Research Article

Exact Explicit Solutions and Conservation Laws for a Coupled Zakharov-Kuznetsov System

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We study a coupled Zakharov-Kuznetsov system, which is an extension of a coupled Korteweg-de Vries system in the sense of the Zakharov-Kuznetsov equation. Firstly, we obtain some exact solutions of the coupled Zakharov-Kuznetsov system using the simplest equation method. Secondly, the conservation laws for the coupled Zakharov-Kuznetsov system will be constructed by using the multiplier approach.

1. Introduction

It is well known that the two-dimensional generalizations of the Korteweg-de Vries (KdV) equation

$$u_t + auu_x + u_{xxx} = 0 \quad (1)$$

are the Kadomtsev-Petviashvili (KP) equation and the Zakharov-Kuznetsov (ZK) equation. The ZK equation

$$u_t + auu_x + b(u_{xx} + u_{yy})_x = 0 \quad (2)$$

governs the behaviour of weakly nonlinear ion-acoustic waves in a plasma comprising cold ions and hot isothermal electrons in the presence of a uniform magnetic field [1]. In [2] a new hierarchy of nonlinear evolution equations was derived, and one particular system of equations

$$u_t = \beta u_{xxx} + \alpha(uv)_x + \gamma(vw)_x, \quad (3a)$$

$$v_t = \beta v_{xxx} + \lambda(wu)_x, \quad (3b)$$

$$w_t = \beta w_{xxx} + \lambda(uv)_x, \quad (3c)$$

where α , β , γ , and λ are constants, was later studied by [3]. This coupled KdV system (3a), (3b), and (3c) was extended to the new coupled ZK system

$$u_t - \alpha(uv)_x - \gamma(vw)_x - \beta(u_{xx} + u_{yy})_x = 0, \quad (4a)$$

$$v_t - \lambda(wu)_x - \beta(v_{xx} + v_{yy})_x = 0, \quad (4b)$$

$$w_t - \lambda(uv)_x - \beta(w_{xx} + w_{yy})_x = 0, \quad (4c)$$

in the sense of the ZK Equation (2) in [1], and travelling wave solutions were determined using the extended tanh-coth method and sech method.

In the last few decades, several powerful methods have been introduced in the literature, which can be used to find exact solutions of nonlinear differential equations arising from physical problems. These methods include the inverse scattering transform method [4], the Darboux transformation [5], the Hirota's bilinear method [6], the Jacobi elliptic function expansion method [7, 8], the multiple-exp method [9], the sine-cosine method [10], the Lie symmetry method [11, 12], and the (G'/G) -expansion method [13].

The purpose of this paper is to employ the simplest equation method [14, 15] to obtain some exact explicit solutions of the coupled Zakharov-Kuznetsov system (4a),

(4b), and (4c). Furthermore, we derive conservation laws for (4a), (4b), and (4c) using the multiplier approach [16–18].

2. Exact Solutions Using Simplest Equation Method

In this section we employ the simplest equation method [14, 15] and obtain some exact explicit solutions of (4a), (4b), and (4c). The simplest equations that will be used in this paper are the Bernoulli and Riccati equations. It is well known that their solutions can be written in elementary functions. See, for example, [19].

By using the transformation

$$z = k_1 t + k_2 x + k_3 y + k_4, \quad (5)$$

where k_i , $i = 1, \dots, 4$, are constants, the coupled Zakharov-Kuznetsov system (4a), (4b), and (4c) transforms to a third-order coupled system of nonlinear ordinary differential equations (ODEs)

$$\begin{aligned} & \beta k_2^3 E'''(z) + \beta k_3^2 k_2 E'''(z) + \alpha k_2 F(z) E'(z) \\ & - k_1 E'(z) + \alpha k_2 E(z) F'(z) + \gamma k_2 G(z) F'(z) \end{aligned} \quad (6a)$$

$$+ \gamma k_2 F(z) G'(z) = 0,$$

$$\begin{aligned} & \beta k_2^3 F'''(z) + \beta k_3^2 k_2 F'''(z) + \lambda k_2 G(z) E'(z) \\ & + \lambda k_2 E(z) G'(z) - k_1 F'(z) = 0, \end{aligned} \quad (6b)$$

$$\begin{aligned} & \beta k_2^3 G'''(z) + \beta k_3^2 k_2 G'''(z) + \lambda k_2 F(z) E'(z) \\ & + \lambda k_2 E(z) F'(z) - k_1 G'(z) = 0. \end{aligned} \quad (6c)$$

We now present the simplest equation method for a system of three ODEs. Consider the solutions of (6a), (6b), and (6c) in the form

$$\begin{aligned} E(z) &= \sum_{i=0}^M A_i (H(z))^i, \\ F(z) &= \sum_{i=0}^M B_i (H(z))^i, \\ G(z) &= \sum_{i=0}^M C_i (H(z))^i, \end{aligned} \quad (7)$$

where $H(z)$ satisfies the Bernoulli or Riccati equation, M is a positive integer that can be determined by balancing procedure [15], and A_i , B_i , and C_i ($i = 0, 1, \dots, M$) are parameters to be determined.

The Bernoulli equation we consider in this paper is

$$H'(z) = aH(z) + bH^2(z), \quad (8)$$

where a and b are constants. Its solution can be written as

$$H(z) = a \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b \cosh[a(z+C)] - b \sinh[a(z+C)]} \right\}. \quad (9)$$

For the Riccati equation

$$H'(z) = aH^2(z) + bH(z) + c, \quad (10)$$

where a , b , and c are constants, we will use the solutions

$$\begin{aligned} H(z) &= -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left[\frac{1}{2}\theta(z+C)\right], \\ H(z) &= -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left(\frac{1}{2}\theta z\right) \\ &+ \frac{\operatorname{sech}(\theta z/2)}{C \cosh(\theta z/2) - (2a/\theta) \sinh(\theta z/2)}, \end{aligned} \quad (11)$$

where $\theta^2 = b^2 - 4ac$.

2.1. Solutions of (4a), (4b), and (4c) Using the Bernoulli Equation as the Simplest Equation. The balancing procedure yields $M = 2$. Thus, the solutions of (6a), (6b), and (6c) are of the form

$$E(z) = A_0 + A_1 H + A_2 H^2, \quad (12a)$$

$$F(z) = B_0 + B_1 H + B_2 H^2, \quad (12b)$$

$$G(z) = C_0 + C_1 H + C_2 H^2. \quad (12c)$$

Substituting (12a), (12b), and (12c) into (6a), (6b), and (6c) and making use of the Bernoulli equation (8) and then equating the coefficients of the functions H^i to zero, we obtain an algebraic system of equations in terms of A_i , B_i , and C_i ($i = 0, 1, 2$). Solving this system of algebraic equations, with the aid of Mathematica, one possible set of values of A_i , B_i , and C_i ($i = 0, 1, 2$) is

$$\begin{aligned} a &= 1, \quad b = 3, \\ \alpha &= -\frac{\gamma B_2^2 + 54 \beta k_2^2 A_2 + 54 \beta k_3^2 A_2}{A_2 B_2}, \\ \lambda &= -\frac{54 \beta (k_2^2 + k_3^2)}{A_2}, \end{aligned}$$

$$A_0 = \frac{A_2 B_0}{B_2}, \quad A_1 = \left(\frac{1}{3}\right) A_2, \quad B_1 = \left(\frac{1}{3}\right) B_2,$$

$$C_0 = B_0, \quad C_1 = \left(\frac{1}{3}\right) B_2, \quad C_2 = B_2,$$

$$k_1 = -\frac{\beta k_2 (108 B_0 k_3^2 - k_2^2 B_2 - k_3^2 B_2 + 108 B_0 k_2^2)}{B_2}. \quad (13)$$

As a result, a solution of (4a), (4b), and (4c) is

$$u(t, x, y) = A_0 + A_1 a \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b \cosh[a(z+C)] - b \sinh[a(z+C)]} \right\} + A_2 a^2 \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b \cosh[a(z+C)] - b \sinh[a(z+C)]} \right\}^2, \quad (14a)$$

$$v(t, x, y) = B_0 + B_1 a \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b \cosh[a(z+C)] - b \sinh[a(z+C)]} \right\} + B_2 a^2 \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b \cosh[a(z+C)] - b \sinh[a(z+C)]} \right\}^2, \quad (14b)$$

$$w(t, x, y) = C_0 + C_1 a \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b \cosh[a(z+C)] - b \sinh[a(z+C)]} \right\} + C_2 a^2 \left\{ \frac{\cosh[a(z+C)] + \sinh[a(z+C)]}{1 - b \cosh[a(z+C)] - b \sinh[a(z+C)]} \right\}^2, \quad (14c)$$

where $z = k_1 t + k_2 x + k_3 y + k_4$.

2.2. Solutions of (4a), (4b), and (4c) Using Riccati Equation as the Simplest Equation. The balancing procedure yields $M = 2$ and so the solutions of (6a), (6b), and (6c) are of the form

$$E(z) = A_0 + A_1 H + A_2 H^2, \quad (15a)$$

$$F(z) = B_0 + B_1 H + B_2 H^2, \quad (15b)$$

$$G(z) = C_0 + C_1 H + C_2 H^2. \quad (15c)$$

Substituting (15a), (15b), and (15c) into (6a), (6b), and (6c) and using (10), we obtain an algebraic system of equations in terms of A_i , B_i , and C_i ($i = 0, 1, 2$) by equating all coefficients of the functions H^i to zero. Solving the resultant system, one possible set of values is

$$a = 1, \quad b = 3, \quad c = 1, \\ \alpha = \frac{\gamma B_2^2 - 6 \beta k_2^2 A_2 - 6 \beta k_3^2 A_2}{B_2 A_2},$$

$$\lambda = \frac{6 \beta (k_2^2 + k_3^2)}{A_2},$$

$$A_0 = \frac{A_2 B_0}{B_2}, \quad A_1 = 3 A_2, \quad B_1 = 3 B_2,$$

$$C_0 = -B_0, \quad C_1 = -3 B_2, \quad C_2 = -B_2,$$

$$k_1 = -\frac{\beta k_2 (-17 k_3^2 B_2 - 17 k_2^2 B_2 + 12 B_0 k_3^2 + 12 B_0 k_2^2)}{B_2}. \quad (16)$$

Consequently, the solutions of (4a), (4b), and (4c) are

$$u(t, x, y) = A_0 + A_1 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[\frac{1}{2} \theta (z+C) \right] \right\} + A_2 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[\frac{1}{2} \theta (z+C) \right] \right\}^2, \quad (17a)$$

$$v(t, x, y) = B_0 + B_1 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[\frac{1}{2} \theta (z+C) \right] \right\} + B_2 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[\frac{1}{2} \theta (z+C) \right] \right\}^2, \quad (17b)$$

$$w(t, x, y) = C_0 + C_1 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[\frac{1}{2} \theta (z+C) \right] \right\} + C_2 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left[\frac{1}{2} \theta (z+C) \right] \right\}^2, \quad (17c)$$

$$u(t, x, y)$$

$$= A_0 + A_1 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2} \theta z \right) + \frac{\operatorname{sech}(\theta z/2)}{C \cosh(\theta z/2) - (2a/\theta) \sinh(\theta z/2)} \right\} + A_2 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2} \theta z \right) + \frac{\operatorname{sech}(\theta z/2)}{C \cosh(\theta z/2) - (2a/\theta) \sinh(\theta z/2)} \right\}^2, \quad (18a)$$

$$v(t, x, y) = B_0 + B_1 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2} \theta z \right) + \frac{\operatorname{sech}(\theta z/2)}{C \cosh(\theta z/2) - (2a/\theta) \sinh(\theta z/2)} \right\} + B_2 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh \left(\frac{1}{2} \theta z \right) + \frac{\operatorname{sech}(\theta z/2)}{C \cosh(\theta z/2) - (2a/\theta) \sinh(\theta z/2)} \right\}^2, \quad (18b)$$

$$\begin{aligned}
w(t, x, y) &= C_0 + C_1 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left(\frac{1}{2}\theta z\right) \right. \\
&\quad \left. + \frac{\operatorname{sech}(\theta z/2)}{C \cosh(\theta z/2) - (2a/\theta) \sinh(\theta z/2)} \right\} \\
&\quad + C_2 \left\{ -\frac{b}{2a} - \frac{\theta}{2a} \tanh\left(\frac{1}{2}\theta z\right) \right. \\
&\quad \left. + \frac{\operatorname{sech}(\theta z/2)}{C \cosh(\theta z/2) - (2a/\theta) \sinh(\theta z/2)} \right\}^2, \quad (18c)
\end{aligned}$$

where $z = k_1 t + k_2 x + k_3 y + k_4$.

3. Conservation Laws of (4a), (4b), and (4c)

In this section we derive conservation laws for the coupled Zakharov-Kuznetsov system (4a), (4b), and (4c). The multiplier approach will be used. For details the reader is referred to [11, 16–18].

In our case we obtain multipliers [17] of the form

$$\Lambda_1 = f_1(y), \quad \Lambda_2 = f_2(y), \quad \Lambda_3 = f_3(y), \quad (19)$$

and corresponding to the above multipliers we then obtain the following conserved vectors [17] of (4a), (4b), and (4c):

$$\begin{aligned}
T_1^t &= f_1(y) u, \\
T_2^x &= \frac{1}{3} \left\{ -3\alpha f_1(y) uv - \beta f_1''(y) u - 3\gamma f_1(y) vw \right. \\
&\quad \left. - 3\beta f_1(y) u_{xx} + \beta f_1'(y) u_y - \beta f_1(y) u_{yy} \right\}, \\
T_3^y &= \frac{1}{3} \left\{ \beta f_1'(y) u_x - 2\beta f_1(y) u_{xy} \right\}, \\
T_2^t &= f_2(y) v, \\
T_2^x &= \frac{1}{3} \left\{ -3\lambda f_2(y) uv - \beta f_2''(y) v - 3\beta f_2(y) v_{xx} \right. \\
&\quad \left. + \beta f_2'(y) v_y - \beta f_2(y) v_{yy} \right\}, \\
T_2^y &= \frac{1}{3} \left\{ \beta f_2'(y) v_x - 2\beta f_2(y) v_{xy} \right\}, \\
T_3^t &= f_3(y) w, \\
T_3^x &= \frac{1}{3} \left\{ -3\lambda f_3(y) uv - \beta f_3''(y) w - 3\beta f_3(y) w_{xx} \right. \\
&\quad \left. + \beta f_3'(y) w_y - \beta f_3(y) w_{yy} \right\}, \\
T_3^y &= \frac{1}{3} \left\{ \beta f_3'(y) w_x - 2\beta f_3(y) w_{xy} \right\}. \quad (20)
\end{aligned}$$

It should be noted that due to the presence of the arbitrary function $f(y)$ in the multipliers there are infinitely many conservation laws for the coupled Zakharov-Kuznetsov system (4a), (4b), and (4c).

4. Concluding Remarks

In this paper we obtained some exact solutions of the coupled Zakharov-Kuznetsov system (4a), (4b), and (4c) by the aid of the simplest equation method. The solutions obtained are solitary waves. Moreover, the conservation laws for the coupled Zakharov-Kuznetsov system (4a), (4b), and (4c) were also derived by using the multiplier approach.

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