$$q = \left[a - \frac{4i\lambda_1\lambda_2 G}{(\lambda_1^2 + \lambda_2^2)F} \right] \exp i(At + Bx), \quad (19)$$

where

$$\begin{aligned} G &= 2L \cosh \Theta_1 - 2iM \sinh \Theta_1 + (L^2 + M^2 + 1) \cos \Theta_2 \\ &\quad + i(1 - L^2 - M^2) \sin \Theta_2, \\ F &= (1 - L^2 - M^2) \lambda_1 \sinh \Theta_1 + i(L^2 + M^2 + 1) \lambda_2 \cosh \Theta_1 \\ &\quad + 2\lambda_1 M \sin \Theta_2 + 2i\lambda_2 L \cos \Theta_2, \end{aligned} \quad (20)$$

with

$$\Theta_1 = \zeta_1 x + \zeta_2 t, \quad \Theta_2 = \eta_1 x + \eta_2 t. \quad (21)$$

Next according to different values of those parameters in solution (19), we will analyse the novel properties of solitons.

3.1. One-Soliton Solutions for (3). When $a = 0$, that is to say, the initial seed for (3) is zero, solution (19) reduces to one-soliton solution as

$$q = \frac{\lambda^2 - \lambda^{*2}}{\lambda \lambda^{*2}} \operatorname{sech}(\chi_1 + \chi_1^*) \exp(\chi_1 - \chi_1^*), \quad (22)$$

with

$$\begin{aligned} \chi_1 &= -i\lambda^2 x + \left(i - \frac{i}{4\lambda^2} - i\lambda^2 \right) t, \\ \chi_1^* &= i\lambda^{*2} x + \left(-i + \frac{i}{4\lambda^{*2}} - i\lambda^{*2} \right) t. \end{aligned} \quad (23)$$

Solution (22) represents a bright soliton whose dynamic features are delineated in Figures 1. Through symbolic computation, we can conclude the following physical quantities for solution (22): the maximum amplitude $|(\lambda^2 - \lambda^{*2})/\lambda \lambda^{*2}|$, the width $1/|\lambda^2 - \lambda^{*2}|$, the envelope velocity $(1/4\lambda^{*2} - 1/4\lambda^2 - \lambda^2 - \lambda^{*2})/(\lambda^2 - \lambda^{*2})$, and the energy of the one-soliton solution

$$E_1 = \int_{-\infty}^{+\infty} [|q(x, t)|^2 - |q(\pm\infty, t)|^2] dx = \frac{128\lambda_1\lambda_2}{(\lambda_1^2 + \lambda_2^2)^3}. \quad (24)$$

So, the amplitude and envelope velocity will increase when the value of λ_1 becomes bigger. As shown in Figures 1, the amplitude is higher and the envelope velocity is bigger in Figure 1(a) than in Figure 1(b).

3.2. Breather and Dark One-Soliton Solutions for (1). When $a \neq 0$, the nonzero initial seed $q = a \exp i(At + Bx)$ describes the nonvanishing boundary conditions. For simplicity, taking $\lambda_1 = \lambda_2$, $B = 2/a^2$, we have the following relations:

$$A = 2 + \frac{2}{a^2} - \frac{a^2}{2}, \quad \zeta_1 = \lambda_2^2, \quad (25a)$$

$$\eta_1 = \frac{2}{a^2}, \quad \zeta_2 = -\frac{1}{\lambda_2^2} - \lambda_2^2, \\ \eta_2 = \frac{a^2}{2} - \frac{a^2}{2}, \quad L = \frac{1}{\lambda_2 a}, \quad M = -\frac{1}{\lambda_2 a} + \frac{\lambda_2 a}{2}. \quad (25b)$$

With the previous conclusions, solution (19) can be converted into:

$$q = \left[a - \frac{4i\lambda_1\lambda_2 G'}{(\lambda_1^2 + \lambda_2^2)F'} \right] \exp i \left[\left(2 + \frac{2}{a^2} - \frac{a^2}{2} \right) t + \frac{2}{a^2} x \right], \quad (26)$$

where

$$G' = D_1 \cosh \Theta'_1 + D_2 \sinh \Theta'_1 + D_3 \cos \Theta'_2 + D_4 \sin \Theta'_2, \\ F = D_5 \cosh \Theta'_1 + D_6 \sinh \Theta'_1 + D_7 \cos \Theta'_2 + D_8 \sin \Theta'_2, \quad (27)$$

with

$$\Theta'_1 = \lambda_2^2 x - \left(\lambda_2^2 + \frac{1}{\lambda_2^2} \right) t, \\ \Theta'_2 = \left(\frac{a^2}{2} - \frac{2}{a^2} \right) t + \frac{2}{a^2} x, \\ D_1 = \frac{2}{a\lambda_2}, \quad D_2 = i(a\lambda_2 - D_1), \\ D_3 = \frac{2}{a^2\lambda_2^2} + \frac{a^2\lambda_2^2}{4}, \quad D_4 = i(2 - D_3), \\ D_5 = i \left(\frac{2}{a^2\lambda_2} + \frac{a^2\lambda_2^3}{4} \right), \quad D_6 = 2\lambda_2 + iD_5, \\ D_7 = \frac{2i}{a}, \quad D_8 = a\lambda_2^2 + iD_7. \quad (28)$$

Figure 2 display the propagation characteristics of solitons via solutions (26). Figures 2(a) and 2(b) depict the dynamic features of breathers; as shown in Figure 2(a), the main feature is the propagation of the breather that is periodic in the space coordinate and aperiodic in the time coordinate; that is to say, we can obtain the Akhmediev breather [23] via

solutions (26) under suitable parameters chosen. In addition, the Akhmediev breather can be regarded as a modulation instability process. Figure 2(b) portrays the propagation of the breather evolving periodically along the straight line with a certain angle of x -axis and t -axis. Figure 2(c) describes the dynamic feature of the dark one-soliton solutions via solutions (26) on the continuous wave background, which is different with Figures 1 via Solutions (22) on the zero background.

4. Conclusions

Our main attention has been focused on (3), which can describe the propagation of femtosecond pulse in single mode optical silica fiber. By using the Darboux transformation method, we have obtained (1) bright one soliton on the zero background; (2) two types of breathers: the Akhmediev soliton which delineates the modulation instability process and the breather evolving periodically along the straight line with a certain angle of x -axis and t -axis; (3) the dark one-soliton solution on the continuous wave background.

Acknowledgments

The express their sincere thanks to each member of their discussion group for their suggestions. This work has been supported by the National Natural Science Foundation of China under Grant nos. 11172194 and 61250011 and by the Natural Science Foundation of Shanxi Province under Grant no. 2012011004-3.

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