Research Article

Existence and Continuity of Solutions to a Class of Pseudodifferential Equations over p -Adic Field

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We study the pseudodifferential operator T^α and the pseudodifferential equations of type $T^\alpha u + u = \delta_{x_k}$ over p -adic field $\mathbb{Q}_p$, where δ_{x_k} is the Dirac delta function. We discuss the existence and uniqueness of solutions to the equations. Furthermore, we give conditions for the continuity of the solutions u_k when u belongs to the space $L_2(\mathbb{Q}_p)$.

1. Introduction

In recent years p -adic analysis has received a lot of attention due to its applications in mathematical physics; see, for example, [1–13] and references therein.

A good example of the applications is the pseudodifferential equations on the field $\mathbb{Q}_p$. In 60s of the last century, Gibbs first gives logic derivative on dyadic field. Since then, Vladimirov et al. [13] extended logic derivative to p -adic field and made the certain correction of the original Gibbs definition. This kind of derivative referred to Vladimirov pseudodifferential operator. Kuzhel and Albeverio et al. used Vladimirov operator to study pseudodifferential equations; and see, for example, [14–16]. However, the Vladimirov pseudodifferential operator, as a kind of operation, is not closed in the test function space $S(\mathbb{Q}_p)$. This makes the definition of Vladimirov operator difficult to be applied to distribution space $S'(\mathbb{Q}_p)$. In 1992, Su [17] defined derivative and integral operator T^α on locally compact Vilenkin groups G . The space of test functions space $S(G)$ and its distribution $S'(G)$ with the operation of T^α is closed. T^α is defined as a pseudodifferential operator with the symbol $\langle \xi \rangle^\alpha$

$$T^\alpha \varphi = \left(\langle \xi \rangle^\alpha \varphi^\wedge(\xi) \right)^\vee, \quad (1)$$

where $\langle \xi \rangle = \max\{1, |\xi|\}$, $\alpha \in \mathbb{R}$. The new definition plays a role in promoting the development of theory for p -adic analysis.

In this paper, we study a class of pseudodifferential equations of type

$$T^\alpha u + u = \delta_{x_k}, \quad u \in L_2(\mathbb{Q}_p), \quad x_k \in \mathbb{Q}_p, \quad (2)$$

where T^α is pseudodifferential operator defined by Su and $\mathbb{Q}_p$ is the field of p -adic numbers.

Our aim is to show that (2) has a unique solution u_k belonging to $L_2(\mathbb{Q}_p)$ if $\alpha > 1/2$ and has no solutions if $\alpha \leq 1/2$.

Moreover, we give the condition for the continuity of the solutions u_k with the index α .

2. Preliminaries

We use the notations indicated in Taiblesons book [18]. Let $p \geq 2$ be a prime number. The field $\mathbb{Q}_p$ of p -adic numbers is a topologically complete space of rational numbers $\mathbb{Q}$ with respect to p -adic norm $|\cdot|_p$ (non-Archimedean norm), which is defined as follows:

if $x = 0$, then $|0|_p = 0$;

if $x \neq 0$ is an arbitrary rational number, we define $|x|_p = p^{-r}$. And x can be represented as $x = p^r(m/n)$, where $r = \text{order}_p x \in \mathbb{Z}$ and integers m and n are relatively primes and not divisible by p .

The p -adic norm $|\cdot|_p$ satisfies the strong triangle inequality $|x + y|_p \leq \max(|x|_p, |y|_p)$.

Any p -adic number $x \neq 0$ in the topologically complete space $\mathbb{Q}_p$ can be presented as series uniquely:

$$x = \sum_{i=m}^{+\infty} x_i p^i,$$

$$0 \leq x_i \leq p-1, \quad i = m, m+1, \dots, \quad m \in \mathbb{Z}, \quad x_m \neq 0. \quad (3)$$

Define the bitwise operation of addition and multiplication of x in $\mathbb{Q}_p$ (either from left to right carry, or not carry); then $\mathbb{Q}_p$ is a locally compact, totally disconnected, and complete topological field.

Denote by $B_0 = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ the ring of integers in $\mathbb{Q}_p$ which is the subring of $\mathbb{Q}_p$. B_0 is the compact subring and compact subspace of locally compact field $\mathbb{Q}_p$. With the addition operation of p -adic field $\mathbb{Q}_p$, there exists the Haar measure dx on $\mathbb{Q}_p$ such that $\int_{B_0} dx = 1$. Denote by

$$\begin{aligned} B_r &= \{x \in \mathbb{Q}_p : |x|_p \leq p^r\}, \\ S_r &= \{x \in \mathbb{Q}_p : |x|_p = p^r\}, \end{aligned} \quad (4)$$

respectively, the ball and the sphere of radius p^r with the center at $0 \in \mathbb{Q}_p$. Obviously, $\int_{B_r} dx = p^r$, $\int_{S_r} dx = p^r(1-p^{-1})$.

A complex-valued function $\varphi : \mathbb{Q}_p \rightarrow \mathbb{C}$ defined on $\mathbb{Q}_p$ is called locally constant if for any $x \in \mathbb{Q}_p$ there exists an integer $l(x) \in \mathbb{Z}$ satisfying

$$\varphi(x+x') = \varphi(x), \quad \forall x' \in B_{l(x)}. \quad (5)$$

Denote by $\varepsilon = \varepsilon(\mathbb{Q}_p)$ the linear space of all the locally constant functions. $S = S(\mathbb{Q}_p)$ is defined as the linear space of all locally constant functions with compact support in $\mathbb{Q}_p$.

The convergence of the point in ε according to the following definition: $\varphi_k \xrightarrow{\varepsilon} 0$, $k \rightarrow \infty$ if and only if for any compact subset $E \subset \mathbb{Q}_p$, $\varphi_k \rightarrow 0$, $k \rightarrow \infty$ holds uniformly on E .

The convergence in S has the following meaning: $\varphi_k \xrightarrow{S} 0$, $k \rightarrow \infty$ if and only if there exists the indices l and N which do not depend on k , such that the functions $\{\varphi_k\}$ with supports in the ball B_N and with constant on the coset of B_l , $\varphi_k \rightarrow 0$, $k \rightarrow \infty$ hold uniformly in B_N . Then, ε and S are complete topological linear spaces. Also denote by S the test function space.

The Fourier transform of $\varphi \in S(\mathbb{Q}_p)$ is defined by the formula

$$\varphi^\wedge(\xi) = \int_{\mathbb{Q}_p} \varphi(x) \overline{\chi_p(\xi x)} dx, \quad \xi \in \mathbb{Q}_p, \quad (6)$$

and inverse Fourier transform $\varphi^\vee$ by

$$\varphi^\vee(x) = \int_{\mathbb{Q}_p} \varphi(\xi) \chi_p(\xi x) d\xi, \quad x \in \mathbb{Q}_p, \quad (7)$$

where $\chi_p(x) = e^{2\pi i \{x\}}$ is an additive character of the field $\mathbb{Q}_p$, with value 1 in B_0 , and $\{x\} = x_{-1}p^{-1} + x_{-2}p^{-2} + \dots + x_m p^m$. The

Fourier transform and inverse Fourier transform map $S(\mathbb{Q}_p)$ onto $S(\mathbb{Q}_p)$.

Denote by $S' = S'(\mathbb{Q}_p)$ the distribution space of test function space S . S' is a complete topological linear space under the dual topology. The convergence in S' according to the following definition: $f_k \xrightarrow{S'} 0$, $k \rightarrow \infty$ if and only if $\langle f_k, \varphi \rangle \rightarrow 0$, $k \rightarrow \infty$ holds for any $\varphi \in S$.

It follows from the definition of $S(\mathbb{Q}_p)$ that any test function $\varphi \in S(\mathbb{Q}_p)$ is continuous on $\mathbb{Q}_p$. This means the Dirac delta function $\langle \delta_x, \varphi \rangle = \varphi(x)$ is well posed for any point $x \in \mathbb{Q}_p$.

Denote by $L_2(\mathbb{Q}_p)$ the set of the measurable functions f on $\mathbb{Q}_p$ with the condition $\int_{\mathbb{Q}_p} |f(x)|^2 dx < \infty$. And set $L_2(\mathbb{Q}_p)$ is a Hilbert space satisfying the scalar product $(f, g)_{L_2(\mathbb{Q}_p)} = \int_{\mathbb{Q}_p} f(x) \overline{g(x)} dx$.

Let E be compact set, and 1_E is the indicative function of set E . Then there exist standard sequences $\Delta_k(x)$ of 1 satisfying $\Delta_k(x) = 1_{B_k}$, $k \in \mathbb{Z}$, $x \in \mathbb{Q}_p$.

3. Pseudodifferential Operator T^α

In 1992, Su [17] has given definitions of the derivative for the p -adic local fields $\mathbb{Q}_p$, including derivatives of the fractional orders and real orders.

Let $\xi \in \mathbb{Q}_p$ and $\langle \xi \rangle = \max\{1, |\xi|\}$. Its role is played by the operator of pseudodifferential operator T^α ($\alpha \in \mathbb{R}$) which is defined as

$$T^\alpha \varphi(x) = \left(\langle \xi \rangle^\alpha \varphi^\wedge(\xi) \right)^\vee(x), \quad (8)$$

for $\varphi \in S(\mathbb{Q}_p)$. It is easy to see that $T^\alpha \varphi \in S(\mathbb{Q}_p)$. With

$$\langle T^\alpha f, \varphi \rangle = \langle f, T^\alpha \varphi \rangle, \quad f \in S'(\mathbb{Q}_p), \quad \varphi \in S(\mathbb{Q}_p), \quad (9)$$

the defined domain of T^α can be extended to the space $S'(\mathbb{Q}_p)$. Thus, we also have $T^\alpha f = (\langle \xi \rangle^\alpha f^\wedge(\xi))^\vee$ with $f \in S'(\mathbb{Q}_p)$ and $T^\alpha f \in S'(\mathbb{Q}_p)$.

Definition 1. If $\alpha > 0$, then T^α is defined as p -adic derivatives of the order α on $S'(\mathbb{Q}_p)$. And if $\alpha < 0$, then T^α is defined as p -adic integral of the order $-\alpha$ on $S'(\mathbb{Q}_p)$. If $\alpha = 0$, $T^0 f = f$ for any $f \in S'$, then T^0 is called the identity operator.

In what follows we consider T^α as an operator in Hilbert space $L_2(\mathbb{Q}_p) = L_2$. Obviously, the set of functions $D(T^\alpha) = \{f \in L_2 : \langle \xi \rangle^\alpha f^\wedge \in L_2\}$ is the domain of definition of T^α on the space L_2

$$T^\alpha f = \left(\langle \xi \rangle^\alpha f^\wedge \right)^\vee. \quad (10)$$

In [19], Qiu and Su have recently studied the spectrum of T^α and constructed the set of eigenfunctions of T^α :

$$\psi_{NjI}(x) = p^{-N/2} \chi(p^{N-1} j x) \Delta_0(p^N x - z_I), \quad (11)$$

$$N \in \mathbb{Z}, \quad I = z_I + B_0 \in \frac{\mathbb{Q}_p}{B_0}, \quad j = 1, 2, \dots, p-1$$

which forms an orthonormal basis in $L_2(\mathbb{Q}_p)$ (p -adic wavelet basis) such that

$$T^\alpha \psi_{NjI}(x) = \begin{cases} p^{(1-N)\alpha} \psi_{NjI}(x), & N < 1, \\ \psi_{NjI}(x), & N \geq 1. \end{cases} \quad (12)$$

Theorem 2. *The function f is continuous in $D(T^\alpha)$ if and only if $\alpha > 1/2$.*

Proof. Let $f \in D(T^\alpha)$; we expand the function f under the p -adic wavelet basis (11) as follows:

$$f(x) = \sum_{N=1}^{\infty} \sum_{j=1}^{p-1} \sum_I (f, \psi_{NjI}) \psi_{NjI}(x) + \sum_{N=-\infty}^0 \sum_{j=1}^{p-1} \sum_I (f, \psi_{NjI}) \psi_{NjI}(x). \quad (13)$$

Evidently, $\psi_{NjI}(x)$ belong to the domain of definition of T^α , and then, the functions $\psi_{NjI}(x)$ are continuous on $\mathbb{Q}_p$. Thus, it is sufficient to prove the continuity of $f(x)$ with verifying that the series (13) converges uniformly.

First, we note that there is at most one z_I such that $\psi_{NjI}(x) \neq 0$ for the fixed N and x . Indeed, if there are z_{I_1} and z_{I_2} satisfying $\psi_{NjI}(x) \neq 0$, then we have $\Delta_0(p^N x - z_{I_1}) = 1$. But $|p^N x - z_{I_1}| \leq 1$ and $|p^N x - z_{I_2}| \leq 1$. With the strong triangle inequality $|z_{I_1} - z_{I_2}| \leq 1$. From the condition $I = z_I + B_0 \in \mathbb{Q}_p/B_0$, we can get the equality $I_1 = I_2$.

Thus, the sum with the parameter I consists of at most one nonzero term for fixed N and x .

Further, following from (11) and (13), we obtain

$$|\psi_{NjI}(x)| \leq p^{-N/2}, \quad |(f, \psi_{NjI})| \leq \|f\|_{L_2(\mathbb{Q}_p)}. \quad (14)$$

The relations (14) obtain the following estimate if $N \geq 1$

$$\left| \sum_{j=1}^{p-1} \sum_I (f, \psi_{NjI}) \psi_{NjI}(x) \right| \leq p^{-N/2} \|f\|_{L_2(\mathbb{Q}_p)} (p-1). \quad (15)$$

The estimate gives the uniform convergency of the first series in (13).

The condition $f \in D(T^\alpha)$ and (12) imply $(f, \psi_{NjI}) = p^{(N-1)\alpha} (T^\alpha f, \psi_{NjI})$ for $N < 1$. For a fixed $N < 1$, using Cauchy-Schwarz equality and (14), we obtain

$$\begin{aligned} & \left| \sum_{j=1}^{p-1} \sum_I (f, \psi_{NjI}) \psi_{NjI}(x) \right| \\ &= \left| \sum_{j=1}^{p-1} \sum_I p^{(N-1)\alpha} (T^\alpha f, \psi_{NjI}) \psi_{NjI}(x) \right| \\ &\leq \left\{ \sum_{j=1}^{p-1} \sum_I |(T^\alpha f, \psi_{NjI})|^2 \right\}^{1/2} \\ &\quad \times \left\{ \sum_{j=1}^{p-1} \sum_I p^{2(N-1)\alpha} |\psi_{NjI}(x)|^2 \right\}^{1/2} \\ &\leq \|T^\alpha f\|_{L_2(\mathbb{Q}_p)} \left\{ \sum_{j=1}^{p-1} p^{-N+2(N-1)\alpha} \right\}^{1/2}. \end{aligned} \quad (16)$$

The estimate obtained above means the second series in (13) is uniformly convergent if $\alpha > 1/2$. Thus, function f is continuous in $D(T^\alpha)$ on $\mathbb{Q}_p$ for $\alpha > 1/2$. Theorem 2 is proved. $\square$

Next when $\alpha \leq 1/2$, we will give an example in which the function belongs to $D(T^\alpha)$ but is not continuous on $\mathbb{Q}_p$.

Example 3. The function

$$f(x) = \sum_{N=-\infty}^{-1} \frac{1}{|N|} p^{(N-1)/2} \psi_{N10}(x) \quad (17)$$

belongs to $D(T^\alpha)$ but $f(x)$ is not continuous on $\mathbb{Q}_p$ for $\alpha \leq 1/2$.

It is easy to see $f \in L_2(\mathbb{Q}_p)$ and its Fourier transform is

$$f^\wedge(\xi) = \sum_{N=-\infty}^{-1} \frac{1}{|N|} p^{(N-1)/2} \psi_{N10}^\wedge(\xi). \quad (18)$$

From (8) and (12), $\langle \xi \rangle^\alpha \psi_{N10}^\wedge(\xi) = p^{\alpha(1-N)} \psi_{N10}^\wedge(\xi)$. Thus

$$\langle \xi \rangle^\alpha f^\wedge(\xi) = \sum_{N=-\infty}^{-1} \frac{1}{|N|} p^{(N-1)/2} p^{\alpha(1-N)} \psi_{N10}^\wedge(\xi). \quad (19)$$

$\langle \xi \rangle^\alpha f^\wedge(\xi) \in L_2(\mathbb{Q}_p)$ for $\alpha \leq 1/2$ and $\langle \xi \rangle^\alpha f^\wedge(\xi) \notin L_2(\mathbb{Q}_p)$ for $\alpha > 1/2$ with the orthonormal basis $\{\psi_{N10}\}_{N<0}$. So, $f(x) \in D(T^\alpha)$ for $\alpha \leq 1/2$.

Next, We will show that $f(x)$ is not continuous on $\mathbb{Q}_p$. First, using (11), we rewrite the definition (17) of f as

$$f(x) = \sum_{N=-\infty}^{-1} \frac{1}{|N|} p^{-1/2} \chi(p^{N-1}x) \Delta_0(p^N x). \quad (20)$$

We consider the sequence $x_n = p^n$, $n \in \mathbb{N}$. It is easy to see $x_n \rightarrow 0$, ($n \rightarrow \infty$) in the p -adic norm $|\cdot|_p$. Moreover, $\Delta_0(p^N x_n) = \Delta_0(p^{N+n}) = 0$ when $N+n \leq -1$. On the other hand, when $N+n \leq 1$, $p^{N-1}x_n$ become an integer p -adic number, and then $\chi(p^{N-1}x_n) = \chi(p^{N+n-1}) = 1$. From the above relations and (20) we can get that

$$\begin{aligned} f(x_n) &= f(p^n) \\ &= p^{-1/2} \left(\frac{\chi(p^{-1})}{n} + \sum_{N=-n+1}^{-1} \frac{1}{|N|} \right) \rightarrow \infty \end{aligned} \quad (21)$$

as $n \rightarrow \infty$.

Therefore, $f(x)$ cannot be continuous at $x = 0$.

4. Pseudodifferential Equation

In this section we will consider the pseudodifferential equation

$$T^\alpha u + u = \delta_{x_k}, \quad u \in L_2(\mathbb{Q}_p), \quad x_k \in \mathbb{Q}_p, \quad (22)$$

where $T^\alpha : L_2(\mathbb{Q}_p) \rightarrow S'(\mathbb{Q}_p)$ is pseudodifferential operator in the distribution sense.

Theorem 4. Equation (22) has a unique solution $u = u_k \in L_2(\mathbb{Q}_p)$ for $\alpha > 1/2$ and has no solutions belonging to $L_2(\mathbb{Q}_p)$ for $\alpha \leq 1/2$.

Proof. Let $\varphi \in D(T^\alpha)$. Similar to the proof of Theorem 2, we give the expansion of φ using the uniformly convergent series with respect to the complex-conjugated p -adic wavelet basis $\{\overline{\psi_{NjI}}\}$. For $\{\overline{\psi_{NjI}}\}$ are continuous functions on $\mathbb{Q}_p$ we can write

$$\varphi(x_k) = \sum_{N=-\infty}^{+\infty} \sum_{j=1}^{p-1} \sum_I (\varphi, \overline{\psi_{NjI}}) \overline{\psi_{NjI}}(x_k) \quad (23)$$

for $x = x_k$.

Consider

$$\begin{aligned} \overline{\psi_{NjI}}(x_k) &= p^{-N/2} \chi(p^{N-1} j x_k) \Delta_0(p^N x_k - z_I) \\ &= p^{-N/2} \chi(-p^{N-1} j x_k) \Delta_0(p^N x_k - z_I). \end{aligned} \quad (24)$$

It is easy to see $\overline{\psi_{NjI}}(x_k) \neq 0$ equals $|p^N x_k - z_I| \leq 1$. Here $I = z_I + B_0 \in \mathbb{Q}_p/B_0$ and by the strong triangle inequality we obtain $|p^N x_k - z_I| \leq 1$ equals $z_I = \{p^N x_k\}$. Then, we obtain the conclusion that

$$\overline{\psi_{NjI}}(x_k) = \begin{cases} 0, & z_I \neq \{p^N x_k\}, \\ p^{-N/2} \chi(-p^{N-1} j x_k), & z_I = \{p^N x_k\}. \end{cases} \quad (25)$$

And then

$$\begin{aligned} \langle \delta_{x_k}, \varphi \rangle &= \varphi(x_k) \\ &= \sum_{N=-\infty}^{+\infty} \sum_{j=1}^{p-1} p^{-N/2} \chi(-p^{N-1} j x_k) (\varphi, \overline{\psi_{Nj\{p^N x_k\}}}) \\ &= \sum_{N=-\infty}^{+\infty} \sum_{j=1}^{p-1} p^{-N/2} \chi(-p^{N-1} j x_k) \langle \psi_{Nj\{p^N x_k\}}, \varphi \rangle. \end{aligned} \quad (26)$$

For $S(\mathbb{Q}_p) \subset D(T^\alpha)$, the equality (26) implies that

$$\delta_{x_k} = \sum_{N=-\infty}^{+\infty} \sum_{j=1}^{p-1} p^{-N/2} \chi(-p^{N-1} j x_k) \psi_{Nj\{p^N x_k\}}, \quad (27)$$

in which series converges uniformly in $S'(\mathbb{Q}_p)$.

We suppose that there is a function $u_k \in L_2(\mathbb{Q}_p)$ which can be represented as a convergent series in L_2 :

$$u_k(x) = \sum_{N=-\infty}^{+\infty} \sum_{j=1}^{p-1} \sum_I c_{NjI} \psi_{NjI}(x). \quad (28)$$

Applying pseudodifferential operator $T^\alpha + I$ on the both sides, we obtain a series with (12)

$$\begin{aligned} T^\alpha u_k + u_k &= \sum_{N=-\infty}^0 \sum_{j=1}^{p-1} \sum_I c_{NjI} (1 + p^{\alpha(1-N)}) \psi_{NjI} \\ &\quad + \sum_{N=1}^{+\infty} \sum_{j=1}^{p-1} \sum_I 2c_{NjI} \psi_{NjI}, \end{aligned} \quad (29)$$

which converges in $S'(\mathbb{Q}_p)$ for $T^\alpha S(\mathbb{Q}_p) \subset L_2(\mathbb{Q}_p)$. Comparing the two series (27) and (29), we obtain

$$c_{Nj\{p^N x_k\}} = \begin{cases} \frac{1}{2} p^{-N/2} \chi(-p^{N-1} j x_k), & N \geq 1, \\ (p^{\alpha(1-N)} + 1)^{-1} p^{-N/2} \chi(-p^{N-1} j x_k), & N < 1. \end{cases} \quad (30)$$

Thus,

$$\begin{aligned} u_k(x) &= \sum_{N=1}^{+\infty} \sum_{j=1}^{p-1} \frac{1}{2} p^{-N/2} \\ &\quad \times \chi(-p^{N-1} j x_k) \psi_{Nj\{p^N x_k\}}(x) \\ &\quad + \sum_{N=-\infty}^0 \sum_{j=1}^{p-1} (p^{\alpha(1-N)} + 1)^{-1} p^{-N/2} \\ &\quad \times \chi(-p^{N-1} j x_k) \psi_{Nj\{p^N x_k\}}(x). \end{aligned} \quad (31)$$

Next, We will show that the series (31) belong to $L_2(\mathbb{Q}_p)$ for $\alpha > 1/2$. For the general term of the first series we obtain

$$\left| \frac{1}{2} p^{-N/2} \chi(-p^{N-1} j x_k) \right|^2 \leq \frac{1}{4} p^{-N}, \quad N \geq 1 \quad (32)$$

that means the first series converges in $L_2(\mathbb{Q}_p)$ for any $\alpha \in \mathbb{R}$.

We estimate the general term of the second series as follows:

$$\begin{aligned} &\left| (p^{\alpha(1-N)} + 1)^{-1} p^{-N/2} \chi(-p^{N-1} j x_k) \right|^2 \\ &\leq p^{-N} (p^{\alpha(1-N)} + 1)^{-2} \leq p^{-2\alpha} p^{-N(1-2\alpha)}, \quad N < 1. \end{aligned} \quad (33)$$

It is easy to see the second series converges in $L_2(\mathbb{Q}_p)$ for $\alpha > 1/2$.

Thus u_k is a unique solution of (31) for $\alpha > 1/2$.

For $p^{\alpha(1-N)} + 1 \leq 2p^{\alpha(1-N)}$ with $N < 1$, we estimate the general term of the second series as follows:

$$\begin{aligned} &\left| (p^{\alpha(1-N)} + 1)^{-1} p^{-N/2} \chi(-p^{N-1} j x_k) \right|^2 \\ &\geq \frac{1}{4} p^{-2\alpha} p^{-N(1-2\alpha)}, \quad N < 1 \end{aligned} \quad (34)$$

that means the series diverges in $L_2(\mathbb{Q}_p)$ for $\alpha \leq 1/2$. Theorem 4 is proved. $\square$

Theorem 5. The solution u_k of (22) is continuous on $\mathbb{Q}_p$ for $\alpha > 1$.

Proof. We will show that the series (31) converges uniformly on $\mathbb{Q}_p$ if $\alpha > 1$ and converges uniformly on the ball not containing x_k if $1/2 < \alpha \leq 1$.

Indeed, the general term (31) of the first series does not exceed $(1/2)p^{-N}$ by (14) when $N \geq 1$. Thus, the first series converges uniformly with $N \geq 1$.

When $N < 1$ the general term (31) of the second series satisfies

$$(p^{\alpha(1-N)})^{-1} p^{-N} \leq p^{-\alpha} p^{-N(1-\alpha)}. \quad (35)$$

The estimate we obtained above implies that for $\alpha > 1$ the subseries of (31) converges uniformly with $N < 1$. Hence, the series (31) converges uniformly for $\alpha > 1$. Theorem 5 is proved. $\square$

Let $\alpha > 1/2$ and fixed points $x_1, x_2, \dots, x_n \in \mathbb{Q}_p$. Then, define $Sp\{u_k\}_1^n$ as the linear span of solutions u_k ($k = 1, 2, \dots, n$) of (22). Hence, we have the following.

Theorem 6. $Sp\{u_k\}_1^n \cap D(T^{\alpha/2}) = \{0\}$ for $1/2 < \alpha \leq 1$ and $Sp\{u_k\}_1^n \subset D(T^{\alpha/2})$ for $\alpha > 1$.

Proof. The solution u_k of (22) is obtained by (31). Considering the expansion (31) and semigroup property (c.f. [19])

$$T^{\alpha_1} T^{\alpha_2} = T^{\alpha_1 + \alpha_2}, \quad \alpha_1, \alpha_2 > 0 \quad (36)$$

of T^α . Obviously, $u_k \in D(T^{\alpha/2})$ if and only if the following series converge in $L_2(\mathbb{Q}_p)$:

$$\begin{aligned} & \sum_{N=1}^{+\infty} \sum_{j=1}^{p-1} \frac{1}{2} p^{-N/2} \chi(-p^{N-1} j x_k) \psi_{Nj\{p^N x_k\}}(x) \\ & + \sum_{N=-\infty}^0 \sum_{j=1}^{p-1} (p^{\alpha(1-N)} + 1)^{-1} p^{-N/2} \\ & \quad \times \chi(-p^{N-1} j x_k) p^{(\alpha/2)(1-N)} \psi_{Nj\{p^N x_k\}}(x). \end{aligned} \quad (37)$$

For the general term of the first series, we have the estimate as follows:

$$\left| \frac{1}{2} p^{-N/2} \chi(-p^{N-1} j x_k) \right|^2 \leq \frac{1}{4} p^{-N}, \quad N \geq 1 \quad (38)$$

that means it converges in $L_2(\mathbb{Q}_p)$ for any $\alpha \in \mathbb{R}$.

Similarly, for the general term of the second series, we have

$$\begin{aligned} & \left| (p^{\alpha(1-N)} + 1)^{-1} p^{-N/2} \chi(-p^{N-1} j x_k) p^{(\alpha/2)(1-N)} \right|^2 \\ & \leq C p^{(\alpha-1)N}, \quad N < 1. \end{aligned} \quad (39)$$

Hence the series converges when $\alpha > 1$. Then $Sp\{u_k\}_1^n \subset D(T^{\alpha/2})$ if $\alpha > 1$.

For $p^{\alpha(1-N)} + 1 \leq 2p^{\alpha(1-N)}$ with $N < 1$, the general term of the second series can be estimated below

$$\begin{aligned} & \left| (p^{\alpha(1-N)} + 1)^{-1} p^{-N/2} \chi(-p^{N-1} j x_k) p^{(\alpha/2)(1-N)} \right|^2 \\ & \geq \frac{1}{4} p^{-\alpha} p^{-N(1-\alpha)}, \quad N < 1 \end{aligned} \quad (40)$$

that means it diverges in $L_2(\mathbb{Q}_p)$ for $\alpha \leq 1$.

Hence $u_k \notin D(T^{\alpha/2})$. Since the estimate (40) does not depend on the choice of u_k . And considering the functions $\{\psi_{Nj\{p^N x_k\}}(x)\}$ ($N < 1$) of the basis $\{\psi_{NjI}(x)\}$ in the u_k ($k = 1, 2, \dots, n$) of (31) are different for any small negative N , we obtain that $Sp\{u_k\}_1^n \cap D(T^{\alpha/2}) = \{0\}$ if $1/2 < \alpha \leq 1$. Theorem 6 is proved. $\square$

5. Conclusions

In this paper, solutions of pseudodifferential equations with the type $T^\alpha u + u = \delta_{x_k}$ over p -adic field $\mathbb{Q}_p$ are considered. First, we give the condition for the continuity of arbitrary function in the domain of definition $D(T^\alpha)$ with the index $\alpha > 1/2$ and give the example to show noncontinuity when $\alpha \leq 1/2$. Then, we obtain the existence of solutions to the pseudodifferential equations. The results show that the equation has a unique solution belonging to $L_2(\mathbb{Q}_p)$ for $\alpha > 1/2$ and has no solutions for $\alpha \leq 1/2$. Furthermore, we show the continuity of solutions when $\alpha > 1$. Finally, the embedding from the linear span of solutions to the domain of definition $D(T^\alpha)$ is obtained.

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