Research Article

Singularities of a Space Curve according to the Relatively Parallel Adapted Frame and Its Visualization

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The relatively parallel adapted frame or Bishop frame is an alternative approach to define a moving frame that is well defined even when the curve has vanished second derivative, and it has been widely used in the areas of biology, engineering, and computer graphics. The main result of this paper is using the relatively parallel adapted frame for classification of singularity type of Bishop spherical Darboux image and Bishop dual which are deeply related with a space curve and making them visualized by computer.

1. Introduction

In 1975, Bishop [1] introduced a new beautiful frame called the relatively parallel adapted frame or Bishop frame, which could provide the desired means to ride along a space curve with C^k ($k \geq 2$) and $\gamma' \neq 0$. After that, many research papers related to the Bishop frame have been treated in the Euclidean space [2–7], Minkowski space [8, 9], and dual space [10]. This special frame is also extended to study canal and tubular surfaces [9]. It has applications in the areas of biology and computer graphics [11, 12]. For example, it may be possible to compute information about the shape of sequences of DNA using a curve defined by the Bishop frame. The Bishop frame may also provide a new way to control virtual cameras in computer animations. Typical computer graphics applications of the parallel transport frame include the generation of ribbons and tubes from 3D space curves and the generation of forward-facing camera orientations given an appropriate initial camera path. If the curve is coarsely refined, but is smooth enough to generate appropriate frame control points from the parallel transport frame algorithm, the resulting frames can be used as control points for any desired degree of smooth spline interpolations using the methods of Shoemake [13]. Rotating camera orientations relative to a stable forward-facing frame can be added by various techniques such as that of Hanson and Ma [14]. In

many (though not all) applications of camera animation, it is desirable to have the camera gaze direction pointing forward along a space curve throughout the motion. Typical examples would include a flight looking through the front window of an airplane cockpit, riding on a roller coaster, or sliding down a bannister. Other applications require the camera gaze direction to remain in some fixed skew orientation relative to the camera path, for example, a passenger looking out an airplane window. All such applications are easily accommodated using the parallel transport mechanism applied to an initial camera orientation.

Computer vision is the automatic analysis of sequences of images for the purpose of recovering three-dimensional surface shape. In recent years, several branches of mathematics, both ancient and modern, have been applied to computer vision. Projective geometry, which in its mathematical form dates back at least two centuries, is used to describe the relationship between points and lines in different images of the same object. Differential geometry, which is even older, though it received its definitive modern look in the first half of the nineteenth century, is used to describe the shape of curves and surfaces in engineering. A minor revolution in mathematical thought and technique occurred during the 1960s, largely through the inventive genius of the French mathematician René Thom. His ideas partly inspired by Whitney gave birth to what is called singularity theory, a

term which includes catastrophes and bifurcations. Today's singularity being a direct descendant of differential calculus is certain to have a great deal of interest to say about geometry and therefore about all the branches of mathematics, physics, engineering, and other disciplines where the geometrical spirit is a guiding light [15]. More recently, developments in singularity theory have enriched the field of geometry by making possible a wealth of detail only dreamed of fifty years ago. Likewise, developments in the speed and power of computers over the last decade have turned other dreams into reality and made possible real-world applications of mathematical theory.

There are several articles on singularities of Frenet curve in Euclidean space and Minkowski space [15–19]. The curves in these articles are three times differentiable curves such that γ' and γ'' must be linearly independent and $\gamma'' \neq \mathbf{0}$. The main tools in these articles are Serret-Frenet formulas and some related functions on curves such as the distance-squared functions and the height functions. With the help of Serret-Frenet formulas, some classical invariants of extrinsic differential geometry can be treated as singularities of these two functions. However, the Frenet-Serret frame is not defined for all points along every curve. A new frame is needed for the kind of mathematical analysis that is typically done with computer graphics. Bishop introduced the relatively parallel adapted frame or Bishop frame, which could provide the desired means to ride along a space curve such that C^2 and $\gamma' \neq \mathbf{0}$. However, to the best of the authors' knowledge, no literature exists regarding the singularities of space curves according to the relatively parallel adapted frame. Thus, the current study hopes to serve such a need, and it is inspired by the works of Bishop [1], Izumiya et al. [17], and Wang and Pei [18]. In this paper, we introduce the notions of Bishop spherical Darboux image, Bishop dual, Bishop height functions, and extended height functions on a space curve embedded in Euclidean 3-space. These definitions are similar to those carried out by Bruce and Izumiya in Euclidean 3-space [15, 17]. Adopting the relatively parallel adapted frame [1] as the basic tool and using the same methods as in [15, 17], we find that the relationships between the singularities of the discriminant of extended height functions and the sets of bifurcations of the height function correspond to Bishop dual and Bishop spherical Darboux image separately and geometric invariants of curves in Euclidean 3-space. We also get some meaningful properties of Bishop helix. The main result of this paper is in Theorem 1 using the relatively parallel adapted frame for classification of singularity type of some objects directly related with space curves.

The rest of this paper is organized as follows. Firstly, we introduce some basic concepts and the main results in the next two sections. Then, we introduce two different families of functions on γ that will be useful to the study of geometric invariants of regular curve. Afterwards, some general results on the singularity theory are used for families of function germs, and the main result is proved. Finally, we give two examples to illustrate the main results, and the conclusion of the work is drawn.

2. Preliminaries and Notations

Let $\gamma = \gamma(s)$ be a regular unit speed Frenet curve in $\mathbb{E}^3$. We know that there exist an accompanying three-frames called Frenet frame for Frenet curve. Denote by $(\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s))$ the moving Frenet frame along the unit speed Frenet curve $\gamma(s)$. Then, the Frenet formulas are given by

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{pmatrix} = \begin{pmatrix} 0 & k(s) & 0 \\ -k(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix}. \quad (1)$$

Here, $k(s)$ and $\tau(s)$ are called curvature and torsion, respectively [15].

We can parallel transport an orthogonal frame along a curve simply by parallel transporting each component of the frame. The parallel transport frame is based on the observation that while $\mathbf{T}(s)$ for a given curve model is unique, we may choose any convenient arbitrary basis $(\mathbf{N}_1(s), \mathbf{N}_2(s))$ for the remainder of the frame, so long as it is in the normal plane perpendicular to $\mathbf{T}(s)$ at each point. If the derivatives of $(\mathbf{N}_1(s), \mathbf{N}_2(s))$ depend only on $\mathbf{T}(s)$ and not each other, we can make $\mathbf{N}_1(s)$ and $\mathbf{N}_2(s)$ vary smoothly throughout the path regardless of the curvature. Therefore, we have the alternative frame equations

$$\begin{pmatrix} \mathbf{T}'(s) \\ \mathbf{N}_1'(s) \\ \mathbf{N}_2'(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & k_2(s) \\ -k_1(s) & 0 & 0 \\ -k_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}_1(s) \\ \mathbf{N}_2(s) \end{pmatrix}. \quad (2)$$

Here, we will call the set $(\mathbf{T}(s), \mathbf{N}_1(s), \mathbf{N}_2(s))$ as Bishop frame and $k_1(s) = \langle \mathbf{T}', \mathbf{N}_1 \rangle$ and $k_2(s) = \langle \mathbf{T}', \mathbf{N}_2 \rangle$ as Bishop curvatures. The relation matrix can be expressed as

$$\begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta(s) & \sin \theta(s) \\ 0 & -\sin \theta(s) & \cos \theta(s) \end{pmatrix} \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}_1(s) \\ \mathbf{N}_2(s) \end{pmatrix}. \quad (3)$$

One can show that

$$k(s) = \sqrt{k_1^2(s) + k_2^2(s)}, \quad \theta(s) = \arctan \left(\frac{k_2(s)}{k_1(s)} \right), \quad (4)$$

$$\text{where } k_1(s) \neq 0, \quad \tau(s) = \frac{d\theta(s)}{ds},$$

so that $k_1(s)$ and $k_2(s)$ effectively correspond to a Cartesian coordinate system for the polar coordinates k and θ with $\theta = \int \tau(s)ds$. Here, Bishop curvatures are also defined by

$$\begin{aligned} k_1(s) &= k(s) \cos \theta(s), \\ k_2(s) &= k(s) \sin \theta(s). \end{aligned} \quad (5)$$

The orientation of the parallel transport frame includes the arbitrary choice of integration constant θ_0 , which disappears from τ (and, hence, from the Frenet frame) due to the differentiation [1]. The unit sphere with center in the origin in the space $\mathbb{E}^3$ is defined by

$$S^2 = \{\mathbf{x} \in \mathbb{E}^3 \mid \langle \mathbf{x}, \mathbf{x} \rangle = 1\}. \quad (6)$$

For any regular unit speed curve $\gamma : I \rightarrow \mathbb{E}^3$,

$$\mathbf{BD}(s) = -k_2(s) \mathbf{N}_1(s) + k_1(s) \mathbf{N}_2(s) \quad (7)$$

is called a Bishop Darboux vector. We define a vector

$$\widetilde{\mathbf{BD}}(s) = \frac{k_2(s)}{k_1(s)} \mathbf{N}_1(s) - \mathbf{N}_2(s) \quad (8)$$

and call it a modified Bishop Darboux vector along γ . We define the Bishop spherical Darboux image of the curve γ as

$$\mathbf{d}(s) : I \rightarrow S^2, \quad \mathbf{d}(s) = \frac{\mathbf{BD}(s)}{\|\mathbf{BD}(s)\|}. \quad (9)$$

In fact,

$$\mathbf{d}(s) = -\frac{k_1(s)}{\sqrt{k_1^2(s) + k_2^2(s)}} \left(\frac{k_2(s)}{k_1(s)} \mathbf{N}_1(s) - \mathbf{N}_2(s) \right). \quad (10)$$

The Bishop rectifying developable of γ is defined by

$$\mathbf{BRD}_\gamma(s, u) = \{\gamma(s) + u\widetilde{\mathbf{BD}}(s), u \in \mathbb{R}, s \in I\}. \quad (11)$$

The Bishop dual of γ is defined by

$$\mathbf{BDU}_\gamma(\mathbf{v}, u) = \{(\mathbf{v}, u) \in S^2 \times \mathbb{R} : u = \langle \gamma(s), \mathbf{v} \rangle, \langle \mathbf{T}, \mathbf{v} \rangle = 0\}. \quad (12)$$

Denote the tangent Bishop spherical indicatrix, the first Bishop spherical indicatrix, and the second Bishop spherical indicatrix in [5] by $\mathbf{BT}(s) = \mathbf{T}(s)$, $\mathbf{FBN}(s) = \mathbf{N}_1(s)$, and $\mathbf{SBN}(s) = \mathbf{N}_2(s)$ separately. A regular unit speed curve $\gamma : I \rightarrow \mathbb{E}^3$ is called a Bishop slant helix according to Bishop frame provided that the unit vector $\mathbf{N}_1(s)$ of γ has constant angle θ with some fixed unit vector $\mathbf{u}$; that is, $\langle \mathbf{N}_1(s), \mathbf{u} \rangle = \text{constant}$. We define a new invariant $\rho(s) = k'_1(s)k_2(s) - k_1(s)k'_2(s)$ of a regular curve in $\mathbb{E}^3$, and we can describe B -slant curve by the invariant $\rho(s)$.

3. Singularity Classification and Its Visualization

The main results of this paper are in the following theorem.

Theorem 1. Let $\gamma : I \rightarrow \mathbb{E}^3$ be a regular unit speed curve with $k_1(s) > 0$. Then one has the following.

(1)

- (a) The Bishop spherical Darboux image is locally diffeomorphic to a line $\{0\} \times \mathbb{R}$ at s_0 if and only if $\rho(s_0) \neq 0$.
- (b) The Bishop spherical Darboux image is locally diffeomorphic to the cusp C at s_0 if and only if $\rho(s_0) = 0$ and $\rho'(s_0) \neq 0$.

(2)

- (a) The Bishop dual $\mathbf{BDU}_\gamma$ is locally diffeomorphic to the plane $\mathbb{R}^2$ at s_0 if and only if $k_1(s_0)\lambda + k_2(s_0)\mu \neq 0$, where λ and μ are real numbers such that $\lambda^2 + \mu^2 = 1$.
- (b) The Bishop dual $\mathbf{BDU}_\gamma$ is locally diffeomorphic to the cuspidal edge $C \times \mathbb{R}$ at s_0 if and only if

$$\mathbf{v} = \mp \frac{k_1(s_0)}{\sqrt{k_1^2(s_0) + k_2^2(s_0)}} \left(\frac{k_2(s_0)}{k_1(s_0)} \mathbf{N}_1(s_0) - \mathbf{N}_2(s_0) \right), \quad (13)$$

$$\rho(s_0) \neq 0.$$

- (c) The Bishop dual $\mathbf{BDU}_\gamma$ is locally diffeomorphic to the SW at s_0 if and only if

$$\mathbf{v} = \mp \frac{k_1(s_0)}{\sqrt{k_1^2(s_0) + k_2^2(s_0)}} \left(\frac{k_2(s_0)}{k_1(s_0)} \mathbf{N}_1(s_0) - \mathbf{N}_2(s_0) \right), \quad (14)$$

$$\rho(s_0) = 0, \quad \rho'(s_0) \neq 0,$$

where the ordinary cusp is $C = \{(x_1, x_2) \mid x_1^2 = x_2^3\}$.

The cuspidal edge is $C \times \mathbb{R} = \{(x_1, x_2) \mid x_1^2 = x_2^3\} \times \mathbb{R}$.

The swallowtail is $SW = \{(x_1, x_2, x_3) \mid x_1 = 3u^4 + u^2v, x_2 = 4u^3 + 2uv, x_3 = v\}$.

Note that when we consider the case $k_1(s) < 0$, we can get analogous results to the previous theorem; so, we omit them. The pictures of cuspidal edge and swallowtail will be seen in Figure 1.

4. Geometric Invariant of Space Curve and Bishop Height Functions

In this section, we will introduce two different families of functions on γ that will be useful to the study of geometric invariants of regular curve. Let $\gamma : I \rightarrow \mathbb{E}^3$ be a regular unit speed curve. Now, we define a family of smooth functions on I as follows:

$$H : I \times S^2 \rightarrow \mathbb{R} \quad \text{by } H(s, \mathbf{v}) = \langle \gamma(s), \mathbf{v} \rangle. \quad (15)$$

We call it Bishop height function. For any $\mathbf{v} \in S^2$, we denote that $h_v(s) = H(s, \mathbf{v})$. We also define a family of functions on I as follows:

$$\widetilde{H} : I \times S^2 \times \mathbb{R} \rightarrow \mathbb{R} \quad \text{by } \widetilde{H}(s, \mathbf{v}, u) = H(s, \mathbf{v}) - u$$

$$= \langle \gamma(s), \mathbf{v} \rangle - u. \quad (16)$$

We call it extended Bishop height function of γ . We denote that $\widetilde{h}_v(s) = \widetilde{H}(s, \mathbf{v})$. Then, we have the following proposition.

Proposition 2. Let $\gamma : I \rightarrow \mathbb{E}^3$ be a regular unit speed curve with $k_1(s) > 0$. Then, one has the following.

FIGURE 1

(A)

- (1) $h'_v(s) = 0$ if and only if there are real numbers λ and μ such that $\mathbf{v} = \lambda \mathbf{N}_1 + \mu \mathbf{N}_2$ and $\lambda^2 + \mu^2 = 1$.
- (2) $h'_v(s) = h''_v(s) = 0$ if and only if $\mathbf{v} = \pm \mathbf{d}(s)$.
- (3) $h'_v(s) = h''_v(s) = h^{(3)}_v(s) = 0$ if and only if $\mathbf{v} = \pm \mathbf{d}(s)$ and $\rho(s) = 0$.
- (4) $h'_v(s) = h''_v(s) = h^{(3)}_v(s) = h^{(4)}_v(s) = 0$ if and only if $\mathbf{v} = \pm \mathbf{d}(s)$ and $\rho(s) = \rho'(s) = 0$.
- (5) $h'_v(s) = h''_v(s) = h^{(3)}_v(s) = h^{(4)}_v(s) = h^{(5)}_v(s) = 0$ if and only if $\mathbf{v} = \pm \mathbf{d}(s)$ and $\rho(s) = \rho'(s) = \rho''(s) = 0$.

(B)

- (1) $\tilde{h}_v(s) = 0$ if and only if $u = \langle \gamma(s), \mathbf{v} \rangle$.
- (2) $\tilde{h}_v(s) = \tilde{h}'_v(s) = 0$ if and only if there are real numbers λ and μ such that $\mathbf{v} = \lambda \mathbf{N}_1 + \mu \mathbf{N}_2$, $\lambda^2 + \mu^2 = 1$ and $u = \langle \gamma(s), \mathbf{v} \rangle$.
- (3) $\tilde{h}_v(s) = \tilde{h}'_v(s) = \tilde{h}''_v(s) = 0$ if and only if $\mathbf{v} = \pm \mathbf{d}(s)$ and $u = \langle \gamma(s), \mathbf{v} \rangle$.
- (4) $\tilde{h}_v(s) = \tilde{h}'_v(s) = \tilde{h}''_v(s) = \tilde{h}^{(3)}_v(s) = 0$ if and only if $\mathbf{v} = \pm \mathbf{d}(s)$, $u = \langle \gamma(s), \mathbf{v} \rangle$, and $\rho(s) = 0$.
- (5) $\tilde{h}_v(s) = \tilde{h}'_v(s) = \tilde{h}''_v(s) = \tilde{h}^{(3)}_v(s) = \tilde{h}^{(4)}_v(s) = 0$ if and only if $\mathbf{v} = \pm \mathbf{d}(s)$, $u = \langle \gamma(s), \mathbf{v} \rangle$, and $\rho(s) = \rho'(s) = 0$.
- (6) $\tilde{h}_v(s) = \tilde{h}'_v(s) = \tilde{h}''_v(s) = \tilde{h}^{(3)}_v(s) = \tilde{h}^{(4)}_v(s) = \tilde{h}^{(5)}_v(s) = 0$ if and only if $\mathbf{v} = \pm \mathbf{d}(s)$, $u = \langle \gamma(s), \mathbf{v} \rangle$, and $\rho(s) = \rho'(s) = \rho''(s) = 0$.

Proof. As some equations are rather long, we omit the variable s .

- (1) Since $h'_v(s) = \langle \gamma'(s), \mathbf{v} \rangle = \langle \mathbf{T}(s), \mathbf{v} \rangle = 0$. We have that there are real numbers λ and μ such that $\mathbf{v} = \lambda \mathbf{N}_1 + \mu \mathbf{N}_2$. By the condition that $\mathbf{v} \in S^2$, we get that $\lambda^2 + \mu^2 = 1$. The converse direction also holds.
- (2) Since $h''_v(s) = \langle k_1(s) \mathbf{N}_1(s) + k_2(s) \mathbf{N}_2(s), \mathbf{v} \rangle = 0$, we have that $k_1(s)\lambda + k_2(s)\mu = 0$. It follows from the fact $\lambda^2 + \mu^2 = 1$ that $\mu = \mp k_1 / \sqrt{k_1^2 + k_2^2}$. Therefore, we have that

$$\mathbf{v} = \mp \frac{k_1}{\sqrt{k_1^2 + k_2^2}} \left(\frac{k_2}{k_1} \mathbf{N}_1 - \mathbf{N}_2 \right) = \pm \mathbf{d}(s). \quad (17)$$

- (3) Since $h^{(3)}_v(s) = \langle k'_1(s) \mathbf{N}_1(s) + k'_2(s) \mathbf{N}_2(s) - (k_1^2 + k_2^2) \mathbf{T}, \mathbf{v} \rangle = 0$, by the conditions of (3), we have that

$$\mp \frac{k_1}{\sqrt{k_1^2 + k_2^2}} \left(\frac{k'_1 k_2 - k_1 k'_2}{k_1} \right) = 0. \quad (18)$$

This is equivalent to the condition $k'_1 k_2 - k_1 k'_2 = \rho(s) = 0$.

- (4) Since $h^{(4)}_v(s) = \langle (k''_1 - k_1^3 - k_1 k_2^2) \mathbf{N}_1(s) + (k''_2 - k_2^3 - k_1^2 k_2) \mathbf{N}_2(s) - 3(k_1 k'_1 + k_2 k'_2) \mathbf{T}, \mathbf{v} \rangle = 0$, by the conditions of (4), we have that

$$\mp \frac{k_1}{\sqrt{k_1^2 + k_2^2}} \left(\frac{(k'_1 k_2 - k_1 k'_2)'}{k_1} \right) = 0. \quad (19)$$

This is equivalent to the condition $\rho(s) = \rho'(s) = 0$.

- (5) Since $h^{(5)}_v(s) = \langle (k^{(3)}_1 - 6k_1^2 k'_1 - k'_1 k_2^2 - 2k_1 k_2 k'_2) \mathbf{N}_1(s) + (k^{(3)}_2 - 6k_2^2 k'_2 - 2k_1 k'_1 k_2 - k'_1 k_2^2) \mathbf{N}_2(s) - (4k_1 k''_1 - k_1^4 -$

$2k_1^2k_2^2 + 4k_2k_2'' - k_2^4 + 3(k_1')^2 + 3(k_2')^2 \rangle \mathbf{T}, \mathbf{v} \rangle = 0$, by the conditions of (5) and a rather long computation, we get that

$$\mp \frac{k_1}{\sqrt{k_1^2 + k_2^2}} \cdot \left(\frac{(k_1^{(3)}k_2 - k_1k_2^{(3)}) - (k_1'k_2 - k_1k_2') (k_1^2 + k_2^2)}{k_1} \right) = 0. \quad (20)$$

This is equivalent to the condition $\rho(s) = \rho'(s) = \rho''(s) = 0$.

(B) The proof of B follows from the proof of A; so, we omit it. $\square$

Proposition 3. Let $\gamma : \mathbf{I} \rightarrow \mathbb{E}^3$ be a regular unit speed curve with $k_1(s) > 0$; then, $\rho(s) = 0$ if and only if each of

$$\mathbf{v} = \mp \frac{k_1(s)}{\sqrt{k_1^2(s) + k_2^2(s)}} \left(\frac{k_2(s)}{k_1(s)} \mathbf{N}_1(s) - \mathbf{N}_2(s) \right) \quad (21)$$

is a constant vector.

Proof. Suppose that $k_1(s) > 0$. By straightforward calculations, we have that

$$\mathbf{v}'(s) = \pm \frac{k_1(k_1'k_2 - k_1k_2')}{(k_1^2 + k_2^2)^{3/2}} \mathbf{N}_1 \pm \frac{k_2(k_1'k_2 - k_1k_2')}{(k_1^2 + k_2^2)^{3/2}} \mathbf{N}_2. \quad (22)$$

Thus, $\mathbf{v}'(s) \equiv 0$ if and only if $k_1'k_2 - k_1k_2' = 0$. $\square$

Corollary 4. Let $\gamma : \mathbf{I} \rightarrow \mathbb{E}^3$ be a regular unit speed curve with $k_1(s) \neq 0$; then, $\rho(s) = 0$ if and only if the Bishop spherical Darboux image is a constant map.

By the previous propositions, we can recognize that the function $\rho(s) = k_1'(s)k_2(s) - k_1(s)k_2'(s)$ has special meanings.

Theorem 5. Let $\gamma : \mathbf{I} \rightarrow \mathbb{E}^3$ be a unit speed curve with nonzero natural curvatures; then, γ is a B-slant helix if and only if $k_2/k_1 = \text{constant}$ (cf., [5]).

We also can slightly extend the main results in [5] to the following proposition.

Proposition 6. Let $\gamma : \mathbf{I} \rightarrow \mathbb{E}^3$ be a regular unit speed curve with $k_1(s) \neq 0$ and $\rho(s) = 0$. Then, one has the following claims.

- (1) $\gamma(s)$ is a B-slant helix.
- (2) The bishop spherical Darboux image is a constant map.
- (3) The tangent spherical indicatrix is a circle in osculating plane.
- (4) The Bishop rectifying developable of γ is a cylindrical surface given by $\gamma(s) + u\mathbf{e}$, where $\mathbf{e} = \mathbf{d}(s)$.

(5) The first Bishop spherical indicatrix is a circle on the unit spheres S^2 , and the direction of the circle is given by the constant vector $\mathbf{d}(s) \equiv \mathbf{e}$.

(6) The second Bishop spherical indicatrix is a circle on the unit spheres S^2 , and the direction of the circle is also given by the constant vector $\mathbf{d}(s) \equiv \mathbf{e}$.

Proof. (1) Suppose that $\rho(s) = k_1'(s)k_2(s) - k_1(s)k_2'(s) = 0$. By straightforward calculations, we have

$$\left(\frac{k_2(s)}{k_1(s)} \right)' = \frac{k_2'(s)k_1(s) - k_2(s)k_1'(s)}{k_1^2(s)} = \frac{-\rho(s)}{k_1^2(s)} = 0. \quad (23)$$

This means that $k_2(s)/k_1(s) = \text{constant}$. By the definition of B-slant helix, we know that $\gamma(s)$ is a B-slant helix.

(2) By Corollary 4, we can claim (2) easily.

(3) By Corollary 3.2 in [5], we also can claim (3).

(4) The claim in (4) is clear by definition.

(5) Suppose that $k_1(s) > 0$. Since

$$\mathbf{e} = -\frac{k_1}{\sqrt{k_1^2 + k_2^2}} \left(\frac{k_2}{k_1} \mathbf{N}_1 - \mathbf{N}_2 \right), \quad (24)$$

we get that

$$\begin{aligned} \langle \mathbf{e}, \mathbf{N}_1 \rangle &= \left\langle -\frac{k_1}{\sqrt{k_1^2 + k_2^2}} \left(\frac{k_2}{k_1} \mathbf{N}_1 - \mathbf{N}_2 \right), \mathbf{N}_1 \right\rangle \\ &= -\frac{k_2/k_1}{\sqrt{1 + (k_2/k_1)^2}} = \text{constant}. \end{aligned} \quad (25)$$

So, $\cos(\alpha(s)) = \langle \mathbf{e}(s), \mathbf{N}_1(s) \rangle$ is constant. This means that the first Bishop spherical indicatrix is a circle on the unit spheres S^2 , and the direction of the circle is given by the constant vector $\mathbf{d}(s) \equiv \mathbf{e}$.

(6) Suppose that $k_1(s) > 0$. By the analogous computation as (5), we get that

$$\cos(\beta(s)) = \langle \mathbf{e}(s), \mathbf{N}_2(s) \rangle = \frac{1}{\sqrt{1 + (k_2/k_1)^2}} \quad (26)$$

is constant. This means that the second Bishop spherical indicatrix is a circle on the unit spheres S^2 , and the direction of the circle is also given by the constant vector $\mathbf{d}(s) \equiv \mathbf{e}$. $\square$

5. Versal Unfoldings and Proof of the Main Result

In this section, we use some general results on the singularity theory for families of function germs. Detailed descriptions can be found in the book [15]. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \rightarrow \mathbb{R}$ be a function germ. We call F an r -parameter unfolding of $f(s)$, where $f(s) = F(s, x_0)$. We say that $f(s)$ has A_k -singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$,

FIGURE 2: Curve.

FIGURE 3: Bishop spherical Darboux image which has a cusp point.

and $f^{(k+1)}(s_0) \neq 0$. We also say that $f(s)$ has $A_{\geq k}$ -singularity at s_0 if $f^{(p)}(s_0) = 0$ for all $1 \leq p \leq k$. Let F be an unfolding of $f(s)$, and let $f(s)$ has A_k -singularity ($k \geq 1$) at s_0 . We denote the $(k-1)$ -jet of the partial derivative $\partial F / \partial x_i$ at s_0 by $j^{(k-1)}((\partial F / \partial x_i)(x, s_0))(s_0) = \sum_{j=1}^{k-1} a_{ji} s^j$ for $i = 1, \dots, r$. Then, F is called a versal unfolding if the $k \times r$ matrix of coefficients (a_{0i}, a_{ji}) has rank k , ($k \leq r$), where $a_{0i} = (\partial F / \partial x_i)(x_0, s_0)$. We now introduce an important set concerning the unfolding. The discriminant set of F is the set

$$\mathfrak{D}_F = \left\{ x \in \mathbb{R}^r \mid F(s, x) = \frac{\partial F}{\partial s}(s, x) = 0 \right\}. \quad (27)$$

The bifurcation set of F is the set

$$\mathfrak{B}_F = \left\{ x \in \mathbb{R}^r \mid \frac{\partial F}{\partial s}(s, x) = \frac{\partial^2 F}{\partial s^2}(s, x) = 0 \right\}. \quad (28)$$

Then, we have the following well-known result (cf., [15]).

Theorem 7. Let $F : (\mathbb{R} \times \mathbb{R}^r, (s_0, x_0)) \rightarrow \mathbb{R}$ be an r -parameter unfolding of $f(s)$ which has the A_k -singularity at s_0 .

Suppose that F is a (p) versal unfolding. Then, one has the following.

- (a) If $k = 1$, then $\mathfrak{D}_F$ is locally diffeomorphic to $0 \times \mathbb{R}^{r-1}$.
- (b) If $k = 2$, then $\mathfrak{D}_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.
- (c) If $k = 3$, then $\mathfrak{D}_F$ is locally diffeomorphic to $SW \times \mathbb{R}^{r-3}$.

FIGURE 4: Bishop Gauss surface and Bishop spherical Darboux image (the red curve).

FIGURE 5: A local part of the diffeomorphic image of Bishop dual and locus of its singularity points which is the red curve and some other cuspidal edge points.

Suppose that F is a (p) versal unfolding. Then, one has the following.

- (a) If $k = 2$, then $\mathfrak{B}_F$ is locally diffeomorphic to $0 \times \mathbb{R}^{r-1}$.
- (b) If $k = 3$, then $\mathfrak{B}_F$ is locally diffeomorphic to $C \times \mathbb{R}^{r-2}$.

We consider a unit speed regular curve $\gamma(s) : I \rightarrow \mathbb{E}^3$ with $k_1(s) > 0$ and the height function H of $\gamma(s)$. By Proposition 2, the discriminant set of $\tilde{H}(s, \mathbf{v})$ is given as follows:

$$\mathfrak{D}_{\tilde{H}} = \left\{ (\mathbf{v}, u) \in S^2 \times \mathbb{R} : u = \langle \gamma(s), \mathbf{v} \rangle, \mathbf{v} = \lambda \mathbf{N}_1(s) + \mu \mathbf{N}_2(s) \mid s \in I, \lambda^2 + \mu^2 = 1, \lambda, \mu \in \mathbb{R} \right\}. \quad (29)$$

The bifurcation set of H is

$$\mathfrak{B}_H = \left\{ \mathbf{v} = \mp \frac{k_1(s)}{\sqrt{k_1^2(s) + k_2^2(s)}} \left(\frac{k_2(s)}{k_1(s)} \mathbf{N}_1(s) - \mathbf{N}_2(s) \right) \right\}. \quad (30)$$

Then, we have the following proposition.

Proposition 8. (1) If $h_{v_0}(s)$ has A_k -singularity ($k = 2, 3$) at s_0 , then $H(s, \mathbf{v})$ is a (p) versal unfolding of $h_{v_0}(s)$.

FIGURE 6: Curve.

(2) If $\tilde{h}_{v_0}(s)$ has A_k -singularity ($k = 2, 3$) at s_0 , then $\tilde{H}(s, \mathbf{v})$ is a (p) versal unfolding of $\tilde{h}_{v_0}(s)$.

Proof. (1) We denote that

$$\gamma(s) = (x_1(s), x_2(s), x_3(s)), \quad \mathbf{v} = (v_1, v_2, \pm \sqrt{1 - v_1^2 - v_2^2}). \quad (31)$$

Under this notation, we have that

$$H(s, \mathbf{v}) = x_1(s)v_1 + x_2(s)v_2 \pm x_3(s)\sqrt{1 - v_1^2 - v_2^2}. \quad (32)$$

Thus, we have that

$$\frac{\partial H}{\partial v_i} = x_i(s) \mp \frac{v_i x_3(s)}{\sqrt{1 - v_1^2 - v_2^2}}, \quad i = 1, 2. \quad (33)$$

We also have that

$$\begin{aligned} \frac{\partial}{\partial s} \frac{\partial H}{\partial v_i} &= x'_i(s) \mp \frac{v_i x'_3(s)}{\sqrt{1 - v_1^2 - v_2^2}}, \quad i = 1, 2, \\ \frac{\partial^2}{\partial s^2} \frac{\partial H}{\partial v_i} &= x''_i(s) \mp \frac{v_i x''_3(s)}{\sqrt{1 - v_1^2 - v_2^2}}, \quad i = 1, 2. \end{aligned} \quad (34)$$

Therefore, the 2-jet of $(\partial H / \partial v_i)(s, v)$ ($i = 1, 2$) at s_0 is given by

$$\begin{aligned} j^2 \left(\frac{\partial H}{\partial v_i}(s, v_0) \right) (s_0) &= \frac{\partial}{\partial s} \frac{\partial H}{\partial v_i}(s - s_0) + \frac{1}{2} \frac{\partial^2}{\partial s^2} \frac{\partial H}{\partial v_i}(s - s_0)^2 \\ &= a_{1i}(s - s_0) + \frac{1}{2} a_{2i}(s - s_0)^2. \end{aligned} \quad (35)$$

It is enough to show that the rank of the matrix A is 2, where

$$A = \begin{bmatrix} x'_1(s) \mp \frac{v_1 x'_3(s)}{\sqrt{1 - v_1^2 - v_2^2}} & x'_2(s) \mp \frac{v_2 x'_3(s)}{\sqrt{1 - v_1^2 - v_2^2}} \\ x''_1(s) \mp \frac{v_1 x''_3(s)}{\sqrt{1 - v_1^2 - v_2^2}} & x''_2(s) \mp \frac{v_2 x''_3(s)}{\sqrt{1 - v_1^2 - v_2^2}} \end{bmatrix}. \quad (36)$$

Denote that $v_3 = \pm \sqrt{1 - v_1^2 - v_2^2}$. Then, we have that

$$\begin{aligned} \det A &= (x''_2 x'_1 - x'_2 x''_1) - \frac{v_1}{v_3} (x''_2 x'_3 - x'_2 x''_3) \\ &\quad - \frac{v_2}{v_3} (x''_3 x'_1 - x'_3 x''_1) \\ &= \frac{1}{v_3} \langle \mathbf{v}, \mathbf{T}(s) \wedge \mathbf{T}'(s) \rangle \\ &= \frac{1}{v_3} \langle \mathbf{v}, \mathbf{T}(s) \wedge k_1(s) \mathbf{N}_1(s) + k_2(s) \mathbf{N}_2(s) \rangle \\ &= \frac{1}{v_3} \langle \mathbf{v}, k_1(s) \mathbf{N}_2(s) - k_2(s) \mathbf{N}_1(s) \rangle \\ &= \frac{1}{v_3} \left\langle \mp \frac{k_1}{\sqrt{k_1^2 + k_2^2}} \left(\frac{k_2}{k_1} \mathbf{N}_1(s) - \mathbf{N}_2(s) \right), \right. \\ &\quad \left. k_1(s) \mathbf{N}_2(s) - k_2(s) \mathbf{N}_1(s) \right\rangle \end{aligned}$$

$$\begin{aligned}
&= \mp \frac{1}{v_3} \frac{k_1}{\sqrt{k_1^2 + k_2^2}} \left(-\frac{k_2^2}{k_1} - k_1 \right) \\
&= \pm \frac{1}{v_3} \sqrt{k_1^2 + k_2^2} \neq 0.
\end{aligned} \quad (37)$$

Note that $\mathbf{v} \in \mathfrak{B}_H$ is a singular point, where

$$\mathbf{v} = \mp \frac{k_1}{\sqrt{k_1^2 + k_2^2}} \left(\frac{k_2}{k_1} \mathbf{N}_1(s) - \mathbf{N}_2(s) \right). \quad (38)$$

(2) Under the same notations as (1), we have that

$$\tilde{H}(s, \mathbf{v}) = x_1(s) v_1 + x_2(s) v_2 \pm x_3(s) \sqrt{1 - v_1^2 - v_2^2} - u. \quad (39)$$

It is enough to show that the rank of the matrix B is 2, where

$$B = \begin{bmatrix} x_1'(s) \mp \frac{u_1 x_3'(s)}{\sqrt{1 - v_1^2 - v_2^2}} & x_2'(s) \mp \frac{u_2 x_3'(s)}{\sqrt{1 - v_1^2 - v_2^2}} & -1 \\ x_1''(s) \mp \frac{u_1 x_3''(s)}{\sqrt{1 - v_1^2 - v_2^2}} & x_2''(s) \mp \frac{u_2 x_3''(s)}{\sqrt{1 - v_1^2 - v_2^2}} & 0 \end{bmatrix}. \quad (40)$$

This follows from the proof of (1). This completes the proof. $\square$

Proof of Theorem 1. (1) The bifurcation set $\mathfrak{B}_H$ of H is

$$\mathfrak{B}_H = \left\{ \mathbf{v} = \mp \frac{k_1(s)}{\sqrt{k_1^2(s) + k_2^2(s)}} \left(\frac{k_2(s)}{k_1(s)} \mathbf{N}_1(s) - \mathbf{N}_2(s) \right) \right\}. \quad (41)$$

The assertion (1) of Theorem 1 follows from Proposition 2, Proposition 8, and Theorem 7.

(2) The discriminant set $\mathfrak{D}_{\tilde{H}}$ of $\tilde{H}$ is

$$\mathfrak{D}_{\tilde{H}} = \left\{ (v, u) \in S^2 \times \mathbb{R} : u = \langle \gamma(s), \mathbf{v} \rangle, \mathbf{v} = \lambda \mathbf{N}_1(s) + \mu \mathbf{N}_2(s) \mid s \in I, \lambda^2 + \mu^2 = 1, \lambda, \mu \in \mathbb{R} \right\}. \quad (42)$$

The assertion (2) of Theorem 1 follows from Proposition 2, Proposition 8, and Theorem 7. $\square$

6. Examples

As applications and illustration of the main results (Theorem 1), we give two examples in this section.

Example 9. Let $\gamma(s)$ be a unit speed curve of $\mathbb{E}^3$ defined by

$$\gamma(s) = \left(\frac{2}{5} \sin 2s - \frac{1}{40} \sin 8s, -\frac{2}{5} \cos 2s + \frac{1}{40} \cos 8s, \frac{4}{15} \sin 3s \right) \quad (43)$$

with respect to an arclength parameter s (Figure 2).

The curvature and torsion of this curve are, respectively, as follows:

$$\begin{aligned}
k(s) &= -4 \sin 3s, \\
\tau(s) &= 4 \cos 3s.
\end{aligned} \quad (44)$$

Using the Bishop curvature equation (5), we obtain

$$\begin{aligned}
k_1(s) &= -4 \sin(3s) \cos\left(\frac{4}{3} \sin 3s\right), \\
k_2(s) &= -4 \sin(3s) \sin\left(\frac{4}{3} \sin 3s\right).
\end{aligned} \quad (45)$$

We can calculate the geometric invariant

$$\rho(s) = -64 \sin^2(3s) \cos(3s). \quad (46)$$

Using (3), we obtain the Bishop frame $\{\mathbf{T}(s), \mathbf{N}_1(s), \mathbf{N}_2(s)\}$ as follows:

$$\begin{aligned}
\mathbf{T}(s) &= \left(-\frac{4}{5} \cos(2s) - \frac{1}{5} \cos(8s), \right. \\
&\quad \left. \frac{4}{5} \sin(2s) - \frac{1}{5} \sin(8s), \frac{4}{5} \cos(3s) \right),
\end{aligned} \quad (47)$$

$\mathbf{N}_1(s)$

$$\begin{aligned}
&= \frac{1}{\sqrt{\sin^2 s (4 \cos^2 s - 1)^2}} \\
&\quad \cdot \left((\sin 8s - \sin 2s) \left(\frac{2}{5} \cos\left(\frac{4}{3} \sin 3s\right) \right. \right. \\
&\quad \left. \left. - \frac{3}{25} \sin\left(\frac{4}{3} \sin 3s\right) \sin 3s \right) \right. \\
&\quad \left. + \frac{24}{125} \sin\left(\frac{4}{3} \sin 3s\right) \cos 3s (\cos 8s - \cos 2s), \right. \\
&\quad \left. \frac{2}{5} \cos\left(\frac{4}{3} \sin 3s\right) (\cos 2s - \cos 8s) - \sin\left(\frac{4}{3} \sin 3s\right) \right. \\
&\quad \cdot \left(\frac{8}{25} \cos 3s (\sin 8s - \sin 2s) \right. \\
&\quad \left. + \left(\frac{12}{25} \cos 2s - \frac{3}{25} \cos 8s \right) \sin 3s \right), \\
&\quad \left. - \frac{3}{5} \cos\left(\frac{4}{3} \sin 3s\right) \sin 3s - \sin\left(\frac{4}{3} \sin 3s\right) \right. \\
&\quad \cdot \left(\left(\frac{8}{25} \cos 2s - \frac{2}{25} \cos 8s \right) (\cos 2s - \cos 8s) \right. \\
&\quad \left. + \left(\frac{8}{25} \sin 2s - \frac{2}{25} \sin 8s \right) (\sin 2s - \sin 8s) \right) \Big),
\end{aligned} \quad (48)$$

FIGURE 7: Bishop Gauss surface and a local part of Bishop spherical Darboux image which is the red curve.

$$\begin{aligned}
 \mathbf{N}_2(s) &= \frac{1}{\sqrt{\sin^2 s (4\cos^2 s - 1)^2}} \\
 &\cdot \left(\frac{2}{5} (\sin 8s - \sin 2s) \sin \left(\frac{4}{3} \sin 3s \right) \right. \\
 &\quad + \cos \left(\frac{4}{3} \sin 3s \right) \left(\frac{3}{25} \sin 8s - \frac{12}{25} \sin 2s \right) \sin 3s \\
 &\quad - \frac{8}{25} \cos 3s (\cos 2s - \cos 8s), \frac{2}{5} (\cos 2s - \cos 8s) \\
 &\quad \cdot \sin \left(\frac{4}{3} \sin 3s \right) + \cos \left(\frac{4}{3} \sin 3s \right) \\
 &\quad \cdot \left(\frac{8}{25} \cos 3s (\sin 8s - \sin 2s) \right. \\
 &\quad \quad \left. + \left(\frac{12}{25} \cos 2s - \frac{3}{25} \cos 8s \right) \sin 3s \right), \\
 &\quad - \frac{3}{5} \sin \left(\frac{4}{3} \sin 3s \right) \sin 3s + \cos \left(\frac{4}{3} \sin 3s \right) \\
 &\quad \cdot \left(\left(\frac{8}{25} \cos 2s - \frac{2}{25} \cos 8s \right) (\cos 2s - \cos 8s) \right. \\
 &\quad \quad \left. - \left(\frac{8}{25} \sin 2s - \frac{2}{25} \sin 8s \right) (\sin 8s - \sin 2s) \right) \Bigg). \quad (49)
 \end{aligned}$$

Using (10) and (44), we obtain the Bishop spherical Darboux image (Figure 3)

$$\begin{aligned}
 \mathbf{d}(s) &= \frac{1}{\sqrt{\sin^2 s (4\cos^2 s - 1)^2}} \\
 &\cdot \left(\sin 3s \left(\frac{3}{25} \sin 8s - \frac{12}{25} \sin 2s \right) + \frac{8}{25} \cos 3s \right.
 \end{aligned}$$

$$\begin{aligned}
 &\cdot (\cos 8s - \cos 2s), \frac{8}{25} \cos 3s (\sin 8s - \sin 2s) + \sin 3s \\
 &\cdot \left(\frac{12}{25} \cos 2s - \frac{3}{25} \cos 8s \right), \frac{2}{5} (1 + \cos 6s) \Bigg). \quad (50)
 \end{aligned}$$

We define Bishop Gauss surface by

$$\text{BGS}(s, \omega) = \cos \omega \mathbf{N}_1(s) + \sin \omega \mathbf{N}_2(s). \quad (51)$$

This surface is deeply related with Bishop spherical Darboux image and Bishop dual. We make both Bishop spherical Darboux image and Bishop Gauss surface visualized by Maple in Figure 4. We can see that Bishop spherical Darboux image lies in Bishop Gauss surface, and it has a cuspidal point.

For the Bishop dual, we should define a diffeomorphic mapping and make its diffeomorphic image visualized in Figure 5. We can see the locus of singular points which is the red curve and some other cuspidal edge points in this surface. The diffeomorphic mapping is defined by

$$\Phi : S^2 \times \mathbb{R} \rightarrow \mathbb{R}^3, \quad \Phi(\mathbf{v}, u) = u \left(v_1, v_2, \pm \sqrt{1 - v_1^2 - v_2^2} \right), \quad (52)$$

where $\mathbf{v} = (v_1, v_2, v_3) \in S^2$ and $u \in \mathbb{R}$. Thus, the diffeomorphic image of Bishop dual is

$$\Phi(\text{BDU}(s, \omega)_\gamma) = \langle \gamma(s), \text{BGS}(s, \omega) \rangle \text{BGS}(s, \omega). \quad (53)$$

We can see the diffeomorphic image of Bishop dual in Figure 5.

Example 10. Let $\gamma(s)$ be a unit speed curve of $\mathbb{E}^3$ defined by

$$\gamma(s) = \left(3 \cos \frac{1}{5}s, 3 \sin \frac{1}{5}s, \frac{4}{5}s \right) \quad (54)$$

FIGURE 8: The diffeomorphic image of Bishop dual and locus of its singularity points which is the red curve.

with respect to an arclength parameter s (Figure 6). The curvature and torsion of this curve are, respectively, as follows:

$$\begin{aligned} k(s) &= \frac{3}{25}, \\ \tau(s) &= \frac{4}{25}. \end{aligned} \quad (55)$$

Using the Bishop curvature equation (5), we obtain the following:

$$\begin{aligned} k_1(s) &= \frac{3}{25} \cos\left(\frac{4}{25}s\right), \\ k_2(s) &= \frac{3}{25} \sin\left(\frac{4}{25}s\right). \end{aligned} \quad (56)$$

We can calculate the geometric invariant $\rho(s) = -36/253$. Using the same method as Example 9, we can get Bishop spherical Darboux image, Bishop Gauss surface, and Bishop dual of this curve. The pictures of Bishop spherical Darboux image, Bishop Gauss surface, and diffeomorphic image of Bishop dual are visualized in Figures 7 and 8. We can see that Bishop spherical Darboux image lies in Bishop Gauss surface, and it is locally diffeomorphic to a line. The diffeomorphic image of Bishop dual has some cuspidal edge points.

7. Conclusions

In this paper, we introduce the notions of Bishop spherical Darboux image, Bishop dual, Bishop height functions, and extended height functions on a space curve embedded in Euclidean 3-space. We use the Bishop-Serret-Frenet formulas and Bishop height functions to study these objects from the singularity viewpoint. We establish the relationships between the singularities of the discriminant of extended height functions and the sets of bifurcations of the height function, which are Bishop dual and Bishop spherical Darboux image

separately, and geometric invariants of a space curve in Euclidean 3-space. We also get some meaningful properties of Bishop helix. Note that the Bishop dual and the Bishop spherical Darboux image according to Bishop frame are very different from those in [15, 17]; however, we found that they have some analogous singularity properties under some different conditions. The main result of this paper is Theorem 1 using the relatively parallel adapted frame for classification of singularity type of some objects deeply related with a space curve and making them visualized by computer. As applications of our main results, we give two examples.

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