Research Article

Soft Rough Approximation Operators on a Complete Atomic Boolean Lattice

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The concept of soft sets based on complete atomic Boolean lattice, which can be seen as a generalization of soft sets, is introduced. Some operations on these soft sets are discussed, and new types of soft sets such as full, keeping infimum, and keeping supremum are defined and supported by some illustrative examples. Two pairs of new soft rough approximation operators are proposed and the relationship among soft set is investigated, and their related properties are given. We show that Järvinen’s approximations can be viewed as a special case of our approximation. If \( B = \wp(U) \), then our soft approximations coincide with crisp soft rough approximations (Feng et al. 2011).

1. Introduction

Most of traditional methods for formal modeling, reasoning, and computing are crisp, deterministic, and precise in character. However, many practical problems within fields such as economics, engineering, environmental science, medical science, and social sciences involve data that contain uncertainties. We cannot use traditional methods because of various types of uncertainties present in these problems.

There are several theories probability theory, fuzzy set theory, theory of interval mathematics, and rough set theory [1], which we can be considered as mathematical tools for dealing with uncertainties. But all these theories have their own difficulties (see [2]). For example, theory of probabilities can deal only with stochastically stable phenomena. To overcome these kinds of difficulties, Molodtsov [2] proposed a completely new approach, which is called soft set theory, for modelling uncertainty.


Rough set theory was initiated by Pawlak [1] for dealing with vagueness and granularity in information systems. This theory handles the approximation of an arbitrary subset of a universe by two definable or observable subsets called lower and upper approximations. It has been successfully applied to machine learning, intelligent systems, inductive reasoning, pattern recognition, mereology, image processing, signal analysis, knowledge discovery, decision analysis, expert systems, and many other fields (see [1, 11]). Since many classes of information granules are lattice ordered [12, 13], lattice theory [14–16] has found renewed interest and applications in diverse areas such as mathematical morphology [17], fuzzy set theory [18, 19], computational intelligence [20], automated decision making [21], and formal concept analysis [22]. In [23, 24] Järvinen studied properties of approximations in a more general setting of complete atomic Boolean lattices. He defined in a lattice theoretical setting two maps which mimic the rough approximation operators and noted that this setting is suitable also for other operators based on binary relations.

It has been found that soft set and rough set are closely related concepts. Based on the equivalence relation on the
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The universe of discourse, Feng et al. [25, 26] investigate that for all \( x \neq 0 \) there exists an atom \( a \in A(\mathcal{B}) \) s.t. \( a \leq x \). Namely, if \( \{a \in A(\mathcal{B}) : a \leq x\} = \emptyset \), then \( x = \bigvee \{a \in A(\mathcal{B}) : a \leq x\} = \bigvee \emptyset = 0 \).

Question 6. Approximations can be viewed as a special case of our \( \mathcal{B} \)-calculus. Further more, if \( c(\mathcal{B}) = \mathcal{B} \), then for all \( y \in \mathcal{B} \) is extensive, order preserving, and idempotent. An element \( c(\mathcal{B}) \) is said to be extensive, if \( c(\mathcal{B}) = \mathcal{B} \Rightarrow c(\mathcal{B}) \leq \mathcal{B} \) is an order embedding on \( \mathcal{B} \). Theorem 1. Let \( \mathcal{B} = (\mathcal{B}, \leq) \) be a complete lattice, \( S, T \subseteq \mathcal{B} \), and \( \{X_i : i \in I\} \subseteq \wp(\mathcal{B}) \).

(i) If \( S \subseteq T \), then \( \bigvee S \subseteq \bigvee T \).

(ii) \( \bigvee (S \cup T) = (\bigvee S) \bigvee (\bigvee T) \).

(iii) \( \bigvee (\bigcup \{X_i : i \in I\}) = \bigvee (\bigvee X_i \in I) \).

Next we recall the concept of Boolean lattices. They are bounded distributive lattices with a complementation operation.

Definition 2 (see [27]). A lattice \( \mathcal{B} = (\mathcal{B}, \leq) \) is called a Boolean lattice, if

(i) \( \mathcal{B} \) is distributive;

(ii) \( \mathcal{B} \) has a least element \( 0 \) and a greatest element \( 1 \), and;

(iii) each \( x \in \mathcal{B} \) has a complement \( x' \in \mathcal{B} \) such that \( x \lor x' = 1 \) and \( x \land x' = 0 \).

Lemma 3 (see [27]). Let \( \mathcal{B} = (\mathcal{B}, \leq) \) be a Boolean lattice; then for all \( x, y \in \mathcal{B} \)

(i) \( 0' = 1 \) and \( 1' = 0 \),

(ii) \( x'' = x \),

(iii) \( (x \lor y)' = x' \land y' \) and \( (x \land y)' = x' \lor y' \),

(iv) \( x \leq y \iff x \land y' = 0 \).

Let us recall some definitions and results that are useful in our consideration given in [23].

Lemma 4 (see [23]). Let \( \mathcal{B} = (\mathcal{B}, \leq) \) be a complete Boolean lattice. Then for all \( \{x_i : i \in I\} \subseteq \mathcal{B} \) and \( y \in \mathcal{B} \)

\[
y \land \left( \bigvee_{i \in I} x_i \right) = \bigvee_{i \in I} (y \land x_i),
\]

\[
y \lor \left( \bigwedge_{i \in I} x_i \right) = \bigwedge_{i \in I} (y \lor x_i).
\]

Definition 5 (see [23]). Let \( \mathcal{B} = (\mathcal{B}, \leq) \) be an ordered set and \( x, y \in \mathcal{B} \) be such that \( x \) is covered by \( y \) (or that \( y \) covers \( x \)) and write \( x \prec y \) if \( x < y \) and there is no element \( z \in \mathcal{B} \) with \( x < z < y \).

Definition 6 (see [23]). Let \( \mathcal{B} = (\mathcal{B}, \leq) \) be a lattice with a least element \( 0 \). Then \( a \in \mathcal{B} \) is called an atom if \( 0 < a \). The set of atoms of \( \mathcal{B} \) is denoted by \( A(\mathcal{B}) \). The lattice \( \mathcal{B} \) is called atomic if every element of \( \mathcal{B} \) is a supremum of the atoms below it; that is, \( x = \bigvee \{a \in A(\mathcal{B}) : a \leq x\} \).

It is obvious that in a lattice \( \mathcal{B} = (\mathcal{B}, \leq) \) with a least element \( 0 \),

\[
a \land x \neq 0 \iff a \leq x
\]

for all \( a \in A(\mathcal{B}) \) and \( x \in \mathcal{B} \). This implies that \( a \land b = 0 \) for all \( a, b \in A(\mathcal{B}) \) s.t. \( a \neq b \). Furthermore, if \( B \) is atomic, then for all \( x \neq 0 \) there exists an atom \( a \in A(\mathcal{B}) \) s.t. \( a \leq x \). Namely, if \( \{a \in A(\mathcal{B}) : a \leq x\} = \emptyset \), then \( x = \bigvee \{a \in A(\mathcal{B}) : a \leq x\} = \bigvee \emptyset = 0 \).
Definition 7 (see [23]). Let \( B = (B, \leq) \) be a complete atomic Boolean lattice. We say that \( \varphi : A(B) \rightarrow B \) is

(i) extensive, if \( a \leq \varphi(a) \) for all \( a \in A(B) \),
(ii) symmetric, if \( a \leq \varphi(b) \) implies \( b \leq \varphi(a) \) for all \( a, b \in A(B) \),
(iii) closed, if \( b \leq \varphi(a) \) implies \( \varphi(b) \leq \varphi(a) \) for all \( a, b \in A(B) \).

Definition 8 (see [23]). Let \( B = (B, \leq) \) be a complete atomic Boolean lattice and let \( \varphi : A(B) \rightarrow B \) be any mapping. Then for any element \( x \in B \), let
\[
\begin{aligned}
x^\varphi &= \bigvee \{ a \in A(B) : \varphi(a) \leq x \}, \\
x^\Delta &= \bigvee \{ a \in A(B) : \varphi(a) \land x \neq 0 \}.
\end{aligned}
\]  

The elements \( x^\varphi \) and \( x^\Delta \) are called the lower and the upper approximations of \( x \) with respect to \( \varphi \), respectively. Two elements \( x \) and \( y \) are called equivalent if they have the same upper and lower approximations. The resulting equivalence classes are called rough sets.

The following results are shown in [23, 24]. The ordered sets \((B^A, \leq)\) and \((B^B, \leq)\) are always complete lattices. They are distributive sublattices of \((B, \leq)\) if \( \varphi \) is extensive and closed. If the map \( \varphi \) is extensive, symmetric, and closed, then the ordered sets \((B^A, \leq)\) and \((B^B, \leq)\) are mutually equal complete atomic Boolean lattices.

Proposition 9 (see [23]). Let \( B = (B, \leq) \) be a complete atomic Boolean lattice and let \( \varphi : A(B) \rightarrow B \) be any mapping. Then for all \( a \in A(B) \) and \( x \in B \),

(i) \( a \leq x^\varphi \iff \varphi(a) \leq x \);
(ii) \( a \leq x^\Delta \iff \varphi(a) \land x \neq 0 \).

Proposition 10 (see [23]). Let \( B = (B, \leq) \) be a complete atomic Boolean lattice and let \( \varphi : A(B) \rightarrow B \) be an extensive mapping. Then for all \( x \in B \),

(i) \( x^\varphi \leq x \);
(ii) \( x \leq x^\Delta \).

Proposition 11 (see [23]). Let \( B = (B, \leq) \) be a complete atomic Boolean lattice and let \( \varphi : A(B) \rightarrow B \) be extensive and closed mapping. Then for all \( x \in B \),

(i) \( x^\varphi = x^{\varphi\varphi} \);
(ii) \( x^{\Delta\Delta} = x^\Delta \).

Proposition 12 (see [23]). Let \( B = (B, \leq) \) be a complete atomic Boolean lattice and let \( \varphi : A(B) \rightarrow B \) be an extensive, symmetric and closed mapping. Then for all \( x \in B \),

(i) \( x^{\varphi\Delta} = x^\varphi \);
(ii) \( x^{\Delta\varphi} = x^\Delta \).

Next, we recall the definitions of Pawlak rough sets, soft sets, and soft rough approximation operators.

Definition 13 (see [29]). An information system (or a knowledge representation system) is a pair \( \gamma = (U, A) \) of nonempty finite sets \( U \) and \( A \), where \( U \) is a set of objects and \( A \) is a set of attributes; each attribute \( a \in A \) is a function \( a : U \rightarrow V_a \), where \( V_a \) is the set of values (called domain) of attribute \( a \).

Let \( U \) be a non-empty finite universe and let \( R \) be an equivalence relation on \( U \). The pair \( (U, R) \) is called a Pawlak approximation space. The equivalence relation \( R \) is often called an indiscernibility relation and related to an information system. Specifically, if \( \gamma = (U, A) \) is an information system and \( B \subseteq A \), then an indiscernibility relation \( R = I(B) \) can be defined by
\[
(x, y) \in I(B) \iff a(x) = a(y) \quad \forall a \in B,
\]
where \( x, y \in U \) and \( a(x) \) denotes the value of attribute \( a \) for object \( x \).

Using the indiscernibility relation \( R \), one can define the following two operations:
\[
\begin{aligned}
R_\gamma X &= \{ x \in U : [x]_R \subseteq X \}, \\
R^* X &= \{ x \in U : [x]_R \cap X \neq \emptyset \}
\end{aligned}
\]
assigning to every subset \( X \subseteq U \) two sets \( R_\gamma X \) and \( R^* X \) called the \( R \)-lower and the \( R \)-upper approximation of \( X \), respectively.

If \( R_\gamma X = R^* X \), then \( X \) is said to be \( R \)-definable; otherwise, \( X \) is said to be \( R \)-rough.

Let us recall now the soft set notion, which is a newly emerging mathematical approach to vagueness.

Definition 14 (see [2]). Let \( U \) be a universal set and let \( E \) be a set of parameters. Let \( A \) be a nonempty subset of \( E \). A soft set over \( A \), with support \( A \), denoted by \( f_A \) on \( U \) is defined by the set of ordered pairs
\[
f_A = \{(e, f_A(e)) : e \in E, f_A(e) \in \wp(U)\}
\]
or is a function \( f_A : E \rightarrow \wp(U) \) s.t
\[
f_A(e) \neq \emptyset, \quad \forall e \in A \subseteq E,
\]
\[
f_A(e) = \emptyset, \quad \text{if} \ e \notin A.
\]

Example 15. Suppose that \( U \) is the set of houses under consideration and \( A \) and \( B \) are both parameter sets. Let there be four houses in the universe \( U \) given by \( U = \{h_1, h_2, h_3, h_4\} \). And \( A = \{\text{expensive, modern}\} \) and \( B = \{\text{modern}\} \). The soft sets \( f_A \) and \( g_B \) describe the "attractiveness of the houses." For the sake of ease of designation, we use \( e \), instead of expensive and \( m \) instead of modern. The soft set \( f_A \) is defined as follows
\[
f(e) \text{ means expensive houses, and } f(m) \text{ means modern houses. The soft set } f_A \text{ is the collection of approximations as below:}
\]
\[
f_A = \{(e, \{h_1, h_2\}, (m, \{h_3, h_4\})\}.
\]
The soft set \(g_B\) is defined as \(g(m)\), which means the modern houses. The soft set \(g_B\) is the collection of approximations as below:

\[
g_B = \{ (m, \{h_1, h_2\}) \}.
\]

\[\text{(9)}\]

**Definition 16** (see [25, 26]). Let \(U\) be a universal set and let \(f_A\) be a soft set over \(U\). Then the pair \(P = (U, f_A)\) is called soft approximation space. We define a pair of operators \(\text{apr}_P, \text{apr}_P^*\) : \(\wp(U) \rightarrow \wp(U)\) as follows:

\[
\text{apr}_P(X) = \{ u \in U : \exists a \in A, \text{ s.t. } u \in f(a) \subseteq X \},
\]

\[
\text{apr}_P^*(X) = \{ u \in U : \exists a \in A, \text{ s.t. } u \in f(a) \cap X \neq \emptyset \}.
\]

\[\text{(10)}\]

The elements \(\text{apr}_P(X)\) and \(\text{apr}_P^*(X)\) are called the soft \(P\)-lower and the soft \(P\)-upper approximations of \(X\).

If \(\text{apr}_P(X) = \text{apr}_P^*(X)\), \(X\) is said to be soft \(P\)-definable; otherwise \(X\) is called a soft \(P\)-rough set.

**Example 17.** Let us consider the following soft set \(S = f_E\) which describes "life expectancy". Suppose that the universe \(U = \{u_1, u_2, u_3, u_4, u_5, u_6\}\) consists of six persons and \(E = \{e_1, e_2, e_3, e_4\}\) is a set of decision parameters. The \(e_i\) (\(i = 1, 2, 3, 4\)) stands for "under stress," "young," "drug addict" and "healthy." Set \(f(e_1) = \{u_2\}\), \(f(e_2) = \{u_1, u_2\}\), \(f(e_3) = \emptyset\); and \(f(e_4) = \{u_1, u_2, u_3, u_6\}\). The soft set \(f_E\) can be viewed as the following collection of approximations:

\[
f_E = \{ \text{under stress, } \{u_2\}; \text{young, } \{u_1, u_2\}; \}
\]

\[
\text{drug addict, } \emptyset; \text{ healthy, } \{u_1, u_2, u_3, u_6\}\}.
\]

\[\text{(11)}\]

On the other hand, "life expectancy" topic can also be described using rough sets as follows: the evaluation will be done in terms of attributes: "sex," "age category," "living area," and "habits," characterized by the value sets \{"man, woman","baby, young, mature age, old"\}, \{"city, village"\}, and \{"smoke, drinking, smoke and drinking, no smoke and no drinking"\}. We denote "smoke and drinking" by SD and "no smoke and no drinking" by NSND. The information will be given by Table 1, where the rows are labeled by attributes and the table entries are the attribute values for each person. From here we obtain the following equivalence classes, induced by the above mentioned attributes:

\[
[u_1]_R = [u_2]_R = [u_1, u_2],
\]

\[
[u_3]_R = [u_5]_R = [u_3, u_5],
\]

\[\text{(12)}\]

\[
[u_4]_R = [u_4], \quad [u_6]_R = [u_6].
\]

Let \(X\) be a target subset of \(U\), that we wish to represent using the above equivalence classes. Hence we analyze the upper and lower approximations of \(X\), in some particular cases.

1. Set \(X = \{u_1, u_2, u_3, u_6\}\). It follows that

\[
R_1 X = \{u_1, u_2, u_3, u_6\},
\]

\[\text{(13)}\]

\[\text{R'} X = \{u_1, u_2, u_3, u_5, u_6\}.\]

Let us calculate now the soft \(P\)-lower and \(P\)-upper approximations of \(X\), where \(P = (U, S)\). We obtain

\[
\text{apr}_P(X) = \{u_1, u_2, u_3, u_6\} = X,
\]

\[\text{(14)}\]

\[
\text{apr}_P^*(X) = \{u_1, u_2, u_3, u_5, u_6\} = X;\]

hence \(X\) is soft \(P\)-definable.

2. Set \(X = \{u_3\}\). It follows that \(R_1 X = \{u_3\}\). On the other hand, \(\text{apr}_P(X) = \text{apr}_P^*(X) = X\), hence \(X\) is soft \(P\)-definable.

The above results show that soft rough set approximation is a worth considering alternative to the rough set approximation. Soft rough sets could provide a better approximation than rough sets do, depending on the structure of the equivalence classes and of the subsets \(F(e)\), where \(e \in E\).

### 3. Soft Sets on a Complete Atomic Boolean Lattice

**Definition 18.** Let \(B = (B, \leq)\) be a complete atomic Boolean lattice and let \(E\) be a set of parameters. Let \(A\) be a non-empty subset of \(E\). A soft set over \(A\), with support \(A\), denoted by \(f_A\) on \(B\) is defined by the set of ordered pairs

\[
f_A = \{(e, f_A(e)) : e \in E, f_A(e) \in B\},
\]

or is a function \(f_A : E \rightarrow B\) such that

\[
f_A(e) \neq 0, \quad \forall e \in A \subseteq E,
\]

\[\text{(16)}\]

\[f_A(e) = 0 \quad \text{if } e \notin A.\]

In other words, a soft set over \(B\) is a parameterized family of elements of \(B\). For each \(e \in A\), \(f(e)\) is considered as \(e\)-approximate element of \(f_A\).

**Definition 19.** Let \(B = (B, \leq)\) be a complete atomic Boolean lattice. Let \(A_1, A_2 \subseteq E\) and let \(f_{A_1}\) and \(g_{A_2}\) be two soft sets over \(B\).
(i) $f_{A_1}$ is a soft subset of $g_{A_2}$, denoted by $f_{A_1} \subseteq g_{A_2}$ if $A_1 \subseteq A_2$ and $f(e) \leq g(e)$ for every $e \in A_1$.

(ii) $f_{A_1}$ and $g_{A_2}$ are called soft equal, denoted by $f_{A_1} = g_{A_2}$ if $f_{A_1} \subseteq g_{A_2}$ and $g_{A_2} \subseteq f_{A_1}$.

Definition 20. Let $B = (B, \leq)$ be a complete atomic Boolean lattice. Let $A \subseteq E$ and let $f_A$ be a soft set over $B$.

(i) $f_A$ is called null, denoted by $0_A$ if $f(e) = 0$ for every $e \in A$.

(ii) $f_A$ is called absolute, denoted by $1_A$ if $f(e) = 1$ for every $e \in A$.

We stipulate that $0_\emptyset$ is also a soft set over $B$ with $0 : \emptyset \to B$.

Let $A \subseteq E$ and let $f_A$ be a soft set over $B$. Obviously,

$$0_A \subseteq f_A \subseteq 1_A. \quad (17)$$

Below, we introduce some operations on soft sets on $B$ and investigate their properties.

Definition 21. Let $B = (B, \leq)$ be a complete atomic Boolean lattice. Let $A_1, A_2 \subseteq E$ and let $f_{A_1}$ and $g_{A_2}$ be two soft sets over $B$.

(i) $h_{A_1}$ is called the intersection of $f_{A_1}$ and $g_{A_2}$, denoted by $h_{A_1} = f_{A_1} \cap g_{A_2}$, if $A_3 = A_1 \cap A_2$ and $h(e) = f(e) \land g(e)$ for every $e \in A_3$.

(ii) $h_{A_1}$ is called the union of $f_{A_1}$ and $g_{A_2}$, denoted by $h_{A_1} = f_{A_1} \cup g_{A_2}$, if $A_3 = A_1 \cup A_2$ and $h(e) = f(e) \lor g(e)$ if $e \in A - B, h(e) = g(e)$ if $e \in B - A$ and $h(e) = f(e) \lor g(e)$ if $e \in A \cap B$.

Definition 22. Let $B = (B, \leq)$ be a complete atomic Boolean lattice. Let $A \subseteq E$ and let $f_A$ be a soft set over $B$. The complement of $f_A$, denoted by $(f_A)^c$ is defined by $(f_A)^c = (f^c, A)$, where $f^c : A \to B$ is a mapping given by $f^c(e) = f(e)^c$ for every $e \in A$.

Proposition 23. Let $B = (B, \leq)$ be a complete atomic Boolean lattice. Let $A_1, A_2, A_3 \subseteq E$ and let $f_{A_1}, g_{A_2}$, and $h_{A_3}$ be three soft sets over $B$. Then

(i) $f_{A_1} \cup f_{A_1} = f_{A_1}$,

(ii) $f_{A_1} \cup g_{A_2} = g_{A_2} \cup f_{A_1}$,

(iii) $(f_{A_1} \cup g_{A_2}) \cap h_{A_3} = f_{A_1} \cap (g_{A_2} \cap h_{A_3})$.

Proof. (i) and (ii) are obvious. We only prove (iii). Put

$$\left( f_{A_1} \cup g_{A_2} \right) \cap h_{A_3} = k_{A_1 \cup A_2 \cup A_3},$$

$$f_{A_1} \cup \left( g_{A_2} \cap h_{A_3} \right) = I_{A_1 \cup A_2 \cup A_3},$$

$$f_{A_1} \cup g_{A_2} = s_{A_1 \cup A_2}, \quad g_{A_2} \cup h_{A_3} = t_{A_2 \cup A_3}.$$

For any $e \in A_1 \cup A_2 \cup A_3$ it follows that $e \in A_1$, or $e \in A_2$, or $e \in A_3$.

Case 1 ($e \in A_3$).

(a) If $e \notin A_1$ and $e \notin A_2$, then $k(e) = h(e) = t(e) = I(e)$.

(b) If $e \notin A_1$ and $e \in A_2$, then $k(e) = s(e) \lor h(e) = f(e) \lor h(e) = t(e) = I(e)$.

(c) If $e \in A_1$ and $e \notin A_2$, then $k(e) = s(e) \land h(e) = f(e) \land h(e) = t(e) = I(e)$.

(d) If $e \in A_1$ and $e \in A_2$, then $k(e) = s(e) \land h(e) = f(e) \lor g(e) \lor h(e) = f(e) \lor t(e) = I(e)$.

Thus $(f_{A_1} \cup g_{A_2}) \cap h_{A_3} = f_{A_1} \cup (g_{A_2} \cap h_{A_3})$.

\[ \square \]

Proposition 24. Let $B = (B, \leq)$ be a complete atomic Boolean lattice. Let $A_1, A_2, A_3 \subseteq E$ and let $f_{A_1}, g_{A_2}$, and $h_{A_3}$ be three soft sets over $B$. Then

(i) $f_{A_1} \cap f_{A_1} = f_{A_1}$,

(ii) $f_{A_1} \cap g_{A_2} = g_{A_2} \cap f_{A_1}$,

(iii) $(f_{A_1} \cap g_{A_2}) \cup h_{A_3} = f_{A_1} \cap (g_{A_2} \cup h_{A_3})$.

Proof. (i) and (ii) are obvious. We only prove (iii). Put

$$\left( f_{A_1} \cap g_{A_2} \right) \cup h_{A_3} = k_{A_1 \cap A_2 \cup A_3},$$

$$f_{A_1} \cap \left( g_{A_2} \cup h_{A_3} \right) = I_{A_1 \cap A_2 \cup A_3},$$

$$f_{A_1} \cap g_{A_2} = s_{A_1 \cap A_2}, \quad g_{A_2} \cup h_{A_3} = t_{A_2 \cup A_3}.$$

For any $e \in A_1 \cap A_2 \cap A_3$, it follows that $e \in A_1$, or $e \in A_2$, and $e \in A_3$. Since $k(e) = (f(e) \land g(e) \land h(e)) = I(e)$, then $(f_{A_1} \cap g_{A_2}) \cup h_{A_3} = f_{A_1} \cap (g_{A_2} \cup h_{A_3})$.

\[ \square \]

Proposition 25. Let $B = (B, \leq)$ be a complete atomic Boolean lattice. Let $A_1, A_2, A_3 \subseteq E$ and let $f_{A_1}, g_{A_2}$, and $h_{A_3}$ be three soft sets over $B$. Then

(i) $(f_{A_1} \cup g_{A_2}) \cap h_{A_3} = (f_{A_1} \cap h_{A_3}) \cup (g_{A_2} \cap h_{A_3})$,

(ii) $(f_{A_1} \cap g_{A_2}) \cap h_{A_3} = (f_{A_1} \cap h_{A_3}) \cap (g_{A_2} \cap h_{A_3})$.

Proof. (i) Put $(f_{A_1} \cup g_{A_2}) \cap h_{A_3} = k_{A_1 \cup A_2 \cup A_3}, (f_{A_1} \cap h_{A_3}) \cap (g_{A_2} \cup h_{A_3}) = I_{A_1 \cap A_2 \cup A_3}$. Obviously, $(A_1 \cup A_2) \cap A_3 = (A_1 \cap A_2) \cup (A_2 \cap A_3)$. For any $e \in (A_1 \cup A_2) \cap A_3$, it follows that $e \in A_1 \cap A_3$ or $e \in A_2 \cap A_3$.

(a) If $e \notin A_1 \cap A_3$ and $e \in A_2 \cap A_3$, then $e \notin A_1$, or $e \in A_2$, and $e \in A_3$. So $k(e) = g(e) \land h(e) = I(e)$.

(b) If $e \in A_1 \cap A_3$ and $e \notin A_2 \cap A_3$, then $e \in A_1$, or $e \notin A_2$, and $e \in A_3$. So $k(e) = f(e) \land h(e) = I(e)$.
(c) If $e \in A_1 \cap A_3$ and $e \in A_2 \cap A_3$, then $e \in A_1$, $e \in A_2$, and $e \in A_3$. So $k(e) = (f(e) \lor g(e)) \land h(e) = (f(e) \lor h(e)) \lor (g(e) \lor h(e)) = l(e)$.

Thus $(f_{A_1} \cup g_{A_2}) \cap h_{A_3} = (f_{A_1} \cap h_{A_3}) \cup (g_{A_2} \cap h_{A_3})$.

(ii) This is similar to the proof of (i). □

**Proposition 26.** Let $B = (B, \leq)$ be a complete atomic Boolean lattice. Let $A_1, A_2 \subseteq E$ and let $f_{A_1}$ and $g_{A_2}$ be two soft sets over $B$.

(i) $(f_{A_1})_{A_2} = f_{A_1}$.

(ii) $f_{A_1} \cup (f_{A_1})_{A_2} = 1_A$.

(iii) $f_{A_1} \cap (f_{A_1})_{A_2} = 0_A$.

(iv) $(f_{A_1} \cup g_{A_2})_{A_3} = (f_{A_1})_{A_3} \cap (g_{A_2})_{A_3}$.

(v) $(f_{A_1} \cap g_{A_2})_{A_3} = (f_{A_1})_{A_3} \cup (g_{A_2})_{A_3}$.

**Proof.** (i) Put $(f_{A_1})_{A_2} = g_{A_2}, (f_{A_1})_{A_2} = h_{A_3}$.

For any $e \in A$, $h(e) = g(e)^4 = (f(e) \lor g(e))^4 = f(e)^4 = f(e)^4$. So, $h(e) = g(e)^4 = (f(e)^4) = f(e)$ (by Lemma 3). This shows that $h_{A_3} = f_{A_3}$; that is $(f_{A_1})_{A_2} = f_{A_1}$.

(ii) Put $f_{A_1} \cup (f_{A_1})_{A_2} = h_{A_3}$.

For any $e \in A$, $h(e) = f(e) \lor f(e)^4 = f(e) \lor f(e)^4 = 1$. Hence $f_{A_1} \cup (f_{A_1})_{A_2} = 1_A$.

(iii) This is similar to the proof of (ii).

(iv) Put $(f_{A_1} \cup g_{A_2})_{A_3} = h_{A_3}, (f_{A_1})_{A_3} \cap (g_{A_2})_{A_3} = l_{A_3}$.

For any $e \in A$, $h(e) = (f(e) \lor g(e)^4), l(e) = f(e)^3 \lor g(e)^4$. Hence $h(e) = l(e)$ by Lemma 3.

(v) This is similar to the proof of (iv). □

**Definition 27.** Let $B = (B, \leq)$ be a complete atomic Boolean lattice and let $f_E$ be a soft set over $B$.

(i) $f_E$ is called full if $\bigvee_{e \in E} f(e) = 1$.

(ii) $f_E$ is keeping infimum, if for any $e_1, e_2 \in E$, there exists $e_3 \in E$ such that $f(e_1) \land f(e_2) = f(e_3)$.

(iii) $f_E$ is keeping supremum, if for any $e_1, e_2 \in E$, there exists $e_3 \in E$ such that $f(e_1) \lor f(e_2) = f(e_3)$.

(iv) $f_E$ is called partition of $B$ if

1. $\bigvee_{e \in E} f(e) = 1$,
2. for every $e \in E, f(e) \neq 0$,
3. for every $e_1, e_2 \in E$ either $f(e_1) = f(e_2)$ or $f(e_1) \land f(e_2) = 0$.

Obviously, every partition soft set is full and $f_E$ is keeping infimum (resp., keeping supremum) if and only if for every $E^* \subseteq E$, there exists $e^* \in E$ such that $\bigwedge_{e \in E^*} f(e) = f(e^*)$ (resp., $\bigvee_{e \in E^*} f(e) = f(e^*)$).

**Example 28.** Let $B = \{0, a, b, c, d, e, f, 1\}$ and let the order $\leq$ be defined as in Figure 1.

The set of atoms of a complete atomic Boolean lattice $B = (B, \leq)$ is $\{a, b, c\}$. Let $A = \{e_1, e_2, e_3, e_4\}$ and let $f_A$ be a soft set over $B$ defined as follows:

\[
\begin{align*}
  f(e_1) &= e, & f(e_2) &= b, \\
  f(e_3) &= c, & f(e_4) &= 0.
\end{align*}
\]

![Figure 1](image)

Obviously, $f_A$ is not a partition since $f(e_4) = 0$. Also, $f_A$ is full since $\bigvee_{e \in A} f(e) = e \lor b \lor c = 1$. Also, $f_A$ is keeping infimum. In fact $f(e_1) \land f(e_2) = f(e_1) \land f(e_4) = f(e_3) \land f(e_4) = f(e_4) = 0$.

$f(e_1) \land f(e_3) = e \land c = e = f(e_1)$ and $f(e_2) \land f(e_3) = b \land c = 0 = f(e_2)$. Consequently, $f_A$ is keeping infimum. On the other hand, $f_A$ is not keeping supremum since $f(e_1) \lor f(e_2) = e \lor b = 1 \neq f(e_1)$ for every $e \in A$.

Let $g_A$ be a soft set over $B$ defined as follows:

$g(e_1) = d, g(e_2) = a, g(e_3) = e, and g(e_4) = 1$; then $g_A$ is a partition, keeping infimum, and keeping supremum.

Next, we investigate the lattice structure of soft sets on a complete atomic Boolean Lattice $B$. We denote

\[
S(B, E) = \{f_E : f_E \text{ is soft set over } B\},
\]

\[
S_1(B, E) = \{f_A : A \subseteq E \text{ and } f_A \text{ is soft set over } B\}. \tag{21}
\]

Obviously,

\[
S_1(B, E) \subseteq S(B, E). \tag{22}
\]

**Theorem 29.** For any $f_A, g_B \in S(B, E)$, define

\[
f_A \leq g_B \iff f_A \subseteq g_B, \quad f_A \lor g_B = f_A \cup g_B, \quad f_A \land g_B = f_A \cap g_B. \tag{23}
\]

Then $S(B, E)$ is a distributive lattice with smallest element $0_{\Sigma} = 0_{\phi}$ and greatest element $1_{\Sigma} = 1_{E}$.

**Proof.** Denote $\Sigma = S(B, E)$. It is easily proved that

\[
0_{\Sigma} = 0_{\phi}, \quad 1_{\Sigma} = 1_{E}. \tag{24}
\]

By Proposition 25 $S(X, E)$ is a distributive lattice with $1_{\Sigma}$ and $0_{\Sigma}$. □
Theorem 30. For any \( f_A, g_B \in S_1(B, E) \), define
\[
 f_A \leq g_B \iff f_A \subseteq g_B, \quad f_A \lor g_B = f_A \sqcup g_B, \quad f_A \land g_B = f_A \sqcap g_B.
\]
Then \( S_1(B, E) \) is a Boolean lattice.

Proof. Denote \( \Sigma_1 = S_1(B, E) \). It is easily proved that \( S_1(B, E) \) is a distributive lattice with \( 0_{\Sigma_1} = 0_E \) and \( 1_{\Sigma_1} = 1_E \).

Let \( f_E \in \Sigma_1 \). Put \( h_E = f_E \lor f_E^c \). Since \( h_E = f_E \sqcup f_E^c \), then for any \( e \in E \),
\[
h(e) = f(e) \lor f^c(e) = f(e) \lor f(e)^c = 1.
\]
So, \( h_E = 1_E = 1_{\Sigma_1} \). This shows that \( f_E \lor f_E^c = 1_{\Sigma_1} \). Similarly, we can prove that \( f_E \land f_E^c = 0_{\Sigma_1} \). Hence \( (f_E)^c = f_E^c \) and therefore \( S_1(B, E) \) is a Boolean lattice.

4. Soft Rough Approximation Operators on a Complete Atomic Boolean Lattice

Definition 31. Let \( B = (B, \leq) \) be a complete atomic Boolean lattice and let \( f_A \) be a soft set over \( B \). For any element \( x \in B \), we define a pair of operators \( x^\vee, x^\wedge : B \to B \) as follows:
\[
x^\vee = \bigvee \{ b \in A(B) : \exists e \in A s.t. b \leq f(e), f(e) \leq x \},
\]
\[
x^\wedge = \bigwedge \{ b \in A(B) : \exists e \in A s.t. b \leq f(e), f(e) \land x \neq 0 \}.
\]

The elements \( x^\vee \) and \( x^\wedge \) are called the soft lower and the soft upper approximations of \( x \) over \( B \). Two elements \( x \) and \( y \) are called soft equivalent if they have the same soft upper and soft lower approximations over \( B \). The resulting equivalence classes are called soft rough sets over \( B \).

Lemma 32. Let \( B = (B, \leq) \) be a complete atomic Boolean lattice and let \( f_A \) be a soft set over \( B \). Then for all \( c \in A(B) \) and \( x \in B \)

(i) \( c \leq x^\vee \iff \exists e \in A s.t. c \leq f(e) \) and \( f(e) \leq x \);

(ii) \( c \leq x^\wedge \iff \exists e \in A s.t. c \leq f(e) \) and \( f(e) \land x \neq 0 \).

Proof. (i) (\( \Rightarrow \)) Suppose that \( c \leq x^\vee \). Assume that for all \( e \in E \) either \( c \not\leq f(e) \) or \( f(e) \not\leq x \). If \( c \not\leq f(e) \) then \( c \not\leq x^\vee \), a contradiction. If \( c \not\leq f(e) \), then \( c \land x^\vee = c \land \bigvee \{ b \in A(B) : \exists e \in A s.t. b \leq f(e) \land f(e) \leq x \} = \bigvee \{ b \in A(B) : \exists e \in A s.t. b \leq f(e) \land f(e) \leq x \} = x \). This implies that \( c \leq (x^\vee)^c \), which is a contradiction.

(\( \Leftarrow \)) Suppose that \( \exists e \in A s.t. c \leq f(e) \) and \( f(e) \leq x \); then \( c \leq \bigvee \{ b \in A(B) : \exists e \in A s.t. b \leq f(e) \land f(e) \leq x \} = x^\vee \).

Condition (ii) can be proved similarly.

Proposition 33. Let \( B = (B, \leq) \) be a complete atomic Boolean lattice and let \( f_A \) be a soft set over \( B \). Then for all \( x \in B \)

(i) \( x^\vee = \sqrt{f(e) : e \in A \land f(e) \leq x} \);

(ii) \( x^\wedge = \sqrt{f(e) : e \in A \land f(e) \land x \neq 0} \).

Proof. (i) Let \( c \in A(B) \), s.t \( c \leq x^\vee \); then \( \exists e \in A s.t. c \leq f(e) \) and \( f(e) \leq x \). So, \( c \leq \sqrt{f(e) : e \in A \land f(e) \leq x} \). On the other hand, let \( c \in A(B) \), s.t \( c \leq f(e) \) and \( f(e) \leq x \). If, in fact, \( c \leq f(e) \) implies \( c \not\leq f(e) \), then \( c \land f(e) \). Therefore \( c \leq f(e)^c \) because \( c \in A(B) \). Thus \( c \leq \sqrt{f(e) : e \in A \land f(e) \leq x} \). So, \( c \leq \sqrt{f(e) \land f(e)^c} : e \in A \land f(e) \leq x \). Hence \( \exists e \in A s.t. c \not\leq f(e) \) and \( f(e) \leq x \). Consequently, \( c \leq x^\vee \).

Condition (ii) can be proved similarly.

Proposition 34. Let \( B = (B, \leq) \) be a complete atomic Boolean lattice and let \( f_A \) be a soft set over \( B \).

(i) \( 0^\vee = 0^\wedge = 0 \) and \( 1^\vee = 1^\wedge = \bigvee_{e \in A} f(e) \);

(ii) \( x \leq y \) implies \( x^\vee \leq y^\vee \) and \( x^\wedge \leq y^\wedge \).

Proof. Obvious.

For all \( S \subseteq B \), we denote \( S^\vee = \{ x^\vee : x \in S \} \) and \( S^\wedge = \{ x^\wedge : x \in S \} \).

Proposition 35. Let \( B = (B, \leq) \) be a complete atomic Boolean lattice and let \( f_A \) be a soft set over \( B \); then

(i) for all \( S \subseteq B \), \( \forall S^\vee = (\forall S)^\wedge \);

(ii) if \( f_A \) is keeping infimum, then for all \( S \subseteq B \), \( \forall S^\wedge = (\forall S)^\vee \);

(iii) \((B^\vee, \leq) \) is a complete lattice; \( 0 \) is the least element and \( 1^\wedge \) is the greatest element of \((B^\wedge, \leq) \);

(iv) if \( f_A \) is keeping infimum, then \((B^\wedge, \leq) \) is a complete lattice; \( 0 \) is the least element and \( 1^\vee \) is the greatest element of \((B^\vee, \leq) \);

(v) if \( f_A \) is keeping infimum, the kernel \( \Theta_x = \{ (x, y) : x^\vee = y^\vee \} \) of the map \( \wedge : B \to B \) is a congruence on the semi lattice \((B, \land) \) such that the \( \Theta_x \)-class of any \( x \) has a least element;

(vi) the kernel \( \Theta_x = \{ (x, y) : x^\wedge = y^\wedge \} \) of the map \( \wedge : B \to B \) is a congruence on the semi lattice \((B, \lor) \) such that the \( \Theta_x \)-class of any \( x \) has a least element.

Proof. (i) Let \( S \subseteq B \). The map \( \wedge : B \to B \) is order preserving, which implies that \( \forall S^\wedge \subseteq (\forall S)^\vee \). Let \( b \in A(B) \) and assume that \( b \leq (\forall S)^\vee \). So, \( \exists e \in A s.t. b \leq f(e) \) and \( f(e) \land \forall S \neq 0 \).

Then \( \forall (f(e) \land \forall S) = \forall f(e) \land \forall S = \forall f(e) \land x \in S \). Thus \( \exists e \in A s.t. b \leq f(e) \) and \( f(e) \land \forall S \neq 0 \). Then \( b \in (A(B) : \exists e \in A s.t. b \leq f(e) \) and \( f(e) \land \forall S \neq 0 \). Then \( b \in (A(B) : \exists e \in A s.t. b \leq f(e) \) and \( f(e) \land \forall S \neq 0 \).

So, \( \forall S^\vee = \sqrt{\{ b \in A(B) : \exists e \in A s.t. b \leq f(e) \land f(e) \land \forall S \neq 0 \}} \).
(28)
5. Another Soft Rough Approximation Operators on a Complete Atomic Boolean Lattice

**Definition 40.** Let $B = (B, \leq)$ be a complete atomic Boolean lattice and let $f_A$ be a soft set over $B$. Define a mapping $\varphi_f : A(B) \to B$ by

$$c \leq \varphi_f(b) \iff \exists e \in A, \ s.t. \ c \leq f(e), \ b \leq f(e)$$

for every $c, b \in A(B)$. Then $\varphi_f$ is called the mapping induced by $f_A$ on $B$.

**Proposition 41.** Let $B = (B, \leq)$ be a complete atomic Boolean lattice and let $f_A$ be a soft set over $B$. Let $\varphi_f : A(B) \to B$ be the mapping induced by $f_A$ on $B$. Then the following properties hold.

(i) $\varphi_f$ is symmetric.

(ii) If $f_A$ is full, then $\varphi_f$ is extensive.

(iii) If $f_A$ is a partition, then $\varphi_f$ is extensive, symmetric, and closed.

**Proof.** (i) Obvious.

(ii) Let $b \in A(B)$. Since $f_A$ is full, then $\exists e \in A, s.t. b \leq f(e)$. Hence $b \leq \varphi_f(b)$.

(iii) If $f_A$ is a partition, then $f_A$ is full and hence $\varphi_f$ is extensive. Since $\varphi_f$ is symmetric, it remains to show that $\varphi_f$ is closed. Let $c, b \in A(B)$ s.t $c \leq \varphi_f(b)$. We show that $\varphi_f(c) \leq \varphi_f(b)$. Since $c \leq \varphi_f(b)$, then $\exists e_1 \in A, \ s.t. c \leq f(e_1)$ and $b \leq f(e_1)$. Suppose that $\varphi_f(c) \not\leq \varphi_f(b)$. So, $\exists d \in A(B), \ s.t d \leq \varphi_f(c)$ and $d \not\leq \varphi_f(b)$. But $d \not\leq \varphi_f(b)$ implies that for every $e \in A$, either $d \not\leq f(e)$ or $b \not\leq f(e)$. Since $d \leq \varphi_f(c)$, then $\exists e_2 \in A, \ s.t. d \leq f(e_2)$ and $c \leq f(e_2)$. Since $f_A$ is a partition and $c \leq f(e_2) \wedge f(e_1)$, then $f(e_1) = f(e_2)$. Hence we show that $\exists e_1 \in A, \ s.t. d \leq f(e_1)$ and $b \leq f(e_1)$, a contradiction. Consequently, $\varphi_f(c) \leq \varphi_f(b)$ and thus $\varphi_f$ is closed.

**Proposition 42.** Let $B = (B, \leq)$ be a complete atomic Boolean lattice and let $f_A$ be a soft set over $B$. Let $\varphi_f : A(B) \to B$ be the mapping induced by $f_A$ on $B$. Then the following properties hold.

(i) If $b \leq f(e)$ for $e \in A$ and $b \in A(B)$, then $f(e) \leq \varphi_f(b)$.

(ii) If $f_A$ is a partition and $b \leq f(e)$ for $e \in A$ and $b \in A(B)$, then $f(e) = \varphi_f(b)$.

(iii) If $f_A$ is keeping supremum, then for all $b \in A(B) \exists e \in A, s.t. \varphi_f(b) = f(e)$.

**Proof.** (i) Let $c \in A(B)$ s.t $c \leq f(e)$. Since $b \leq f(e)$, then $c \leq \varphi_f(b)$. Hence $f(e) \leq \varphi_f(b)$.

(ii) Suppose that $f_A$ is a partition and assume that $b \leq f(e)$ for $e \in A$ and $b \in A(B)$. By (i) $f(e) \leq \varphi_f(b)$. On the other hand, let $c \in A(B)$ s.t $c \leq \varphi_f(b)$. Then $\exists e_1 \in A, \ s.t. c \leq f(e_1)$ and $b \leq f(e_1)$. So, $b \leq f(e_1) \wedge f(e_1)$, and since $f_A$ is a partition, then $f(e) = f(e_1)$. Hence $c \leq f(e)$ and therefore $\varphi_f(b) \leq f(e)$. Consequently, $\varphi_f(b) = f(e)$.

(iii) Suppose that $f_A$ is keeping supremum and $b \in A(B)$. Let $c \in A(B)$ s.t $c \leq \varphi_f(b)$. Then $\exists e \in A, \ s.t. c \leq f(e)$ and $b \leq f(e)$. So, $f(e) \leq \varphi_f(b)$ by (i). Hence, $\varphi_f(b) = \vee_{e \in A}(f(e) : c \leq \varphi_f(b))$. Since $f_A$ is keeping supremum, then $\vee_{e \in A}(f(e) : c \leq \varphi_f(b)) = f(e)$ for $e \in A$. Therefore $\varphi_f(b) = f(e)$.

**Definition 43.** Let $B = (B, \leq)$ be a complete atomic Boolean lattice and let $f_A$ be a soft set over $B$. Let $\varphi_f : A(B) \to B$ be the mapping induced by $f_A$ on $B$. We define a pair of soft approximation operators $\vee_f, \Delta_f : B \to B$ as follows:

$$\vee_f = \bigvee \{b \in A(B) : \varphi_f(b) \leq x\},$$

$$\Delta_f = \bigvee \{b \in A(B) : \varphi_f(b) \wedge x \neq 0\}.$$  

The elements $\vee_f$ and $\Delta_f$ are called the soft lower and the soft upper approximations of $x$ with respect to the mapping $\varphi_f$ induced by $f_A$, respectively. Two elements $x$ and $y$ are called equivalent if they have the same soft upper and lower approximations with respect to the mapping $\varphi_f$ induced by $f_A$ on $B$. The resulting equivalence classes are called soft rough sets with respect to the mapping $\varphi_f$ induced by $f_A$ on $B$.

**Proposition 44.** Let $B = (B, \leq)$ be a complete atomic Boolean lattice and let $f_A$ be a soft set over $B$. Let $\varphi_f : A(B) \to B$ be the mapping induced by $f_A$.

(i) $b \leq \vee_f \iff \varphi_f(b) \leq x$.

(ii) $b \leq \Delta_f \iff \varphi_f(b) \wedge x \neq 0$.

(iii) If $f_A$ is full, then $\vee_f \leq x \leq \Delta_f$.

(iv) $0^\Delta = 0$ and $1^\vee = 1$. If $f_A$ is full, then $0^\vee = 0^\Delta = 0$ and $1^\vee = 1^\Delta = 1$.

(v) $x \leq y$ implies $\vee_f \leq y^\vee$ and $\Delta_f \leq x^\Delta$.

(vi) The mappings $\vee_f, \Delta_f : B \to B$ and $\vee_f, \Delta_f : B \to B$ are mutually dual.

(vii) For all $S \subseteq B$, $\forall S^\Delta = (\forall S)^\vee$.

(viii) For all $S \subseteq B$, $\Delta S^\Delta = (\forall S)^\vee$.

(ix) $(B^\Delta, \leq)$ is a complete lattice; 0 is the least element and $1^\Delta$ is the greatest element of $(B^\Delta, \leq)$.

(x) The pair $\langle \vee_f, \Delta_f \rangle$ is a dual Galois connection on $B$.

(xi) $(B^\vee, \geq) \equiv (B^\Delta, \leq)$.

**Proof.** It follows by Propositions 41, 9, 10, 11, and 12; see [23].

In the following we study the relation between the above two pairs of soft rough approximation operators given in Definitions 31 and 40.

**Proposition 45.** Let $B = (B, \leq)$ be a complete atomic Boolean lattice and let $f_A$ be a soft set over $B$. Let $\varphi_f : A(B) \to B$ be the mapping induced by $f_A$. Then the following properties hold.

(i) If $f_A$ is full, then $\Delta_f \leq \Delta_f$.

(ii) If $f_A$ is full and keeping supremum, then $\vee_f \leq \vee_f$.
(iii) If \( f_A \) is a partition, then
\[ (a) \ x \land f = x \land, \]
\[ (b) \ x \lor f = x \lor. \]

**Proof.** (i) Let \( b \in A(B) \) s.t. \( b \leq x^\land \). Then \( \phi_f(b) \leq x \). Since \( f_A \) is full, then \( \exists e \in A, \ s.t. \ b \leq f(e) \). By Proposition 42(i) \( f(e) \leq \phi_f(b) \). Thus \( b \leq f(e) \leq x \) and hence \( b \leq x^\land \). Consequently, \( x^\land \leq x^\land \).

(ii) If \( x = 0 \), then \( x^\land = 0 \lor = 0 = x^\lor \). If \( x \neq 0 \) and \( f_A \) is keeping supremum, then by Proposition 38(3) \( x^\land = 1 \). Hence \( x^\land \leq x^\land \).

(iii) (a) If \( f_A \) is a partition, then it is full. So \( x^\lor \leq x^\land \) by (i). On the other hand, let \( b \in A(B) \) s.t. \( b \leq x^\land \). So \( \exists e \in A, \ s.t. \ b \leq f(e) \leq x \). Since \( f_A \) is a partition and \( b \leq f(e) \), then by Proposition 34(ii) \( f(e) = \phi_f(b) \). This implies that \( b \leq x^\lor \) and therefore \( x^\land \leq x^\land \). Consequently, \( x^\land \leq x^\land \).

(b) This is similar to the proof of (a).

**Example 46.** Let \( B = \{0, a, b, c, d, e, f, 1\} \) and let the order \( \leq \) be defined as in Figure 1. Let \( A = \{e_1, e_2, e_3, e_4\} \) and let \( f_A \) be a soft set over \( B \) defined as follows:
\[ f(e_1) = a, \quad f(e_2) = b, \]
\[ f(e_3) = d, \quad f(e_4) = 0. \]

Obviously, \( f_A \) is not full. Also \( \phi_f(a) = \bigvee \{b \in A(B) : b \leq f_A(a)\} = a \lor = b = d, \phi_f(b) = a \lor = b = d, \) and \( \phi_f(c) = 0 \).

Let \( x = b, y = a \). So \( x^\lor = \bigvee \{d \in A(B) : \phi_f(d) \leq b\} = c, \) and \( x^\land = \bigvee \{d \in A(B) : \phi_f(d) \land b \neq 0\} = a \lor = b = d \). On the other hand
\[ y^\lor = c \) and \( y^\land = d \). Hence \( x^\lor \leq y^\lor \) and \( x^\land \leq y^\land \), but \( x \neq y \).

**Example 47.** Let \( B = \{0, a, b, c, d, e, f, 1\} \) and let the order \( \leq \) be defined as in Figure 1. Let \( A = \{e_1, e_2, e_3, e_4\} \) and let \( g_A \) be a soft set over \( B \) defined as follows:
\[ g(e_1) = a, \quad g(e_2) = e, \]
\[ g(e_3) = c, \quad g(e_4) = f. \]

Obviously, \( g_A \) is full. Also \( \phi_g(a) = a \lor = e, \phi_g(b) = b \lor = c = f, \) and \( \phi_g(c) = a \lor = b \lor = c = 1 \). Let \( x = f \), then \( x^\lor = b \) and \( x^\land = a \lor = b \lor = c = 1 \). Hence \( x^\lor \leq x \leq x^\land \).

In the following, we give a relation between soft rough approximation operators and Järvinen’s approximation operators on a complete atomic Boolean lattice.

**Definition 48.** Let \( B = (B, \leq) \) be a complete atomic Boolean lattice. Let \( \phi : A(B) \to B \) be extensive, symmetric, and closed mapping. Define a mapping \( f_\phi : A \to B \) by \( f_\phi(e) = \phi(e) \) for every \( e \in A \), where \( A = A(B) \). Then \( f_\phi(A) \) is called the soft set induced by \( \phi \) on \( B \).

**Theorem 49.** Let \( B = (B, \leq) \) be a complete atomic Boolean lattice. Let \( f_\phi(A) \) be the soft set induced by \( \phi \) on \( B \). Then for every \( x \in B, x^\lor = x^\land \) and \( x^\lor = x^\land \), where
\[ x^\lor = \bigvee \{b \in A(B) : \exists e \in A, s.t. b \leq (f_\phi)(e), (f_\phi)(e) \leq x \}, \]
\[ x^\land = \bigvee \{b \in A(B) : \phi(b) \leq x \}, \]
\[ x^\lor = \bigvee \{b \in A(B) : \exists e \in A, s.t. b \leq (f_\phi)(e), (f_\phi)(e) \land x \neq 0 \}, \]
\[ x^\land = \bigvee \{b \in A(B) : \phi(b) \land x \neq 0 \}. \] (34)

**Proof.** Obvious.

**Theorem 50.** Let \( B = (B, \leq) \) be a complete atomic Boolean lattice. Let \( f_A \) be a partition soft set over \( B \). Then for every \( x \in B, x^\lor = x^\land \) and \( x^\lor = x^\land \).

**Proof.** It follows immediately by Propositions 41(iii) and 45(iii).

**Remark 51.** Theorems 49 and 50 illustrate that Järvinen’s approximations can be viewed as a special case of our soft rough approximations on a complete atomic Boolean lattice.

6. Conclusion

In this paper, we introduced the concept of soft sets on a complete atomic Boolean lattice as a generalization of soft sets and obtained the lattice structure of these soft sets. Two pairs of soft rough approximation operators on a complete atomic Boolean lattice were considered, and their properties were given. We show that Järvinen’s approximations can be viewed as a special case of our soft rough approximations. We may mention that soft rough sets on a complete atomic Boolean lattice can be used in object evaluation and group decision making. It should be noted that the use of soft rough sets could, to some extent, automatically reduce the noise factor caused by the subjective nature of the expert’s evaluation. We will investigate these problems in future papers.

**References**


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