Research Article

A Preconditioner for FETI-DP Method of Stokes Problem with Mortar-Type $CR(h) - P_0(h)$ Discretization

Chunmei Wang^{1,2}

¹ Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China

² Nanjing Normal University Taizhou College, Taizhou 225300, China

Correspondence should be addressed to Chunmei Wang; chunmeisuzhou@163.com

Received 19 October 2012; Revised 21 February 2013; Accepted 21 February 2013

Academic Editor: Salvatore Alfonzetti

Copyright © 2013 Chunmei Wang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

FETI-DP method is developed for incompressible Stokes problem discretized by mortar-type $CR(h) - P_0(h)$ mixed elements. A preconditioner is proposed, and it is proved that the condition number of the preconditioned FETI-DP operator is bounded by $(1 + \log(H/h))^2$, where H and h are mesh sizes. Finally, the numerical tests are presented to verify the theoretical results.

1. Introduction

FETI-DP method is a nonoverlapping domain decomposition method which has been implemented for large scale engineering applications. It has been developed for elliptic problems with heterogeneous coefficients, Stokes problems, and compressible elasticity problems (cf. [1–4]). Recently, extensions to irregular subdomains and inexact subdomain solvers have been done in [5–7]. Li [4] developed a FETI-DP method for the Stokes problem by adding primal continuity constraints to accelerate the convergence of the method. Li and Widlund [8] developed BDDC methods for incompressible Stokes equations which established a close connection between the BDDC and FETI-DP methods for the Stokes case. Kim et al. [9] developed a BDDC method discretized by mortar finite element methods for geometrically nonconforming partitions, and the BDDC preconditioner was shown to be closely related to the Neumann-Dirichlet version of the FETI-DP method.

Nonconforming discretizations are important for multiphysics simulations, contact-impact problems, generations of meshes and partitions aligned with jumps in diffusion coefficients, hp -adaptive methods, and special discretizations in the neighborhood of singularities. In this paper, we use the inf-sup stable $CR(h) - P_0(h)$ mixed elements in each subdomain. In the event of applying the mortar condition, it is necessary to know functions on the interface. However,

the degrees of freedom (d.o.f) of Crouzeix-Raviart element are associated with the edge midpoints and functions on the interface depend on the nodal values corresponding to interface nodes and some subdomain interior nodes lying closest to the interface. We adopt nonstandard mortar condition introduced in [10] which is only associated with nodal values on the interface.

The inf-sup constant for mortar element space is crucial in the analysis of the approximation order. If the constant is independent of mesh size and subdomain size, then the optimal order of approximation follows independently of the number of subdomains and mesh size as in the case of elliptic problems.

The fundamental idea of this paper is the same as [4], with additional technical complications caused by nonmatching grids across subdomain interfaces and the Crouzeix-Raviart element discretization for velocity space. We propose a preconditioner and analyze the condition number bounded by $(1 + \log(H/h))^2$, where H and h are mesh sizes.

The paper is organized as follows. In Section 2, a domain decomposition procedure is described for solving incompressible Stokes problem. In Section 3, the matrix form of the mortar condition is introduced. In Section 4, we derive an FETI-DP operator for the Stokes problem and propose a preconditioner for the FETI-DP operator. Section 5 gives several technical lemmas. Section 6 is devoted to the proof of the condition number of the preconditioned FETI-DP operator.

The numerical tests are presented to verify the theoretical results in Section 7.

Throughout the paper, the following notations are used: $u \asymp v$, $u \leq v$, and $u \geq v$ mean that there exists positive constants c and C independent of the subdomain size and mesh size, such that

$$cu \leq v \leq Cu, \quad u \leq Cv, \quad u \geq cv, \quad \text{respectively.} \quad (1)$$

2. A Domain Decomposition Procedure for Stokes Problems

We consider the following two-dimensional incompressible Stokes problem:

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p &= \mathbf{f}, \quad \text{in } \Omega, \\ -\nabla \cdot \mathbf{u} &= 0, \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{0}, \quad \text{on } \partial\Omega, \end{aligned} \quad (2)$$

where Ω is a bounded polygonal domain in R^2 , $\mathbf{u}$ represents the velocity of fluid, p is the pressure, and $\mathbf{f}$ is the external force.

We first decompose Ω into a nonoverlapping subdomain partition $\{\Omega_i\}_{i=1}^N$. We consider the subdomains to be geometrically conforming. With each subdomain Ω_i , we assume that the triangulation $T_h(\Omega_i)$ is quasiuniform. Let $h_i = \max_{\tau \in T_h(\Omega_i)} \text{diam } \tau$ be the mesh size parameter. Let $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$ denote the interface between the two neighboring subdomains Ω_i and Ω_j , and $\Gamma = \cup_{i=1}^N \partial\Omega_i \setminus \partial\Omega$ be the interface skeleton. We denote the sets of the Crouzeix-Raviart element nodes and the P1 conforming element nodes that are contained in Ω , $\partial\Omega$, Ω_i , $\partial\Omega_i$, and Γ_{ij} by Ω_h^{CR} , $\partial\Omega_h^{\text{CR}}$, $\Omega_{i,h}^{\text{CR}}$, $\partial\Omega_{i,h}^{\text{CR}}$, $\Gamma_{ij,h}^{\text{CR}}$ and Ω_h , $\partial\Omega_h$, $\Omega_{i,h}$, $\partial\Omega_{i,h}$, $\Gamma_{ij,h}$, respectively. The resulting triangulation can be nonmatching across the subdomain interface. We note that each interface Γ_{ij} inherits two independent triangulations. One side of Γ_{ij} associated with Ω_i is defined as a mortar side denoted by $\gamma_{m(i)}$, and the other side associated with Ω_j is defined as a nonmortar side denoted by $\delta_{m(j)}$.

For each subdomain Ω_i , we introduce the following spaces:

$$\begin{aligned} H_D^1(\Omega_i) &= \{v \in H^1(\Omega_i) : v = 0 \text{ on } \partial\Omega_i \cap \partial\Omega\}, \\ L_0^2(\Omega_i) &= \left\{p \in L^2(\Omega_i) : \int_{\Omega_i} p dx = 0\right\}, \\ \Pi^0 &= \left\{p^0 : p^0|_{\Omega_i} \text{ is constant and } \int_{\Omega} p^0 dx = 0\right\}. \end{aligned} \quad (3)$$

Then, the variational form of the problem (2) is as follows: find $(\mathbf{u}, p_I, p^0) \in \prod_{i=1}^N [H_D^1(\Omega_i)]^2 \times \prod_{i=1}^N L_0^2(\Omega_i) \times \Pi^0$, such that

$$\begin{aligned} \sum_{i=1}^N (\nabla \mathbf{u}, \nabla \mathbf{v})_{\Omega_i} - \sum_{i=1}^N (p_I + p^0, \nabla \cdot \mathbf{v})_{\Omega_i} &= \sum_{i=1}^N (\mathbf{f}, \mathbf{v})_{\Omega_i}, \\ \forall \mathbf{v} &\in \prod_{i=1}^N [H_D^1(\Omega_i)]^2, \end{aligned}$$

$$\begin{aligned} -\sum_{i=1}^N (\nabla \cdot \mathbf{u}, q_I)_{\Omega_i} &= 0, \quad \forall q_I \in \prod_{i=1}^N L_0^2(\Omega_i), \\ -\sum_{i=1}^N (\nabla \cdot \mathbf{u}, q^0)_{\Omega_i} &= 0, \quad \forall q^0 \in \Pi^0, \\ \mathbf{u}|_{\Omega_i} - \mathbf{u}|_{\Omega_j} &= 0, \quad \forall \Gamma_{ij} \subset \Gamma. \end{aligned} \quad (4)$$

Here, $(\cdot, \cdot)_{\Omega_i}$ denotes the inner product in $[L^2(\Omega_i)]^n$ for $n = 1, 2$.

We consider the inf-sup stable $\text{CR}(h_i) - P_0(h_i)$ elements in each subdomain Ω_i . Let $\mathbf{W}^h(\Omega_i) = \{\mathbf{v}_i \in [L^2(\Omega_i)]^2 : \mathbf{v}_i|_{\tau} \in [P_1(\tau)]^2, \mathbf{v}_i \text{ is continuous at the midpoints of the edges of } \tau, \text{ and } \mathbf{v}_i(m) = \mathbf{0}, m \text{ is a midpoint on } \partial\Omega \cap \partial\tau, \forall \tau \in T_h(\Omega_i)\}$,

$$Q_i = \{q_i \in L_0^2(\Omega_i) : q_i|_{\tau} \in P_0(\tau), \forall \tau \in T_h(\Omega_i)\}, \quad (5)$$

where $P_m(\tau)$ is a set of polynomials of degree less than or equal to m in τ .

The Lagrange multiplier space corresponding to Γ_{ij} is defined as follows:

$$\begin{aligned} \mathbf{M}(\delta_{m(j)}) &= \left\{ \boldsymbol{\lambda} \in [L^2(\Gamma_{ij})]^2 : \boldsymbol{\lambda}|_{\tau} \text{ is a constant vector}, \right. \\ &\quad \left. \forall \tau \in T_h(\delta_{m(j)}) \right\}. \end{aligned} \quad (6)$$

Then we take the Lagrange multiplier space

$$\mathbf{M} = \prod_{\delta_{m(j)} \subset \Gamma} \mathbf{M}(\delta_{m(j)}). \quad (7)$$

We introduce the nonstandard mortar condition (cf. [10]):

$$Q_m(J_m(\mathbf{v}_i)) = Q_m(\mathbf{v}_j), \quad \text{on } \Gamma_{ij} = \gamma_{m(i)} = \delta_{m(j)}, \quad (8)$$

where $\mathbf{v}_k \in \mathbf{W}^h(\Omega_k)$ ($k = i, j$), $Q_m : [L^2(\Gamma_{ij})]^2 \rightarrow \mathbf{M}(\delta_{m(j)})$ is the L^2 orthogonal projection defined as $(Q_m \mathbf{u}, \boldsymbol{\lambda})_{[L^2(\delta_{m(j)})]^2} = (\mathbf{u}, \boldsymbol{\lambda})_{[L^2(\delta_{m(j)})]^2}$, $\forall \boldsymbol{\lambda} \in \mathbf{M}(\delta_{m(j)})$. The operator J_m is defined in the following. We can also write (8) as follows:

$$\int_{\delta_{m(j)}} (J_m \mathbf{v}_i - \mathbf{v}_j) \cdot \boldsymbol{\lambda}_{ij} ds = 0, \quad \forall \boldsymbol{\lambda}_{ij} \in \mathbf{M}(\delta_{m(j)}). \quad (9)$$

We use e to denote a triangle edge. Let $Z_h(\gamma_{m(i)}) = \prod_{e \in T_h(\gamma_{m(i)})} P_1(e)$ be the space of piecewise linear functions defined on the triangulation $T_h(\gamma_{m(i)})$, and let $T_{h/2}(\gamma_{m(i)})$ be the triangulation obtained as a result of dividing each edge of $T_h(\gamma_{m(i)})$ into two equal segments. Let $Y_{h/2}(\gamma_{m(i)})$ be the conforming space of piecewise linear continuous functions on the triangulation $T_{h/2}(\gamma_{m(i)})$. The midpoint, left and right endpoints of each edge $e \in T_h(\gamma_{m(i)})$ are denoted by x_m^e , x_l^e , and x_r^e , respectively. The length of e is denoted by $|e|$.

We now define the operator $J_m : \mathbf{W}^h(\gamma_{m(i)}) = \mathbf{W}^h(\Omega_i)|_{\gamma_{m(i)}} \rightarrow [Z_h(\gamma_{m(i)})]^2$. It is based on the definition of another interpolation operator $I_m : \mathbf{W}^h(\gamma_{m(i)}) \rightarrow [Y_{h/2}(\gamma_{m(i)})]^2$.

Definition 1. For $\mathbf{u} \in \mathbf{W}^h(\gamma_{m(i)})$, $I_m \mathbf{u} \in [Y_{h/2}(\gamma_{m(i)})]^2$ is defined by the nodal values as

$$I_m \mathbf{u}(x) = \begin{cases} \mathbf{u}(x), & x \in \gamma_{m(i),h}^{\text{CR}}, \\ \frac{|e_l|}{|e_l| + |e_r|} \mathbf{u}(x_m^{e_l}) + \frac{|e_r|}{|e_l| + |e_r|} \mathbf{u}(x_m^{e_r}), & x \in \gamma_{m(i),h}, \\ \mathbf{u}(x_m^{e_e}) + \frac{|e_e|}{|e_e| + |e'_e|} (\mathbf{u}(x_m^{e_e}) - \mathbf{u}(x_m^{e'_e})), & x \in \partial\gamma_{m(i),h}. \end{cases} \quad (10)$$

Here, e_l and e_r are the left- and the right-neighboring edges of $x \in \gamma_{m(i),h}$, respectively. e_e represents a triangle edge of $T_h(\gamma_{m(i)})$, touching $\partial\gamma_{m(i)}$, and e'_e is the corresponding neighboring edge.

Each component of the interpolation is done basically by first joining the neighboring edge midpoints using straight lines, and then simply extending the two end straight lines toward the end of the mortar $\gamma_{m(i)}$ (cf. Figure 1). The operator J_m can now be defined using I_m .

Definition 2. For $\mathbf{u} \in \mathbf{W}^h(\gamma_{m(i)})$, $J_m \mathbf{u} \in [Z_h(\gamma_{m(i)})]^2$ is a vector whose components are piecewise linear functions on the edges $\{e\}$ of $\gamma_{m(i)}$ defined by its values at the two endpoints $x_l^e, x_r^e \in \bar{\gamma}_{m(i),h}$ of each edge e . If e is an interior edge of $\gamma_{m(i)}$, then

$$J_m \mathbf{u}(x) = \begin{cases} \mathbf{u}(x_m^e) + \frac{1}{2} \{I_m \mathbf{u}(x_l^e) - I_m \mathbf{u}(x_r^e)\}, & x = x_l^e, \\ \mathbf{u}(x_m^e) + \frac{1}{2} \{I_m \mathbf{u}(x_r^e) - I_m \mathbf{u}(x_l^e)\}, & x = x_r^e. \end{cases} \quad (11)$$

It is easy to see that, if e is a boundary edge of $\gamma_{m(i)}$, then $J_m \mathbf{u}(x) = I_m \mathbf{u}(x)$ for $x = x_l^e, x_r^e$.

We define the following spaces:

$$\begin{aligned} \mathbf{W}^h(\Omega) &= \prod_{i=1}^N \mathbf{W}^h(\Omega_i), \\ \widetilde{\mathbf{W}}^h(\Omega) &= \left\{ \mathbf{v} \in \mathbf{W}^h(\Omega) : \int_{\Gamma_{ij}} (\mathbf{v}_i - \mathbf{v}_j) ds = \mathbf{0}, \forall \Gamma_{ij} \subset \Gamma \right\}, \\ \mathbf{V} &= \left\{ \mathbf{v} \in \widetilde{\mathbf{W}}^h(\Omega), \mathbf{v} \text{ satisfies (9)} \right\}, \\ Q_I &= \prod_{i=1}^N Q_i, \\ P &= \left\{ q \in L_0^2(\Omega) : q|_{\Omega_i} \in Q_i, \forall i = 1, \dots, N \right\}, \end{aligned} \quad (12)$$

where $\mathbf{V}$ and P are the spaces for velocity and pressure, respectively.

FIGURE 1: Here each component of $\mathbf{u} \in \mathbf{W}^h(\gamma_{m(i)})$ is a (discontinuous) piecewise linear function shown (using solid lines) in the lower figure. The dashed lines (in both upper and lower figures) correspond to each component of $I_m \mathbf{u}$. The solid lines in the upper figure correspond to each component of $J_m \mathbf{u}$.

3. Matrix form of Mortar Conditions

For $\mathbf{v} = (\mathbf{v}_1^t, \dots, \mathbf{v}_N^t)^t \in \widetilde{\mathbf{W}}^h(\Omega)$, $\mathbf{v}_i = (\mathbf{v}_I^i, \mathbf{v}_\Delta^i, \mathbf{v}_c^i)^t$, where the symbols I , Δ , and c represent the d.o.f corresponding to the interior nodes, nodes related to $\int_{\Gamma_{ij}} \mathbf{v} ds = \mathbf{0}$, and nodes related to the remaining parts of edges, respectively. We define the spaces $\mathbf{W}_I$, $\mathbf{W}_\Delta$, and $\mathbf{W}_c$ which consists of $\mathbf{v}_I$, $\mathbf{v}_\Delta$, and $\mathbf{v}_c$, respectively.

Let B_i^{ij} be a matrix with entries

$$(B_i^{ij})_{lk} = \int_{\delta_{m(j)}} \boldsymbol{\psi}_l \cdot \boldsymbol{\phi}_k ds, \quad \forall l = 1, \dots, L, \quad \forall k = 1, \dots, K, \quad (13)$$

where $\{\boldsymbol{\psi}_l\}_{l=1}^L$ is basis for $\mathbf{M}(\delta_{m(j)})$ and $\{\boldsymbol{\phi}_k\}_{k=1}^K$ is basis for $J_m \mathbf{W}^h(\Omega_i)|_{\delta_{m(j)}}$. Let B_j^{ij} be a matrix with entries

$$(B_j^{ij})_{lk} = - \int_{\delta_{m(j)}} \boldsymbol{\psi}_l \cdot \boldsymbol{\phi}_k ds, \quad \forall l = 1, \dots, L, \quad \forall k = 1, \dots, P, \quad (14)$$

where $\{\boldsymbol{\psi}_l\}_{l=1}^L$ is basis for $\mathbf{M}(\delta_{m(j)})$ and $\{\boldsymbol{\phi}_k\}_{k=1}^P$ is basis for $\mathbf{W}^h(\Omega_j)|_{\delta_{m(j)}}$. Then we write (9) into a matrix form as

$$B_i^{ij} \mathbf{v}_i|_{\delta_{m(j)}} + B_j^{ij} \mathbf{v}_j|_{\delta_{m(j)}} = \mathbf{0}. \quad (15)$$

Let $\mathbf{W}_i = \mathbf{W}^h(\Omega_i)|_{\partial\Omega_i}$, $i = 1, \dots, N$. The standard discrete Stokes harmonic extension operator $SH_i : \mathbf{W}_i \rightarrow \mathbf{W}^h(\Omega_i)$ is defined by the following: given a velocity $\mathbf{u}_i \in \mathbf{W}_i$, find

$SH_i \mathbf{u}_i \in \mathbf{W}^h(\Omega_i)$, $p_I^i \in Q_i$ and $p^0 \in \Pi^0$, such that on each subdomain Ω_i ,

$$\begin{aligned} \sum_{i=1}^N (\nabla \mathbf{u}_i, \nabla \mathbf{v}_i)_{\Omega_i} - \sum_{i=1}^N (p_I^i + p^0, \nabla \cdot \mathbf{v}_i)_{\Omega_i} &= \mathbf{0}, \\ \forall \mathbf{v}_i \in \mathbf{W}^h(\Omega_i), \\ - \sum_{i=1}^N (\nabla \cdot \mathbf{u}_i, q_I^i)_{\Omega_i} &= 0, \quad \forall q_I^i \in Q_i, \\ - \sum_{i=1}^N (\nabla \cdot \mathbf{u}_i, q^0)_{\Omega_i} &= 0, \quad \forall q^0 \in \Pi^0, \\ SH_i \mathbf{u}_i &= \mathbf{u}_i, \quad \text{on } \partial\Omega_i. \end{aligned} \quad (16)$$

Define $E_{ij} : \mathbf{M}(\delta_{m(j)}) \rightarrow \mathbf{M}$ to be an extension operator by zero and $R_{ij}^l : \mathbf{W}_l \rightarrow \mathbf{W}_l|_{\delta_{m(j)}}$ for $l = i, j$ to be a restriction operator, and let $B_i = \sum_{\delta_{m(j)} \subset \Gamma} E_{ij} B_i^{ij} R_{ij}^i + \sum_{\delta_{m(i)} \subset \Gamma} E_{ij} B_i^{ij} R_{ij}^i$. Then (15) becomes

$$B\mathbf{w} = \mathbf{0}, \quad (17)$$

where

$$\begin{aligned} B &= (B_1, \dots, B_N), \\ \mathbf{w} &= (\mathbf{w}_1^t, \dots, \mathbf{w}_N^t)^t, \quad \mathbf{w}_i = \mathbf{v}_i|_{\partial\Omega_i}, \quad i = 1, \dots, N. \end{aligned} \quad (18)$$

Let $B_{i,c}$ and $B_{i,\Delta}$ be matrices that consist of the columns of B_i corresponding to the d.o.f related to $\int_{\Gamma_{ij}} \mathbf{w} ds = 0$ and the d.o.f related to the remaining parts of the edges, respectively. Then, (17) can also be written into

$$B_\Delta \mathbf{w}_\Delta + B_c \mathbf{w}_c = \mathbf{0}, \quad (19)$$

where $B_\Delta = (B_{1,\Delta}, \dots, B_{N,\Delta})$ and $B_c = \sum_{i=1}^N B_{i,c} L_c^i$, where $L_c^i : \mathbf{W}_c \rightarrow \mathbf{W}_c^i$ is the restriction map that gives $L_c^i \mathbf{w}_c = \mathbf{w}_c^i$, $\forall i = 1, \dots, N$, for $\mathbf{w}_c \in \mathbf{W}_c$. We will use any of these two expressions (17) and (19) for the sake of convenience.

4. FETI-DP Method

4.1. FETI-DP Operator. In this section, we formulate a FETI-DP operator with the mortar condition (19).

We define the following spaces:

$$\mathbf{W} = \left\{ \mathbf{v} \in \prod_{i=1}^N \mathbf{W}_i : \int_{\Gamma_{ij}} (\mathbf{v}_i - \mathbf{v}_j) ds = 0 \right\}. \quad (20)$$

To solve the Stokes problem efficiently and correctly, we will consider the following primal constraints:

$$\int_{\delta_{m(j)}} (J_m \mathbf{v}_i - \mathbf{v}_j) ds = \mathbf{0}, \quad \forall \delta_{m(j)} \subset \Gamma. \quad (21)$$

Note that (21) holds by replacing $\lambda_{ij} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in (9), because constant multipliers belong to the Lagrange

multiplier space $\mathbf{M}(\delta_{m(j)})$. These primal constraints were introduced by Li (cf. [4]). The primal constraints enlarge the size of the coarse problem, so that it may lead to a fast convergence of the FETI-DP iteration.

We rewrite (21) as

$$R^t (B_\Delta \mathbf{w}_\Delta + B_c \mathbf{w}_c) = \mathbf{0}, \quad (22)$$

where the matrix R is a Boolean matrix, the number of columns of the matrix R equals to twice the number of Γ_{ij} , and the number of rows of the matrix R equals to the d.o.f of the space $\mathbf{M}$. For $\lambda \in \mathbf{M}$, at each interior nodal point of Γ_{ij} , $\lambda|_{\Gamma_{ij}}$ has two components corresponding to horizontal and vertical parts of velocity function. For $\lambda \in \mathbf{M}$, $R^t \lambda = \mathbf{0}$ means that for all Γ_{ij} , the sums of $\lambda|_{\Gamma_{ij}}$ corresponding to the horizontal and vertical parts of velocity function are zero.

Let N be the Lagrange multiplier space corresponding to the constraints (22), and for $\mu \in N$, $\mu|_{\Gamma_{ij}}$ has two components that correspond to the constraints for horizontal and vertical velocity. By introducing Lagrange multipliers λ and μ to enforce the constraints (19) and (22), the following is induced from the Galerkin approximation to (4) as follows: find $(\mathbf{u}_I, p_I, \mathbf{u}_\Delta, \mathbf{u}_c, p^0, \mu, \lambda) \in \mathbf{W}_I \times Q_I \times \mathbf{W}_\Delta \times \mathbf{W}_c \times \Pi^0 \times N \times \mathbf{M}$, such that

$$\begin{pmatrix} A_{II} & G_{II} & A_{I\Delta} & A_{Ic} & G_{I0} & \mathbf{0} & \mathbf{0} \\ G_{II}^t & \mathbf{0} & G_{\Delta I}^t & G_{cI}^t & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ A_{\Delta I} & G_{\Delta I} & A_{\Delta\Delta} & A_{\Delta c} & G_{\Delta 0} & B_\Delta^t R & B_\Delta^t \\ A_{cI} & G_{cI} & A_{c\Delta} & A_{cc} & G_{c0} & B_c^t R & B_c^t \\ G_{I0}^t & \mathbf{0} & G_{\Delta 0}^t & G_{c0}^t & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & R^t B_\Delta & R^t B_c & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & B_\Delta & B_c & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_\Delta \\ \mathbf{u}_c \\ p^0 \\ \mu \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{f}_I \\ \mathbf{0} \\ \mathbf{f}_\Delta \\ \mathbf{f}_c \\ \mathbf{0} \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}. \quad (23)$$

Here, $\begin{pmatrix} A_{II} & A_{I\Delta} & A_{Ic} \\ A_{\Delta I} & A_{\Delta\Delta} & A_{\Delta c} \\ A_{cI} & A_{c\Delta} & A_{cc} \end{pmatrix}$ is a stiffness matrix induced from $\sum_{i=1}^N (\nabla \mathbf{u}, \nabla \mathbf{v})_{\Omega_i}$, $(G_{II}^t \ G_{\Delta I}^t \ G_{cI}^t)$ is a matrix induced from $\sum_{i=1}^N (-\nabla \cdot \mathbf{v}, p_I)_{\Omega_i}$, and $(G_{I0}^t \ G_{\Delta 0}^t \ G_{c0}^t)$ is a matrix induced from $\sum_{i=1}^N (-\nabla \cdot \mathbf{v}, p^0)_{\Omega_i}$. Since $p^0|_{\Omega_i}$ is constant and $\mathbf{v}_I|_{\partial\Omega_i} = \mathbf{0}$ for $\mathbf{v}_I \in \mathbf{W}_I$, we have $G_{I0} = \mathbf{0}$ by the divergence theorem.

Let

$$\begin{aligned}
 K_{\Delta\Delta} &= \begin{pmatrix} A_{II} & G_{II} & A_{I\Delta} \\ G_{II}^t & \mathbf{0} & G_{\Delta I}^t \\ A_{\Delta I} & G_{\Delta I} & A_{\Delta\Delta} \end{pmatrix}, \\
 K_{\Delta c} &= \begin{pmatrix} A_{Ic} & G_{I0} & \mathbf{0} \\ G_{cI}^t & \mathbf{0} & \mathbf{0} \\ A_{\Delta c} & G_{\Delta 0} & B_{\Delta}^t R \end{pmatrix}, \\
 K_{cc} &= \begin{pmatrix} A_{cc} & G_{c0} & B_c^t R \\ G_{c0}^t & \mathbf{0} & \mathbf{0} \\ R^t B_c & \mathbf{0} & \mathbf{0} \end{pmatrix}, \\
 \tilde{B}_{\Delta} &= (\mathbf{0} \ \mathbf{0} \ B_{\Delta}), \quad \tilde{B}_c = (B_c \ 0 \ \mathbf{0}), \\
 \tilde{\mathbf{f}}_{\Delta} &= \begin{pmatrix} \mathbf{f}_I \\ \mathbf{0} \\ \mathbf{f}_{\Delta} \end{pmatrix}, \quad \tilde{\mathbf{f}}_c = \begin{pmatrix} \mathbf{f}_c \\ \mathbf{0} \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{z}_{\Delta} = \begin{pmatrix} \mathbf{u}_I \\ p_I \\ \mathbf{u}_{\Delta} \end{pmatrix}, \\
 \mathbf{z}_c &= \begin{pmatrix} \mathbf{u}_c \\ p^0 \\ \mu \end{pmatrix},
 \end{aligned} \tag{24}$$

where $\mathbf{z}_c$ is a primal variable, then (23) can be written as

$$\begin{pmatrix} K_{\Delta\Delta} & K_{\Delta c} & \tilde{B}_{\Delta}^t \\ K_{\Delta c}^t & K_{cc} & \tilde{B}_c^t \\ \tilde{B}_{\Delta} & \tilde{B}_c & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{z}_{\Delta} \\ \mathbf{z}_c \\ \lambda \end{pmatrix} = \begin{pmatrix} \tilde{\mathbf{f}}_{\Delta} \\ \tilde{\mathbf{f}}_c \\ \mathbf{0} \end{pmatrix}. \tag{25}$$

In Section 5, we will show that the matrix $K_{\Delta\Delta}$ is invertible (cf. Lemma 9). After eliminating $\mathbf{z}_{\Delta}$, we obtain the following equations for $\mathbf{z}_c$ and λ :

$$\begin{pmatrix} -F_{cc} & F_{cl} \\ F_{cl}^t & F_{ll} \end{pmatrix} \begin{pmatrix} \mathbf{z}_c \\ \lambda \end{pmatrix} = \begin{pmatrix} -\mathbf{d}_c \\ \mathbf{d}_l \end{pmatrix}, \tag{26}$$

where

$$\begin{aligned}
 F_{ll} &= \tilde{B}_{\Delta} K_{\Delta\Delta}^{-1} \tilde{B}_{\Delta}^t, \\
 F_{cl} &= K_{\Delta c}^t K_{\Delta\Delta}^{-1} \tilde{B}_{\Delta}^t - \tilde{B}_c^t, \\
 F_{cc} &= K_{cc} - K_{\Delta c}^t K_{\Delta\Delta}^{-1} K_{\Delta c}, \\
 \mathbf{d}_l &= \tilde{B}_{\Delta} K_{\Delta\Delta}^{-1} \tilde{\mathbf{f}}_{\Delta}, \\
 \mathbf{d}_c &= \tilde{\mathbf{f}}_c - K_{\Delta c}^t K_{\Delta\Delta}^{-1} \tilde{\mathbf{f}}_{\Delta}.
 \end{aligned} \tag{27}$$

The matrix F_{cc} which is a coarse problem in the FETI-DP method is invertible (cf. Lemma 10). Hence by eliminating $\mathbf{z}_c$, we obtain the following equation for λ :

$$F_{DP} \lambda = \mathbf{d}_l - F_{cl}^t F_{cc}^{-1} \mathbf{d}_c, \tag{28}$$

where $F_{DP} = F_{ll} + F_{cl}^t F_{cc}^{-1} F_{cl}$, and we call it the FETI-DP operator. Since we add the primal constraints to the coarse problem, λ is not uniquely determined in the space $\mathbf{M}$. We define a subspace of $\mathbf{M}$ as follows:

$$\tilde{\mathbf{M}} = \{\lambda \in \mathbf{M} : R^t \lambda = \mathbf{0}\}. \tag{29}$$

The matrix F_{DP} is symmetric and positive definite (s.p.d) on $\tilde{\mathbf{M}}$, and the solution λ of (28) is uniquely determined in $\tilde{\mathbf{M}}$.

4.2. Preconditioner. For $\mathbf{w}_i \in \mathbf{W}_i$, we define $S_i \mathbf{w}_i$ by

$$\begin{pmatrix} A_{II}^i & G_{II}^i & A_{I\Delta}^i & A_{Ic}^i \\ (G_{II}^i)^t & \mathbf{0} & (G_{\Delta I}^i)^t & (G_{cI}^i)^t \\ A_{\Delta I}^i & G_{\Delta I}^i & A_{\Delta\Delta}^i & A_{\Delta c}^i \\ A_{cI}^i & G_{cI}^i & A_{c\Delta}^i & A_{cc}^i \end{pmatrix} \begin{pmatrix} \mathbf{u}_I^i \\ p_I^i \\ \mathbf{w}_{\Delta}^i \\ \mathbf{w}_c^i \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \\ S_i \begin{pmatrix} \mathbf{w}_{\Delta}^i \\ \mathbf{w}_c^i \end{pmatrix} \end{pmatrix}, \tag{30}$$

where the subscript i for a matrix denotes the submatrix corresponding to subdomain Ω_i . Since the upper left 2×2 matrix

$$\begin{pmatrix} A_{II}^i & G_{II}^i \\ (G_{II}^i)^t & \mathbf{0} \end{pmatrix} \tag{31}$$

represents the local Stokes problem with a Dirichlet boundary condition, it is invertible, so that the Schur complementary S_i is well defined. We then assemble local Schur complementary matrices and define

$$S = \text{diag}(S_1, \dots, S_N) \tag{32}$$

and it can be seen easily that S is an s.p.d operator on $\mathbf{W}$. Hence, we define

$$\|\mathbf{w}\|_{\mathbf{W}} = \left(\sum_{i=1}^N \langle S_i \mathbf{w}_i, \mathbf{w}_i \rangle \right)^{1/2} \tag{33}$$

as a norm for $\mathbf{w} \in \mathbf{W}$. Here, $\langle \cdot, \cdot \rangle$ denotes the L^2 -inner product of vectors.

We define the following spaces:

$$\mathbf{W}_{ij}^0 = \left\{ \mathbf{w}_{ij} \in \mathbf{W}^h(\Omega_i) \Big|_{\delta_{m(j)}} : \int_{\delta_{m(j)}} \mathbf{w}_{ij} ds = \mathbf{0} \right\}, \tag{34}$$

$$\mathbf{W}^0 = \prod_i \prod_{\delta_{m(j)} \subset \Gamma} \mathbf{W}_{ij}^0.$$

For a function $\mathbf{w}_{ij} \in \mathbf{W}_{ij}^0$, let $\tilde{\mathbf{w}}_{ij}$ be the zero extension of $\mathbf{w}_{ij}$ into $\mathbf{W}_i$. For $\mathbf{w} \in \mathbf{W}^0$ we define an extension $\tilde{\mathbf{w}} \in \mathbf{W}$ by

$$\tilde{\mathbf{w}} = (\tilde{\mathbf{w}}_1, \dots, \tilde{\mathbf{w}}_N) \quad \text{with} \quad \tilde{\mathbf{w}}_i = \sum_{\delta_{m(j)} \subset \Gamma} \tilde{\mathbf{w}}_{ij}, \quad \forall i = 1, \dots, N, \tag{35}$$

and define a norm on $\mathbf{W}^0$ by

$$\|\mathbf{w}\|_{\mathbf{W}^0} = \|\tilde{\mathbf{w}}\|_{\mathbf{W}}. \tag{36}$$

We introduce the following subspaces with the norms induced from the spaces $\mathbf{W}$ and $\mathbf{W}^0$:

$$\widetilde{\mathbf{W}} = \left\{ \mathbf{w} \in \mathbf{W} : \int_{\delta_{m(j)}} (J_m \mathbf{w}_i - \mathbf{w}_j) ds = \mathbf{0}, \forall \delta_{m(j)} \subset \Gamma \right\}, \quad (37)$$

$$\widetilde{\mathbf{W}}_\Delta = \left\{ \mathbf{w} \in \widetilde{\mathbf{W}} : \int_{\partial\Omega_i} \mathbf{w}_i \cdot \mathbf{n}_i = 0, i = 1, \dots, N \right\}, \quad (38)$$

$$\widetilde{\mathbf{W}}^0 = \{ \mathbf{w} \in \mathbf{W}^0 : \widetilde{\mathbf{w}} \in \widetilde{\mathbf{W}} \}, \quad (39)$$

where $\mathbf{n}_i$ denotes the outward unit normal vector on the subdomain boundary $\partial\Omega_i$.

Recall the definition of $\widetilde{\mathbf{M}}$ in (29), and let $\langle \cdot, \cdot \rangle_m$ be a duality pairing between $\widetilde{\mathbf{M}}$ and $\widetilde{\mathbf{W}}^0$ defined as

$$\langle \boldsymbol{\lambda}, \mathbf{w} \rangle_m = \sum_i \sum_{\delta_{m(j)} \subset \Gamma} \int_{\delta_{m(j)}} \boldsymbol{\lambda}_{ij} \cdot \mathbf{w}_{ij} ds. \quad (40)$$

Then we define a dual norm for $\boldsymbol{\lambda} \in \widetilde{\mathbf{M}}$ by

$$\|\boldsymbol{\lambda}\|_{\widetilde{\mathbf{M}}}^2 = \max_{\mathbf{w} \in \widetilde{\mathbf{W}}^0 \setminus \{0\}} \frac{\langle \boldsymbol{\lambda}, \mathbf{w} \rangle_m^2}{\|\mathbf{w}\|_{\mathbf{W}^0}^2}. \quad (41)$$

Now, we will find an operator $\widehat{F}_{\text{DP}}$ which gives

$$\langle \widehat{F}_{\text{DP}} \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle = \|\boldsymbol{\lambda}\|_{\widetilde{\mathbf{M}}}^2 \quad (42)$$

and propose $\widehat{F}_{\text{DP}}^{-1}$ as a preconditioner for F_{DP} in (28). Define $R_{ij} : \mathbf{W}^0 \rightarrow \mathbf{W}_{ij}^0$ as a restriction operator and $E_{ij}^i : \mathbf{W}_{ij}^0 \rightarrow \mathbf{W}_i$ as an extension operator by zero. Then for $\mathbf{w} \in \widetilde{\mathbf{W}}^0$,

$$\begin{aligned} \|\mathbf{w}\|_{\mathbf{W}^0}^2 &= \|\widetilde{\mathbf{w}}\|_{\mathbf{W}}^2 = \sum_{i=1}^N \langle S_i \widetilde{\mathbf{w}}_i, \widetilde{\mathbf{w}}_i \rangle \\ &= \sum_{i=1}^N \left\langle S_i \left(\sum_{\delta_{m(j)} \subset \Gamma} E_{ij}^i R_{ij} \mathbf{w} \right), \sum_{\delta_{m(j)} \subset \Gamma} E_{ij}^i R_{ij} \mathbf{w} \right\rangle. \end{aligned} \quad (43)$$

Let $\widehat{S} = \sum_{i=1}^N (\sum_{\delta_{m(j)} \subset \Gamma} E_{ij}^i R_{ij})^t S_i (\sum_{\delta_{m(j)} \subset \Gamma} E_{ij}^i R_{ij})$. Moreover, we have

$$\langle \boldsymbol{\lambda}, \mathbf{w} \rangle_m = \langle \widehat{B} \mathbf{w}, \boldsymbol{\lambda} \rangle, \quad (44)$$

where $\widehat{B} = \text{diag}_{i=1, \dots, N} \text{diag}_{\delta_{m(j)} \subset \Gamma} \widehat{B}_i^{ij}$ and $\widehat{B}_i^{ij}$ is a matrix obtained from B_i^{ij} after deleting the columns corresponding to the d.o.f. of $\int_{\Gamma_{ij}} \mathbf{w} ds = \mathbf{0}$. Note that $\widehat{B}_i^{ij}$ is invertible. We define the following L^2 -orthogonal projections:

$$P_{\widetilde{\mathbf{W}}^0} : \mathbf{W}^0 \rightarrow \widetilde{\mathbf{W}}^0, \quad P_{\widetilde{\mathbf{M}}} : \mathbf{M} \rightarrow \widetilde{\mathbf{M}}. \quad (45)$$

For $\boldsymbol{\lambda} \in \widetilde{\mathbf{M}}$ and $\mathbf{w} \in \widetilde{\mathbf{W}}^0$, we have

$$\langle \boldsymbol{\lambda}, \mathbf{w} \rangle_m = \langle \widehat{B}_p \mathbf{w}, \boldsymbol{\lambda} \rangle, \quad \|\mathbf{w}\|_{\mathbf{W}^0}^2 = \langle \widehat{S}_p \mathbf{w}, \mathbf{w} \rangle, \quad (46)$$

where

$$\widehat{S}_p = P_{\widetilde{\mathbf{W}}^0} \widehat{S} P_{\widetilde{\mathbf{W}}^0}, \quad \widehat{B}_p = P_{\widetilde{\mathbf{M}}} \widehat{B} P_{\widetilde{\mathbf{W}}^0}. \quad (47)$$

Then it can be shown that the operators $\widehat{S}_p : \widetilde{\mathbf{W}}^0 \rightarrow \widetilde{\mathbf{W}}^0$, $\widehat{B}_p : \widetilde{\mathbf{W}}^0 \rightarrow \widetilde{\mathbf{M}}$ are invertible and $\widehat{S}_p$ is s.p.d on $\widetilde{\mathbf{W}}^0$. Hence, using (46), the maximum in (41) occurs when $\mathbf{w} \in \widetilde{\mathbf{W}}^0$ satisfies $\widehat{S}_p \mathbf{w} = \widehat{B}_p^t \boldsymbol{\lambda}$. Therefore, we have

$$\|\boldsymbol{\lambda}\|_{\widetilde{\mathbf{M}}}^2 = \langle \widehat{B}_p \widehat{S}_p^{-1} \widehat{B}_p^t \boldsymbol{\lambda}, \boldsymbol{\lambda} \rangle. \quad (48)$$

Let

$$\widehat{F}_{\text{DP}}^{-1} = (\widehat{B}_p \widehat{S}_p^{-1} \widehat{B}_p^t)^{-1} = (\widehat{B}_p^t)^{-1} \widehat{S}_p (\widehat{B}_p)^{-1} \quad (49)$$

be the preconditioner for the F_{DP} .

Define the L^2 -orthogonal projections

$$P_{\widetilde{\mathbf{W}}^0}^{ij} : \mathbf{W}^0|_{\delta_{m(j)}} \rightarrow \widetilde{\mathbf{W}}^0|_{\delta_{m(j)}}, \quad P_{\widetilde{\mathbf{M}}}^{ij} : \mathbf{M}|_{\delta_{m(j)}} \rightarrow \widetilde{\mathbf{M}}|_{\delta_{m(j)}}. \quad (50)$$

Then the projection operators $P_{\widetilde{\mathbf{W}}^0}$ and $P_{\widetilde{\mathbf{M}}}$ are composed of diagonal blocks of $P_{\widetilde{\mathbf{W}}^0}^{ij}$ and $P_{\widetilde{\mathbf{M}}}^{ij}$, respectively. Moreover, it can be shown easily that

$$P_{\widetilde{\mathbf{M}}}^{ij} \widehat{B}_i^{ij} P_{\widetilde{\mathbf{W}}^0}^{ij} : \widetilde{\mathbf{W}}^0|_{\delta_{m(j)}} \rightarrow \widetilde{\mathbf{M}}|_{\delta_{m(j)}} \quad (51)$$

is invertible. Hence, it follows that

$$\widehat{B}_p^{-1} = \text{diag}_{i=1, \dots, N} \text{diag}_{\delta_{m(j)} \subset \Gamma} \widehat{B}_i^{ij}, \quad (52)$$

where $\widehat{B}_{ij} = P_{\widetilde{\mathbf{M}}}^{ij} \widehat{B}_i^{ij} P_{\widetilde{\mathbf{W}}^0}^{ij}$, and we obtain

$$\widehat{F}_{\text{DP}}^{-1} = \sum_{i=1}^N \left(\sum_{\delta_{m(j)} \subset \Gamma} E_{ij}^i \widehat{B}_{ij}^{-1} R_{ij} \right)^t S_i \left(\sum_{\delta_{m(j)} \subset \Gamma} E_{ij}^i \widehat{B}_{ij}^{-1} R_{ij} \right). \quad (53)$$

5. Technical Tools

In this section, we state and prove a few technical lemmas necessary for the proof of Theorem 17 of Section 6.

Let $Y_{h/2}(\Omega_i)$ be the conforming space of piecewise linear continuous functions on the triangulation $T_{h/2}(\Omega_i)$ which is constructed by joining the midpoints of the edges of the elements of $T_h(\Omega_i)$, and let $Y_{h/2}^0(\Omega_i)$ be the subspace of $Y_{h/2}(\Omega_i)$ consisting of functions with zero traces on $\partial\Omega_i$.

For each open edge $\varepsilon \subset \partial\Omega_i$, we introduce a local equivalence map (isomorphism) $M_i^\varepsilon : \mathbf{W}_\varepsilon^h(\Omega_i) \rightarrow [Y_{h/2}(\Omega_i)]^2$ (cf. [11]), where $\mathbf{W}_\varepsilon^h(\Omega_i)$ is a subspace of $\mathbf{W}^h(\Omega_i)$ formed by all vectors whose component are zero in $\partial\Omega_{i,h}^{\text{CR}} \setminus \varepsilon_h^{\text{CR}}$. This mapping is a slightly modified version of a similar mapping introduced in [12].

Definition 3. For given $\mathbf{u} \in \mathbf{W}_\varepsilon^h(\Omega_i)$, we introduce $M_i^\varepsilon \mathbf{u} \in [Y_{h/2}(\Omega_i)]^2$ by defining the values of $M_i^\varepsilon \mathbf{u}$ at the nodal points of the triangulation $T_{h/2}(\Omega_i)$.

- (i) For $p \in \Omega_{i,h}^{CR} \cup \partial\Omega_{i,h}^{CR}$, let $M_i^\varepsilon \mathbf{u}(p) = \mathbf{u}(p)$.
- (ii) For $p \in \Omega_{i,h}$, then $M_i^\varepsilon \mathbf{u}(p) = (1/N(p)) \sum_{\tau_j^h} \mathbf{u}|_{\tau_j^h}(p)$, where the sum is taken over all triangles τ_j^h with the common vertex p , and $N(p)$ is the number of elements with p as an vertex.
- (iii) For $p \in \partial\Omega_{i,h} \setminus \varepsilon_h$, let $M_i^\varepsilon \mathbf{u}(p) = \mathbf{0}$.
- (iv) For $p \in \varepsilon_h$, then $M_i^\varepsilon \mathbf{u}(p) = (|p p_r|/|p_l p_r|) \mathbf{u}(p_l) + (|p_l p|/|p_l p_r|) \mathbf{u}(p_r)$, where p_l, p_r are the left- and right-neighboring CR nodal points of p and $|ab|$ is the length of the segment with a, b as its endpoints.

Note that each component of $M_i^\varepsilon \mathbf{u}$ is piecewise linear between the CR nodes of ε_h^{CR} , and $M_i^\varepsilon \mathbf{u} = \mathbf{0}$ on $\partial\Omega_i \setminus \varepsilon$. The mapping $M_i^\varepsilon \mathbf{u}$ has the following properties (cf. [11]):

$$\begin{aligned} |\mathbf{u}|_{H_h^1(\Omega_i)} &\asymp |M_i^\varepsilon \mathbf{u}|_{H^1(\Omega_i)}, \quad \|\mathbf{u}\|_{L^2(\Omega_i)} \asymp \|M_i^\varepsilon \mathbf{u}\|_{L^2(\Omega_i)}, \\ \|M_i^\varepsilon \mathbf{u} - \mathbf{u}\|_{L^2(\Omega_i)} &\leq h_i |\mathbf{u}|_{H_h^1(\Omega_i)}, \\ \|M_i^\varepsilon \mathbf{u} - \mathbf{u}\|_{L^2(\varepsilon)} &\leq h_i^{1/2} |\mathbf{u}|_{H_h^1(\Omega_i)}. \end{aligned} \quad (54)$$

Lemma 4 (see [13]). Let $\mathbf{u} \in \mathbf{W}_j$ and $\mathbf{u} = \mathbf{0}$ in $\partial\Omega_{j,h}^{CR} \setminus \delta_{m(j),h}^{CR}$ for a slave edge $\delta_{m(j)} \subset \partial\Omega_j$. Then we have

$$|SH_j \mathbf{u}|_{H_h^1(\Omega_j)} \leq \|M_j^{\delta_{m(j)}} SH_j \mathbf{u}\|_{H_{00}^{1/2}(\delta_{m(j)})}, \quad (55)$$

where $SH_j \mathbf{u}$ is the discrete harmonic extension of $\mathbf{u}$.

Lemma 5 (see [11]). Let $\delta_{m(j)} \subset \partial\Omega_j$ be a slave, then for any $\mathbf{u} \in \mathbf{W}^h(\Omega_j)$, we have

$$\begin{aligned} |\mathbf{u}^{\delta_{m(j)}}|_{H_h^1(\Omega_j)}^2 &\leq \left(1 + \log\left(\frac{H_j}{h}\right)\right)^2 \\ &\quad \times \left(H_j^{-2} \|\mathbf{u}\|_{L^2(\Omega_j)}^2 + |\mathbf{u}|_{H_h^1(\Omega_j)}^2\right), \end{aligned} \quad (56)$$

where $\mathbf{u}^{\delta_{m(j)}}$ is a discrete harmonic function taking the same vector values as those of $\mathbf{u}$ at CR nodal points on $\delta_{m(j),h}^{CR}$ and is equal to zero vector at the remaining CR nodal points on $\partial\Omega_j$ and $H_j = \text{diam } \Omega_j$.

We also need a special mortar operator defined over trace spaces: $\Pi_{\delta_{m(j)}} : [L^2(\delta_{m(j)})]^2 \rightarrow \mathbf{W}_0(\delta_{m(j)})$, where $\mathbf{W}_0(\delta_{m(j)})$ is a set of vectors whose components are continuous and equal to zero at the endpoints of $\delta_{m(j)}$ and are piecewise linear over all segments that have their ends in $\delta_{m(j),h}^{CR}$. Each $\Pi_{\delta_{m(j)}} \mathbf{u} \in \mathbf{W}_0(\delta_{m(j)})$ is uniquely determined by its values in $\delta_{m(j),h}^{CR}$.

Definition 6 (see [11]). Let $\Pi_{\delta_{m(j)}} : [L^2(\delta_{m(j)})]^2 \rightarrow \mathbf{W}_0(\delta_{m(j)})$ be defined by

$$\Pi_{\delta_{m(j)}} \prod \mathbf{u}(x) = Q_m \mathbf{u}(x), \quad \forall x \in \delta_{m(j),h}^{CR}. \quad (57)$$

The L^2 and $H_{00}^{1/2}$ stability of $\Pi_{\delta_{m(j)}}$ is stated in the next lemma.

Lemma 7 (see [11]). For $\Pi_{\delta_{m(j)}}$ defined previously, we have

$$\begin{aligned} \|\Pi_{\delta_{m(j)}} \mathbf{u}\|_{L^2(\delta_{m(j)})} &\leq \|\mathbf{u}\|_{L^2(\delta_{m(j)})} \quad \forall \mathbf{u} \in [L^2(\delta_{m(j)})]^2, \\ \|\Pi_{\delta_{m(j)}} \mathbf{u}\|_{H_{00}^{1/2}(\delta_{m(j)})} &\leq \|\mathbf{u}\|_{H_{00}^{1/2}(\delta_{m(j)})} \quad \forall \mathbf{u} \in [H_{00}^{1/2}(\delta_{m(j)})]^2. \end{aligned} \quad (58)$$

Lemma 8 (see [10]). Let $\mathbf{u} \in \mathbf{W}^h(\Omega_j)$ be a discrete harmonic vector in Ω_j , with $\mathbf{u} = \mathbf{0}$ at $x \in \partial\Omega_{j,h}^{CR} \setminus \delta_{m(j),h}^{CR}$. Then we have

$$|\mathbf{u}|_{H_h^1(\Omega_j)} \leq \|\Pi_{\delta_{m(j)}} \mathbf{u}\|_{H_{00}^{1/2}(\delta_{m(j)})}. \quad (59)$$

Lemma 9 (see [14]). The matrix $K_{\Delta\Delta}$ is invertible on $W_I \times Q_I \times W_\Delta$.

Lemma 10 (see [14]). Assume that the domain Ω has the triangulation to satisfy that

$$\begin{aligned} -\sum_i (p^0, \nabla \cdot \mathbf{v}_\Delta)_{\Omega_i} + \sum_{i,j} \lambda \cdot \int_{\Gamma_{ij}} (\mathbf{v}_\Delta^i - \mathbf{v}_\Delta^j) ds &= 0, \\ \forall \mathbf{v}_\Delta \in \mathbf{W}_\Delta, \\ -\sum_i (p^0, \nabla \cdot \mathbf{v}_c)_{\Omega_i} + \sum_{i,j} \lambda \cdot \int_{\Gamma_{ij}} (\mathbf{v}_c^i - \mathbf{v}_c^j) ds &= 0, \\ \forall \mathbf{v}_c \in \mathbf{W}_c \end{aligned} \quad (60)$$

for the solution $\begin{pmatrix} p^0 \\ \lambda \end{pmatrix}$. Then the coarse problem matrix F_{cc} is invertible.

Remark 11. Most triangulations of the domain Ω satisfy the assumption of Lemma 10 because the number of velocity unknowns $\mathbf{v}_\Delta$ and $\mathbf{v}_c$ is usually greater than the number of unknowns p^0 and λ .

Lemma 12 (see [14]). We have $B(\widetilde{\mathbf{W}}_\Delta) = B(\widetilde{\mathbf{W}}) = \widetilde{\mathbf{M}}$.

Lemma 13 (see [14]). For $\lambda \in \widetilde{\mathbf{M}}$, we have

$$\langle F_{DP} \lambda, \lambda \rangle = \max_{\mathbf{w} \in \widetilde{\mathbf{W}}_\Delta \setminus \{0\}} \frac{\langle B \mathbf{w}, \lambda \rangle^2}{\|\mathbf{w}\|_{\mathbf{W}}^2}. \quad (61)$$

Lemma 14 (see [15]). For any $\lambda \in \mathbf{M}$, we have

$$\langle \widehat{F}_{DP} \lambda, \lambda \rangle^{1/2} = \sup_{\mathbf{w} \in \widetilde{\mathbf{W}}_\Delta \setminus \{0\}} \sum_{\delta_{m(j)} \subset \Gamma} \frac{\int_{\delta_{m(j)}} \mathbf{w} \lambda ds}{\|\mathbf{w}\|_{\mathbf{W}}}. \quad (62)$$

Proof. The proof follows from the definitions of $\mathbf{W}$ and $\widetilde{\mathbf{W}}_\Delta$, (40), and a standard algebraic argument; see [16]. $\square$

Definition 15 (see [17]). Given $\mathbf{u} \in \mathbf{W}_i$, we define an operator $P_i : \mathbf{W}_i \rightarrow \widetilde{\mathbf{W}}$ as follows:

- (i) for $p \in \gamma_{m(i),h}^{CR}$, let $P_i \mathbf{u}(p) = \mathbf{u}(p)$;

- (ii) for $p \in \delta_{m(j),h}^{\text{CR}}$, let $P_i \mathbf{u}(p) = Q_m J_m \mathbf{u}(p)$;
- (iii) for $p \in \delta_{m(i),h}^{\text{CR}}$, for any slave side $\delta_{m(i)} \subset \partial\Omega_i$, or if the edge midpoint p is on $\partial\Omega$ or on any remaining master or slave side, let $P_i \mathbf{u}(p) = \mathbf{0}$.

We define a projection operator $P : \widetilde{\mathbf{W}} \rightarrow \widetilde{\mathbf{W}}_\Delta$ by

$$(P\mathbf{u})|_{\delta_{m(j)}} = Q_m (J_m \mathbf{u}_i - \mathbf{u}_j) \text{ on } \delta_{m(j)}. \quad (63)$$

where $\mathbf{u}_i$ and $\mathbf{u}_j$ are, respectively, the restriction of $\mathbf{u} \in \widetilde{\mathbf{W}}$ to the mortar side $\gamma_{m(i)}$ and the slave side $\delta_{m(j)}$ of an interface Γ_{ij} .

Lemma 16. For all $\mathbf{u} \in \widetilde{\mathbf{W}}$, we have

$$\|P\mathbf{u}\|_{\mathbf{W}} \leq \left(1 + \log\left(\frac{H}{h}\right)\right) \|\mathbf{u}\|_{\mathbf{W}}, \quad (64)$$

where $H = \max_i H_i$ and $h = \min_i h_i$.

Proof. Let $P\mathbf{u} = (P_1 \mathbf{u}, \dots, P_N \mathbf{u}) \in \widetilde{\mathbf{W}}$, and we have $\|P\mathbf{u}\|_{\mathbf{W}}^2 = \sum_{i=1}^N \|P_i \mathbf{u}\|_{\mathbf{W}_i}^2$. Next we will estimate the term $\|P_i \mathbf{u}\|_{\mathbf{W}_i}^2$.

Note that $P_i \mathbf{u}$ can be nonzero only over Ω_i and the neighboring subdomain Ω_j that share a common edge Γ_{ij} with Ω_i , such that the slave side $\delta_{m(j)}$ is associated with Ω_j . Let N_i denote the set of indices of such subdomains. Thus

$$\|P_i \mathbf{u}\|_{\mathbf{W}_i}^2 = |SH_i P_i \mathbf{u}|_{H_h^1(\Omega_i)}^2 + \sum_{j \in N_i} |SH_j P_i \mathbf{u}|_{H_h^1(\Omega_j)}^2. \quad (65)$$

Then considering Ω_i , we have $P_i \mathbf{u} = \mathbf{u} - \sum_{\delta_{m(i)} \subset \partial\Omega_i} \mathbf{u}^{\delta_{m(i)}}$ with $\mathbf{u}^{\delta_{m(i)}}$ being a discrete harmonic function which equals to $\mathbf{u}$ at the CR nodes of $\delta_{m(i),h}^{\text{CR}}$ and to zero at all remaining nodes of $\partial\Omega_{i,h}^{\text{CR}}$. Thus, by Lemma 5,

$$\begin{aligned} |SH_i P_i \mathbf{u}|_{H_h^1(\Omega_i)}^2 &\leq |SH_i \mathbf{u}|_{H_h^1(\Omega_i)}^2 + \sum_{\delta_{m(i)} \subset \partial\Omega_i} |\mathbf{u}^{\delta_{m(i)}}|_{H_h^1(\Omega_i)}^2 \\ &\leq \left(1 + \log\left(\frac{H_i}{h_i}\right)\right)^2 \left(H_i^{-2} \|SH_i \mathbf{u}\|_{L^2(\Omega_i)}^2 + |SH_i \mathbf{u}|_{H_h^1(\Omega_i)}^2\right). \end{aligned} \quad (66)$$

Next we consider those Ω_j for which $j \in N_i$, consequently the slave side $\delta_{m(j)}$ of the edge Γ_{ij} is an edge of Ω_j , and the mortar side $\gamma_{m(i)}$ is an edge of Ω_i . Then by Lemma 4, we get

$$|SH_j P_i \mathbf{u}|_{H_h^1(\Omega_j)} \leq \|M_j^{\delta_{m(j)}} SH_j P_i \mathbf{u}\|_{H_{00}^{1/2}(\delta_{m(j)})}. \quad (67)$$

We also need to consider the function $\Pi_{\delta_{m(j)}} M_i^{\gamma_{m(i)}} \mathbf{u}^{\gamma_{m(i)}}$. From Definitions 3 and 15, we have

$$\begin{aligned} M_j^{\delta_{m(j)}} SH_j P_i \mathbf{u}(p) &= SH_i P_i \mathbf{u}(p) = Q_m SH_i P_i \mathbf{u}(p) \\ &= Q_m J_m SH_i \mathbf{u}(p), \quad \forall p \in \delta_{m(j),h}^{\text{CR}}. \end{aligned} \quad (68)$$

Thus we get

$$\begin{aligned} \|M_j^{\delta_{m(j)}} SH_j P_i \mathbf{u}\|_{H_{00}^{1/2}(\delta_{m(j)})} &\leq \|M_j^{\delta_{m(j)}} SH_j P_i \mathbf{u} \\ &\quad - \Pi_{\delta_{m(j)}} M_i^{\gamma_{m(i)}} \mathbf{u}^{\gamma_{m(i)}}\|_{H_{00}^{1/2}(\delta_{m(j)})} \\ &\quad + \|\Pi_{\delta_{m(j)}} M_i^{\gamma_{m(i)}} \mathbf{u}^{\gamma_{m(i)}}\|_{H_{00}^{1/2}(\delta_{m(j)})}. \end{aligned} \quad (69)$$

The pervious second term can be estimated as

$$\begin{aligned} \|\Pi_{\delta_{m(j)}} M_i^{\gamma_{m(i)}} \mathbf{u}^{\gamma_{m(i)}}\|_{H_{00}^{1/2}(\delta_{m(j)})}^2 &\leq \|M_i^{\gamma_{m(i)}} \mathbf{u}^{\gamma_{m(i)}}\|_{H_{00}^{1/2}(\delta_{m(j)})}^2 \\ &\leq |M_i^{\gamma_{m(i)}} \mathbf{u}^{\gamma_{m(i)}}|_{H^1(\Omega_i)}^2 \leq |\mathbf{u}^{\gamma_{m(i)}}|_{H_h^1(\Omega_i)}^2 \\ &\leq \left(1 + \log\left(\frac{H_i}{h_i}\right)\right)^2 \left(H_i^{-2} \|SH_i \mathbf{u}\|_{L^2(\Omega_i)}^2 + |SH_i \mathbf{u}|_{H_h^1(\Omega_i)}^2\right), \end{aligned} \quad (70)$$

by Lemmas 5 and 7 and the trace theorem. The first term is bounded using an inverse inequality, (68) and Definition 6 as follows:

$$\begin{aligned} &\|M_j^{\delta_{m(j)}} SH_j P_i \mathbf{u} - \Pi_{\delta_{m(j)}} M_i^{\gamma_{m(i)}} \mathbf{u}^{\gamma_{m(i)}}\|_{H_{00}^{1/2}(\delta_{m(j)})}^2 \\ &\leq h_j^{-1} \|M_j^{\delta_{m(j)}} SH_j P_i \mathbf{u} - \Pi_{\delta_{m(j)}} M_i^{\gamma_{m(i)}} \mathbf{u}^{\gamma_{m(i)}}\|_{L^2(\delta_{m(j)})}^2 \\ &\leq \sum_{p \in \delta_{m(j),h}^{\text{CR}}} |M_j^{\delta_{m(j)}} SH_j P_i \mathbf{u}(p) - \Pi_{\delta_{m(j)}} M_i^{\gamma_{m(i)}} \mathbf{u}^{\gamma_{m(i)}}(p)|^2 \\ &= \sum_{p \in \delta_{m(j),h}^{\text{CR}}} |Q_m (J_m SH_i \mathbf{u} - M_i^{\gamma_{m(i)}} \mathbf{u}^{\gamma_{m(i)}})(p)|^2 \\ &\leq h_j^{-1} \|Q_m (J_m SH_i \mathbf{u} - M_i^{\gamma_{m(i)}} \mathbf{u}^{\gamma_{m(i)}})\|_{L^2(\delta_{m(j)})}^2. \end{aligned} \quad (71)$$

Next, by using the fact that Q_m is an orthogonal L^2 projection, we get

$$\begin{aligned} &\|Q_m (M_i^{\gamma_{m(i)}} \mathbf{u}^{\gamma_{m(i)}} - J_m SH_i \mathbf{u})\|_{L^2(\delta_{m(j)})} \\ &\leq \|M_i^{\gamma_{m(i)}} \mathbf{u}^{\gamma_{m(i)}} - J_m SH_i \mathbf{u}\|_{L^2(\delta_{m(j)})}. \end{aligned} \quad (72)$$

Finally, the fact that $J_m SH_i \mathbf{u} = J_m \mathbf{u}^{Y_{m(i)}}$ yields

$$\begin{aligned} & \|M_i^{Y_{m(i)}} \mathbf{u}^{Y_{m(i)}} - J_m SH_i \mathbf{u}\|_{L^2(\delta_{m(j)})} \\ & \leq \|M_i^{Y_{m(i)}} \mathbf{u}^{Y_{m(i)}} - \mathbf{u}^{Y_{m(i)}}\|_{L^2(\delta_{m(j)})} \\ & \quad + \|\mathbf{u}^{Y_{m(i)}} - J_m \mathbf{u}^{Y_{m(i)}}\|_{L^2(\delta_{m(j)})}. \end{aligned} \quad (73)$$

Applying the property of $M_i^{Y_{m(i)}} \mathbf{u}$ to the first term and Lemma 3.2 from [10] to the second term, Lemma 5 and the quasiuniform triangulation, we get

$$\begin{aligned} h_j^{-1} \|M_i^{Y_{m(i)}} \mathbf{u}^{Y_{m(i)}} - J_m SH_i \mathbf{u}\|_{L^2(\delta_{m(j)})}^2 & \leq \frac{h_i}{h_j} |\mathbf{u}^{Y_{m(i)}}|_{H_h^1(\Omega_i)}^2 \\ & \leq \left(1 + \log\left(\frac{H_i}{h_i}\right)\right)^2 \left(H_i^{-2} \|SH_i \mathbf{u}\|_{L^2(\Omega_i)}^2 + |SH_i \mathbf{u}|_{H_h^1(\Omega_i)}^2\right) \\ & \leq \left(1 + \log\left(\frac{H_i}{h_i}\right)\right)^2 |SH_i \mathbf{u}|_{H_h^1(\Omega_i)}^2. \end{aligned} \quad (74)$$

Finally, summing over all edges in $\partial\Omega_i$ and then over all subdomain ends the proof. $\square$

6. Condition Number Estimate

In this section, we give the condition number estimate of the preconditioned operator in the following main theorem of this paper.

Theorem 17. *For any $\lambda \in \mathbf{M}$, it holds that*

$$\langle \hat{F}_{DP} \lambda, \lambda \rangle \leq \langle F_{DP} \lambda, \lambda \rangle \leq \left(1 + \log\left(\frac{H}{h}\right)\right)^2 \langle \hat{F}_{DP} \lambda, \lambda \rangle, \quad (75)$$

where $H = \max_i H_i$ and $h = \min_i h_i$.

Proof. In the proof of this theorem we use results from this section and the algebraic arguments from [16, Section 6.4].

Lower Bound. From Lemma 13, the zero extension $E(\mathbf{w}_0) \in \bar{\mathbf{W}}_\Delta$ for $\mathbf{w}_0 \in \bar{\mathbf{W}}^0$, and (41), we obtain

$$\begin{aligned} \langle F_{DP} \lambda, \lambda \rangle & = \sup_{\mathbf{w} \in \bar{\mathbf{W}}_\Delta \setminus \{0\}} \frac{\langle B\mathbf{w}, \lambda \rangle^2}{\|\mathbf{w}\|_{\mathbf{W}}^2} \\ & \geq \sup_{\mathbf{w}_0 \in \bar{\mathbf{W}}^0 \setminus \{0\}} \frac{\langle BE(\mathbf{w}_0), \lambda \rangle^2}{\|E(\mathbf{w}_0)\|_{\mathbf{W}}^2} \\ & = \langle \hat{F}_{DP} \lambda, \lambda \rangle. \end{aligned} \quad (76)$$

Upper Bound. For any $\mathbf{w} \in \bar{\mathbf{W}}$,

$$\begin{aligned} \langle B\mathbf{w}, \lambda \rangle & = \sum_{\delta_{m(j)} \subset \Gamma} \int_{\delta_{m(j)}} (J_m \mathbf{u}_i - \mathbf{u}_j) \cdot \lambda_{ij} ds \\ & = \sum_{\delta_{m(j)} \subset \Gamma} \int_{\delta_{m(j)}} Q_m (J_m \mathbf{u}_i - \mathbf{u}_j) \cdot \lambda_{ij} ds \\ & = \sum_{\delta_{m(j)} \subset \Gamma} \int_{\delta_{m(j)}} (P\mathbf{w})|_{\delta_{m(j)}} \cdot \lambda_{ij} ds. \end{aligned} \quad (77)$$

Hence by Lemmas 13, 14, and 16, we conclude that

$$\begin{aligned} \langle F_{DP} \lambda, \lambda \rangle & = \sup_{\mathbf{w} \in \bar{\mathbf{W}}_\Delta \setminus \{0\}} \frac{\langle B\mathbf{w}, \lambda \rangle^2}{\|\mathbf{w}\|_{\mathbf{W}}^2} \\ & = \sup_{\mathbf{w} \in \bar{\mathbf{W}}_\Delta \setminus \{0\}} \frac{\left(\sum_{\delta_{m(j)} \subset \Gamma} \int_{\delta_{m(j)}} (P\mathbf{w})|_{\delta_{m(j)}} \cdot \lambda_{ij} ds\right)^2}{\|\mathbf{w}\|_{\mathbf{W}}^2} \\ & \leq \langle \hat{F}_{DP} \lambda, \lambda \rangle \sup_{\mathbf{w} \in \bar{\mathbf{W}}_\Delta \setminus \{0\}} \frac{\|P\mathbf{w}\|_{\mathbf{W}}^2}{\|\mathbf{w}\|_{\mathbf{W}}^2} \\ & \leq \left(1 + \log\left(\frac{H}{h}\right)\right)^2 \langle \hat{F}_{DP} \lambda, \lambda \rangle. \end{aligned} \quad (78)$$

$\square$

7. Numerical Tests

Consider the following Stokes problem:

$$\begin{aligned} -\Delta \mathbf{u} + \nabla p & = \mathbf{f} \quad \text{in } \Omega, \\ -\nabla \cdot \mathbf{u} & = 0 \quad \text{in } \Omega, \\ \mathbf{u} & = \mathbf{0} \quad \text{on } \partial\Omega, \end{aligned} \quad (79)$$

where $\Omega = [0, 1] \times [0, 1] \subset \mathbb{R}^2$ and $\mathbf{f}$ is given by the exact solution of the problem (79) as follows:

$$\mathbf{u} = \begin{pmatrix} \sin^3(\pi x) \sin^2(\pi y) \cos(\pi y) \\ -\sin^2(\pi x) \sin^3(\pi y) \cos(\pi x) \end{pmatrix}, \quad p = x^2 - y^2. \quad (80)$$

Let N denote the number of subdomains. We only consider the uniform partition of Ω . $N = 4 \times 4$ means that Ω is partitioned into 4×4 square subdomains. For all subdomains, we take the same number of nodes n , including endpoints, in horizontal and vertical edges with $n = 4k + 1$ for some positive integer k . We solve (79) on both matching and nonmatching grids. For matching grids, we make uniform triangulations in each subdomain with $(n - 1)/2 + 1$ nodes on horizontal and vertical edges of subdomain. For nonmatching grids, we take $(n - 1)/2 + 1$ random quasi-uniform nodes on each horizontal and vertical edges of subdomain and generate nonuniform structured triangulations.

TABLE 1: CG iterations (condition numbers) when $N = 4 \times 4$.

n	Matching		Nonmatching	
	F_{DP}	$\hat{F}_{DP}^{-1}F_{DP}$	F_{DP}	$\hat{F}_{DP}^{-1}F_{DP}$
5	21(9.82)	18(5.65)	39(1.94e + 1)	26(3.95)
9	35(3.57e + 1)	21(7.48)	104(2.41e + 2)	29(6.42)
17	47(8.48e + 1)	22(1.01e + 1)	133(9.90e + 2)	33(8.75)
33	53(1.78e + 2)	24(1.29e + 1)	157(2.48e + 3)	35(11.3)
65	68(3.38e + 2)	25(1.57e + 1)	199(5.98e + 3)	37(13.5)

Now, we solve the FETI-DP operator with and without preconditioner varying N and n on both matching and nonmatching triangulations. The Conjugate Gradient (CG) iteration is stopped when the relative residual is less than 10^{-6} .

In Tables 1–3, the number of CG iterations and the corresponding condition numbers are shown with varying N and n . In Table 1, $N = 4 \times 4$ and $n - 1$ increase by double. On both matching and nonmatching grids, the preconditioner performs well and the condition numbers seem to behave $\log^2$ -growth as n increases. Especially on nonmatching grids, the preconditioner is much more efficient. In Tables 2 and 3, the CG iteration becomes stable as N increases with $n = 5$ and $n = 9$.

Moreover, we have observed the convergence behaviors of the approximated solution. The H^1 and L^2 -errors for velocity and pressure are examined. $\mathbf{u}^h$ and p^h denote the approximated solutions for the velocity and pressure, and $\|\mathbf{u} - \mathbf{u}^h\|_{1,*}$ means the square root of $\sum_{i=1}^N \|\mathbf{u} - \mathbf{u}^h\|_{1,\Omega_i}^2$. The errors are shown in Table 4 for various N and n with matching grids. Three cases are considered: when $n - 1$ increases by double with $N = 4 \times 4$, when N increases by double in both edges of Ω with $n = 5$ and $n = 9$. For all cases, we can see that the H^1 -error for velocity $\|\mathbf{u} - \mathbf{u}^h\|_{1,*}$, L^2 -error for pressure $\|p - p^h\|_0$ reduce by half and L^2 -error for velocity $\|\mathbf{u} - \mathbf{u}^h\|_0$ reduces by quarter. For the finite elements $CR(h) - P_0(h)$, these convergence behaviors are optimal.

For the case of nonmatching grids, the errors are shown in Tables 5–7 with various N and n . In Table 5, we observe that the error $\|\mathbf{u} - \mathbf{u}^h\|_{1,*}$ and $\|p - p^h\|_0$ reduce by half and the error $\|\mathbf{u} - \mathbf{u}^h\|_0$ reduces by quarter as $n - 1$ increases by double with $N = 4 \times 4$. When $n = 5$ and $n = 9$, as N increases, the errors also show the optimal convergence behaviors in Tables 6 and 7.

If the inf-sup constant for the space $V \times P$ is independent of N and n , then the optimality of approximation can be shown. Let β^* and β be the inf-sup constants for the space $V \times P$ and the $CR(h) - P_0(h)$ finite elements, respectively, and β_0 be the inf-sup constant for the space $V \times \Pi^0$. Then the constant β^* depends on β and β_0 from the trick conceived by Boland and Nicolaides [18]. Hence, if the constant β_0 is independent of n and N , then the same holds for β^* . In [19], for $V \times \Pi^0$ which is obtained from the Hood-Taylor finite elements, it was shown that the constant β_0 is independent of n , but not shown for N . Following the proofs in [19], we can obtain the same results for the space $V \times \Pi^0$ of the

TABLE 2: CG iterations (condition numbers) when $n = 5$.

N	Matching		Nonmatching	
	F_{DP}	$\hat{F}_{DP}^{-1}F_{DP}$	F_{DP}	$\hat{F}_{DP}^{-1}F_{DP}$
4×4	15(8.32)	12(6.68)	21(11.7)	16(6.87)
8×4	16(8.68)	13(6.71)	22(13.1)	16(6.84)
16×16	14(8.85)	12(6.66)	21(13.8)	16(6.43)
32×32	13(8.96)	12(6.61)	21(15.9)	17(7.12)

TABLE 3: CG iterations (condition numbers) when $n = 9$.

N	Matching		Nonmatching	
	F_{DP}	$\hat{F}_{DP}^{-1}F_{DP}$	F_{DP}	$\hat{F}_{DP}^{-1}F_{DP}$
4×4	30(38.9)	15(12.7)	61(32.1)	21(13.5)
8×4	32(40.2)	16(12.5)	64(33.5)	22(14.1)
16×16	31(40.6)	16(12.3)	66(38.1)	23(14.4)
32×32	29(40.8)	17(12.8)	68(35.8)	23(14.3)

TABLE 4: H^1 - and L^2 -errors on matching grids.

$N = 4 \times 4$	$n = 5$	$n = 9$	$\ \mathbf{u} - \mathbf{u}^h\ _{1,*}$	$\ \mathbf{u} - \mathbf{u}^h\ _0$	$\ p - p^h\ _0$
n	N	N			
5	4×4		$6.60e - 1$	$6.34e - 3$	$1.68e - 1$
9	8×8	4×4	$3.44e - 1$	$1.61e - 3$	$8.51e - 2$
17	16×16	8×8	$1.73e - 1$	$4.13e - 4$	$4.23e - 2$
33	32×32	16×16	$8.71e - 2$	$1.05e - 4$	$2.09e - 2$
65		32×32	$4.35e - 2$	$2.54e - 5$	$1.06e - 2$

TABLE 5: H^1 - and L^2 -errors on nonmatching grids: $N = 4 \times 4$.

n	$\ \mathbf{u} - \mathbf{u}^h\ _{1,*}$	$\ \mathbf{u} - \mathbf{u}^h\ _0$	$\ p - p^h\ _0$
5	$6.41e - 1$	$7.08e - 3$	$1.81e - 1$
9	$3.28e - 1$	$1.79e - 3$	$8.91e - 2$
17	$1.67e - 1$	$4.39e - 4$	$4.36e - 2$
33	$8.43e - 2$	$1.12e - 4$	$2.24e - 2$
65	$4.32e - 2$	$2.98e - 5$	$1.15e - 2$

TABLE 6: H^1 - and L^2 -errors on nonmatching grids: $n = 5$.

N	$\ \mathbf{u} - \mathbf{u}^h\ _{1,*}$	$\ \mathbf{u} - \mathbf{u}^h\ _0$	$\ p - p^h\ _0$
4×4	$3.76e - 1$	$2.11e - 3$	$1.08e - 1$
8×8	$1.85e - 1$	$5.19e - 4$	$5.21e - 2$
16×16	$9.08e - 2$	$1.28e - 4$	$2.59e - 2$
32×32	$4.56e - 2$	$3.15e - 5$	$1.28e - 2$

TABLE 7: H^1 - and L^2 -errors on nonmatching grids: $n = 9$.

N	$\ \mathbf{u} - \mathbf{u}^h\ _{1,*}$	$\ \mathbf{u} - \mathbf{u}^h\ _0$	$\ p - p^h\ _0$
4×4	$7.21e - 1$	$7.73e - 3$	$2.91e - 1$
8×8	$3.58e - 1$	$1.92e - 3$	$1.47e - 1$
16×16	$1.81e - 1$	$4.52e - 4$	$5.19e - 2$
32×32	$8.94e - 2$	$1.15e - 4$	$2.61e - 2$

$CR(h) - P_0(h)$ finite elements. We have no proof that β_0 is independent of N . Instead, we compute the constant β_0 numerically as N increases. The results are given in Table 8

TABLE 8: Inf-sup constant β_0 when $n = 5$ and $n = 9$.

N	$n = 5$		$n = 9$	
	Nonmatching	Matching	Nonmatching	Matching
4×4	0.7537	0.7543	0.7868	0.7870
8×8	0.7050	0.7052	0.7109	0.7111
16×16	0.6765	0.6768	0.6798	0.6801
32×32	0.6584	0.6586	0.6611	0.6613

TABLE 9: Inf-sup constant β_0 when $N = 4 \times 4$.

n	Nonmatching	Matching
5	0.7537	0.7543
9	0.7050	0.7052
17	0.6765	0.6768
33	0.6584	0.6586
65	0.6586	0.6587

both for matching and nonmatching grids when $n = 5$ and $n = 9$. We observe that the constant β_0 becomes stable as N increases. Table 9 gives the constant β_0 as n increases with $N = 4 \times 4$. This confirms that the constant β_0 is independent of n .

Acknowledgments

The author of this paper would like to express her gratitude to Professor Jinru Chen, for his guidance and encouragement, and to the reviewers, for valuable comments and suggestions. This work was supported by the National Science Foundation (NSF) of China (Grants No. 11071124 and 11226334) and the Program of Natural Science Research of Jiangsu Higher Education Institutions of China (Grant No. 12KJB110013).

References

- [1] H. H. Kim, "A FETI-DP formulation of three dimensional elasticity problems with mortar discretization," *SIAM Journal on Numerical Analysis*, vol. 46, no. 5, pp. 2346–2370, 2008.
- [2] A. Klawonn, O. B. Widlund, and M. Dryja, "Dual-primal FETI methods for three-dimensional elliptic problems with heterogeneous coefficients," *SIAM Journal on Numerical Analysis*, vol. 40, no. 1, pp. 159–179, 2002.
- [3] A. Klawonn and O. B. Widlund, "Dual-primal FETI methods for linear elasticity," *Communications on Pure and Applied Mathematics*, vol. 59, no. 11, pp. 1523–1572, 2006.
- [4] J. Li, "A dual-primal FETI method for incompressible Stokes equations," *Numerische Mathematik*, vol. 102, no. 2, pp. 257–275, 2005.
- [5] A. Klawonn and O. Rheinbach, "Inexact FETI-DP methods," *International Journal for Numerical Methods in Engineering*, vol. 69, no. 2, pp. 284–307, 2007.
- [6] A. Klawonn, O. Rheinbach, and O. B. Widlund, "An analysis of a FETI-DP algorithm on irregular subdomains in the plane," *SIAM Journal on Numerical Analysis*, vol. 46, no. 5, pp. 2484–2504, 2008.
- [7] J. Li and O. B. Widlund, "On the use of inexact subdomain solvers for BDDC algorithms," *Computer Methods in Applied Mechanics and Engineering*, vol. 196, no. 8, pp. 1415–1428, 2007.
- [8] J. Li and O. B. Widlund, "BDDC algorithms for incompressible Stokes equations," *SIAM Journal on Numerical Analysis*, vol. 44, no. 6, pp. 2432–2455, 2006.
- [9] H. H. Kim, M. Dryja, and O. B. Widlund, "A BDDC method for mortar discretizations using a transformation of basis," *SIAM Journal on Numerical Analysis*, vol. 47, no. 1, pp. 136–157, 2008/09.
- [10] T. Rahman, P. Bjørstad, and X. J. Xu, "The Crouzeix-Raviart Finite element on nonmatching grids with an approximate mortar condition," *SIAM Journal on Numerical Analysis*, vol. 46, no. 1, pp. 496–516, 2008.
- [11] L. Marcinkowski, "The mortar element method with locally nonconforming elements," *BIT Numerical Mathematics*, vol. 39, no. 4, pp. 716–739, 1999.
- [12] M. Sarkis, "Nonstandard coarse spaces and Schwarz methods for elliptic problems with discontinuous coefficients using nonconforming elements," *Numerische Mathematik*, vol. 77, no. 3, pp. 383–406, 1997.
- [13] L. Marcinkowski, "Additive Schwarz method for mortar discretization of elliptic problems with P_1 nonconforming finite elements," *BIT Numerical Mathematics*, vol. 45, no. 2, pp. 375–394, 2005.
- [14] H. H. Kim and C.-O. Lee, "A Neumann-Dirichlet preconditioner for a FETI-DP formulation of the two-dimensional Stokes problem with mortar methods," *SIAM Journal on Scientific Computing*, vol. 28, no. 3, pp. 1133–1152, 2006.
- [15] L. Marcinkowski, "A preconditioner for a FETI-DP method for mortar element discretization of a 4th order problem in 2D," *Electronic Transactions on Numerical Analysis*, vol. 38, pp. 1–16, 2011.
- [16] A. Toselli and O. B. Widlund, *Domain Decomposition Method-salgorithms and Theory*, vol. 34 of *Computational Mathematics*, Springer, Berlin, Germany, 2005.
- [17] L. Marcinkowski and T. Rahman, "Neumann-Neumann algorithms for a mortar Crouzeix-Raviart element for 2nd order elliptic problems," *BIT Numerical Mathematics*, vol. 48, no. 3, pp. 607–626, 2008.
- [18] J. M. Boland and R. A. Nicolaides, "Stability of finite elements under divergence constraints," *SIAM Journal on Numerical Analysis*, vol. 20, no. 4, pp. 722–731, 1983.
- [19] F. B. Belgacem, "The mixed mortar finite element method for the incompressible Stokes problem: convergence analysis," *SIAM Journal on Numerical Analysis*, vol. 37, no. 4, pp. 1085–1100, 2000.