Research Article

Stability Analysis of a Class of Second Order Sliding Mode Control Including Delay in Input

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This paper deals with a class of second order sliding mode systems. Based on the derivative of the sliding surface, sufficient conditions are given for stability. However, the discontinuous control signal depend neither on the derivative of sliding surface nor on its estimate. Time delay in control input is also an important issue in sliding mode control for engineering applications. Therefore, also sufficient conditions are given for the time delay size on the discontinuous input signal, so that this class of second order sliding mode systems might have amplitude bounded oscillations. Moreover, amplitude of such oscillations may be estimated. Some numerical examples are given to validate the results. At the end, some conclusions are given on the possibilities of the results as well as their limitations.

1. Introduction

Sliding mode control has been effectively used in engineering for more than three decades. One of the major concerns for some applications is chattering in output or control signal. Along with other techniques, higher order sliding modes have been proposed to alleviate chattering. Most used algorithms for high sliding motion are the twisting controller [1], sub-optimal controller [2], and the super twisting controller [3]. These algorithms need to measure or estimate the derivative of the sliding surface. The super twisting algorithm is used to get robust differentiators, so that it can be used together with the other algorithms to get such a derivative. Recently [4], homogeneity has been used to ease parameter design for second order sliding mode controllers. Some recent examples of engineering applications using second order sliding mode controllers are [5–8].

When delays arise, sliding mode controller performance deteriorates. Such delays may occur in states and/or inputs and have been treated using several approaches (see e.g., [9–16]). In [12], a good review is given for different control techniques up to 2003. When only a delayed measure is available for the control signal, two techniques have been used to compensate delay effects: predictors and integral sliding mode control. Some instances are [11, 12, 14], respectively.

On the other hand, concerning analysis for existence and amplitude of steady oscillations due to delays, very few works have been reported, such as [17], where sufficient conditions are given for steady oscillations to exist in a first order sliding mode system; describing function and Poincaré map approach are used in [18] to analyze oscillations for a system with hysteresis. In [19], a different approach for perturbed first sliding mode systems is reported. For second order sliding mode systems with delay, [20] reported sufficient conditions for steady oscillations to occur in a class of systems; in [21], an analysis is given about oscillations for the suboptimal algorithm considering delay in the input.

Motivated by the aforementioned analysis, this paper presents sufficient stability conditions for a class of sliding mode system that does not require a measure of the derivative of the sliding surface in order to provide the control input. Also, sufficient conditions are given for stable oscillations to exist if a delay occurs in the input, as well as an estimate for their amplitude. The class of systems treated here is a more general class of systems than the one reported in [20] and using a different approach. In Section 2, the problem is formulated. Section 3 presents the main results by way of two propositions and their proofs, giving sufficient conditions for finite time sliding motion stability without delay and stability

for oscillations when delay in input occurs. Next, in Section 4, some numerical examples are given and conclusions for the reported results are related.

2. Problem Statement

Let us consider a second order sliding mode system in the form

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= f(t, x_1, x_2) - k \operatorname{sign}(x_1(t)),\end{aligned}\quad (1)$$

where $f(t, x_1, x_2)$ is smooth and $k > 0$. The first issue to consider is getting sufficient conditions such that the system has a stable equilibrium point in $x_1 = 0, x_2 = 0$ and reached in finite time, considering a domain of attraction D

$$\mathbf{x} = [x_1 \ x_2]^T \in D. \quad (2)$$

Now, if conditions for stability are met for (1), when a delay is present in the control input as follows:

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= f(t, x_1, x_2) - k \operatorname{sign}(x_1(t-h)),\end{aligned}\quad (3)$$

with $\mathbf{x} \in D, \forall t \in [-h, 0)$, x_1 might have steady oscillations if the time delay is sufficiently small, such that the system remains in the attraction domain. Otherwise, its dynamics will be unstable. Hence, it is important to get sufficient conditions for delay size, such that the system remains in the domain of attraction and an estimate for the amplitude of the oscillations. Next section shows sufficient conditions to reach the surface in finite time without delay and for existence of bounded amplitude oscillations in presence of delay.

3. Sufficient Conditions for Second Order Sliding Modes without Delay and Amplitude Bounded Oscillations with Delay

In the following proposition, sufficient conditions are given for finite time reaching and, stability of the equilibrium point $x_1 = 0, x_2 = 0$ in system (1) considering a domain of attraction.

Proposition 1. Consider system (1) with $f(t, 0, 0) = 0$. If $k - |f(t, x_1, x_2)| \geq c_0 > 0$ and, moreover, after x_1 first zero crossing, the following is met thereafter:

$$\left| \frac{c_n}{c_{n+1}} \right| > \frac{k - |f_{n \max}(t, x_1, x_2)|}{k - |f_{n+1 \max}(t, x_1, x_2)|}, \quad (4)$$

where t_n is the n th zero crossing time, c_n and c_{n+1} are the values for $x_2(t_n)$ and $x_2(t_{n+1})$ in the n th and $n + 1$ th zero crossing time for x_1 . Also, $|f_{n \max}(t, x_1, x_2)|$ is the biggest value of $|f(t, x_1, x_2)|$ in the interval $t_n \leq t < t_{n+1}$ and $|f_{n+1 \max}(t, x_1, x_2)|$ is the biggest value of $|f(t, x_1, x_2)|$ in the interval $t_{n+1} \leq t < t_{n+2}$. Then, system (1) has a stable equilibrium point in $x_1 = 0, x_2 = 0$ which will be reached in finite time.

Proof. As a first step, let us demonstrate that there exists a finite time for $x_1(t)$ zero crossing. Without loss of generality, let us consider $x_1(0) > 0$, therefore, in a neighborhood t^+ of $t = 0$ we will have

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= f(t, x_1, x_2) - k.\end{aligned}\quad (5)$$

From hypothesis, $k - |f(0, x_1(0), x_2(0))| \geq c_0 > 0$ and hence

$$\dot{x}_2 \leq -c_0. \quad (6)$$

Therefore,

$$x_2(t) \leq -c_0 t + x_2(0). \quad (7)$$

And so

$$x_1(t) \leq -\frac{c_0 t^2}{2} + x_2(0)t + x_1(0). \quad (8)$$

Hence, in a finite time, $t_1, x_1(t_1) = 0$, where

$$t_1 \leq \frac{x_2(0) + \sqrt{x_2^2(0) + 2c_0 x_1(0)}}{c_0}. \quad (9)$$

Now, as a second step, let us demonstrate by induction that in a finite time the equilibrium point is reached. Without loss of generality, assume that the n th $x_1(t)$ zero crossing goes from positive to negative, and therefore $x_2(t_n) = c_n$ with $c_n < 0$. Hence, for $t_n \leq t \leq t_{n+1}$, where $t = t_{n+1}$ is the next time for zero crossing, we will have

$$\begin{aligned}\dot{x}_1 &= x_2, \\ \dot{x}_2 &= f(t, x_1, x_2) + k.\end{aligned}\quad (10)$$

And therefore

$$x_2(t) \geq (k - |f_{n \max}(t, x_1, x_2)|)(t - t_n) + c_n. \quad (11)$$

So that

$$x_1(t) \geq \frac{(k - |f_{n \max}(t, x_1, x_2)|)(t - t_n)^2}{2} + c_n(t - t_n). \quad (12)$$

From the above, for the following zero crossing, $x_1(t_{n+1}) = 0$,

$$(t_{n+1} - t_n) \leq \frac{-2c_n}{(k - |f_{n \max}(t, x_1, x_2)|)}. \quad (13)$$

To assure finite time for equilibrium point reaching, it only has to be guaranteed that $(t_{n+1} - t_n) > (t_{n+2} - t_{n+1})$ since if $(t_{n+1} - t_n) = T$ and hence $(t_{n+2} - t_{n+1}) \leq a_n T$, where $0 < a_n \leq a < 1$, total time will be

$$\begin{aligned}\sum_{n=1}^{\infty} (t_{n+1} - t_n) &\leq T + aT + a^2T + \dots = T \sum_{n=1}^{\infty} a^{n-1} \\ &= T \frac{1}{1-a}.\end{aligned}\quad (14)$$

From (13), and a similar development for negative to positive zero crossing, the following can be deduced:

$$(t_{n+2} - t_{n+1}) + (t_{n+1} - t_n) \leq \frac{|2c_{n+1}|}{(k - |f_{n+1 \max}(t, x_1, x_2)|)} + \frac{|2c_n|}{(k - |f_{n \max}(t, x_1, x_2)|)}. \quad (15)$$

Now, considering that to comply with the supposed sufficient condition for reaching the equilibrium point in finite time $(t_{n+1} - t_n) = T$ and $(t_{n+2} - t_{n+1}) \leq a_n T$, the following must be complied as well:

$$(1 + a_n)(t_{n+1} - t_n) \leq \frac{|2c_{n+1}|}{(k - |f_{n+1 \max}(t, x_1, x_2)|)} + \frac{|2c_n|}{(k - |f_{n \max}(t, x_1, x_2)|)}. \quad (16)$$

So that, making

$$\frac{1}{(t_{n+1} - t_n)} \times \left[\frac{|2c_{n+1}|}{(k - |f_{n+1 \max}(t, x_1, x_2)|)} + \frac{|2c_n|}{(k - |f_{n \max}(t, x_1, x_2)|)} \right] < 2, \quad (17)$$

it is guaranteed that $1 + a_n < 2$.

Now, using again (13) together with (17)

$$\frac{|c_{n+1}|}{(k - |f_{n+1 \max}(t, x_1, x_2)|)} + \frac{|c_n|}{(k - |f_{n \max}(t, x_1, x_2)|)} < \frac{|2c_n|}{(k - |f_{n \max}(t, x_1, x_2)|)}. \quad (18)$$

Therefore, it is sufficient

$$\frac{|c_{n+1}|}{(k - |f_{n+1 \max}(t, x_1, x_2)|)} < \frac{|c_n|}{(k - |f_{n \max}(t, x_1, x_2)|)}. \quad (19)$$

In other words

$$\frac{|c_n|}{|c_{n+1}|} > \frac{(k - |f_{n \max}(t, x_1, x_2)|)}{(k - |f_{n+1 \max}(t, x_1, x_2)|)}. \quad (20)$$

□

Remark 2. If $|f_{n \max}(t, x_1, x_2)|$ and $|f_{n+1 \max}(t, x_1, x_2)|$ are not known, instead of (20) we may use as a more conservative condition $k - |f(t, x_1, x_2)| \geq c_0$, together with

$$\frac{|c_n|}{|c_{n+1}|} > 1. \quad (21)$$

When a delay exists in the discontinuous control signal, the equilibrium point cannot be reached. If the dynamics are

in the domain of attraction, amplitude bounded oscillations are obtained. Next proposition presents sufficient conditions for an upper bound on the delay in order to get amplitude bounded oscillations.

Proposition 3. Considering system (1), complying with conditions of Proposition 1, a sufficient condition such that system (3) has steady oscillations is

$$h \leq \min \left\{ \frac{M_{x_1}^{'+}}{M_{x_2}^{'+}}, \frac{M_{x_1}^{'-}}{M_{x_2}^{'-}} \right\}, \quad (22)$$

where

$$\begin{aligned} M_{x_2}^{'+} &= \sup_{t>0, \mathbf{x} \in D^+, 0<\delta<k} \{x_2 \mid f(t, x_1, x_2) + k = 2\delta\} \\ M_{x_2}^{'-} &= \inf_{t>0, \mathbf{x} \in D^-, 0<\delta<k} \{x_2 \mid f(t, x_1, x_2) - k = -2\delta\} \\ 0 < M_{x_1}^{'+} &\leq M_{x_1}^{+} = \sup \{x_1 > 0 \mid \langle x_1 \ x_2 = M_{x_2}^{'+} \rangle \in D\} \\ 0 > M_{x_1}^{'-} &\geq M_{x_1}^{-} = \inf \{x_1 < 0 \mid \langle x_1 \ x_2 = M_{x_2}^{'-} \rangle \in D\}, \end{aligned} \quad (23)$$

with the domain of attraction being

$$\mathbf{x} = [x_1 \ x_2]^T \in D,$$

$$D^+ = \{D \mid x_1 > 0, x_2 \geq 0\} \quad (24)$$

$$D^- = \{D \mid x_1 < 0, x_2 \leq 0\}.$$

Proof. First, let us get an estimate for a bound on x_2 in the domain of attraction. For $0 \leq t_0 < t_1 < t_2$, without loss of generality, let us consider $0 < x_1(t_1) \leq M_{x_1}^{+}$, $x_1(t_2) > x_1(t_1) > 0 > -\beta = x_1(t_0)$, $\beta \rightarrow 0$, and $t_1 - t_0 = h_{\max} < t_2 - t_0$. Where $h_{\max}$ is the maximum delay such that the system remains in the domain of attraction, so that $h \leq h_{\max}$. Then, in the interval $t_1 < t \leq t_2$, we have $\text{sign}(x_1(t)) = \text{sign}(x_1(t - h)) = 1$ and in the interval $t_0 < t < t_1$, $\text{sign}(x_1(t - h)) = -\text{sign}(x_1(t)) = -1$.

Therefore, with $h = h_{\max}$ and while $t_0 < t < t_1$, we will have $x_1(t) > 0$ and $x_1(t - h) < 0$, so that $\dot{x}_2 = f(t, x_1, x_2) + k$. Moreover, since $k - |f(t, x_1, x_2)| \geq c_0 > 0$, it can be seen that $x_2(t) \geq c_0(t - t_0) + x_2(t_0)$, and accordingly x_1 will grow while $\text{sign}(x_1(t - h)) = -1$, but when $t > t_1$ and since $x_1 > 0$, we have $\text{sign}(x_1(t - h)) = 1$, and therefore $\dot{x}_2 = f(t, x_1, x_2) - k$. Hence, $x_2(t) \leq -c_0(t - t_1) + x_2(t_1)$.

Therefore, there exists $t = t_2$, where $x_2(t_2) = 0$, and accordingly when $t = t_2 \leq t_1 + x_2(t_1)/c_0$, x_1 arrives to a maximum and in the interval $t_2 < t < t_3$, it will diminish until being zero at

$$t = t_3 \leq \frac{x_2(t_1) + \sqrt{x_2(t_1)^2 + 2c_0[t_1 x_2(t_1) + x_1(t_1)]}}{c_0} + t_1. \quad (25)$$

This happens provided the system is in the domain of attraction, so the maximum value must occur for $\mathbf{x} \in D^+$.

An estimate for the maximum x_2 reached value can be gotten noticing that $0 < c_0 \leq k$, since $k - |f(t, x_1, x_2)| \geq c_0 > 0$. Therefore, the biggest possible value for $|f(t, x_1, x_2)|$, inside the domain of attraction, occurs when $c_0 \rightarrow 0$, making feasible that $|f(t, x_1, x_2)| \rightarrow k$. On the other hand, if $c_0 \rightarrow k$, $|f(t, x_1, x_2)|$ might only take the smallest possible value, $|f(t, x_1, x_2)| \rightarrow 0$. Hence, from the aforementioned, $0 < f(t, x_1, x_2) + k < 2k$, and the maximum value for x_2 , $M_{x_2}^{'+}$ inside the domain of attraction may be estimated as

$$M_{x_2}^{'+} = \sup_{t>0, \mathbf{x} \in D^+, 0 < \delta < k} \{x_2 \mid f(t, x_1, x_2) + k = 2\delta\}. \quad (26)$$

Now, with a similar reasoning as above, it can be shown that once $x_1(t) < 0$, a time will come such that $\text{sign}(x_1(t-h)) = -1$, and if the dynamics are in the domain of attraction, x_1 will reach a minimum and begin moving toward the opposite direction remaining bounded. Therefore, an estimate for the minimum value of x_2 is

$$M_{x_2}^{'-} = \inf_{t>0, \mathbf{x} \in D^-, 0 < \delta < k} \{x_2 \mid f(t, x_1, x_2) - k = -2\delta\}. \quad (27)$$

Now, since in the interval $t_0 \leq t \leq t_1$ (see Figure 1), we have $\dot{x}_1 = x_2 \leq M_{x_2}^{'+}$, and considering that

$$\begin{aligned} 0 < M_{x_1}^{'+} &\leq M_{x_1}^+ \\ &= \sup \{x_1 > 0 \mid \langle x_1 \ x_2 = M_{x_2}^{'+} \rangle \in D\}, \end{aligned} \quad (28)$$

we have, $x_1(t_1) \leq M_{x_1}^{'+} \leq M_{x_2}^{'+} h_{\max}$. Therefore, it is clear that there exists $h' \leq h_{\max}$ such that $M_{x_1}^{'+} = M_{x_2}^{'+} h'$. And then for $h \leq h'$, we have $h \leq M_{x_1}^{'+}/M_{x_2}^{'+}$. So that for this delay, the system dynamics remains in the domain of attraction for positive values of x_1 .

Similarly, when $x_1 < 0$, $\dot{x}_1 = x_2 \geq M_{x_2}^{'-}$, and for $h_{\max} = t_4 - t_3$, where t_4 is the time when $x_1(t_4) \geq M_{x_1}^{'+} \geq M_{x_2}^{'-} h_{\max}$.

Considering $0 > M_{x_1}^{'-} \geq M_{x_1}^- = \inf \{x_1 < 0 \mid \langle x_1 \ x_2 = M_{x_2}^{'-} \rangle \in D\}$ and since $M_{x_2}^{'-} < 0$, there exists $h'' \leq h_{\max}$ such that $M_{x_1}^{'-} = M_{x_2}^{'-} h''$. Therefore, for $h \leq h''$, $h \leq M_{x_1}^{'-}/M_{x_2}^{'-}$, so that, for this delay, the system dynamics remains in the domain of attraction for negative values of x_1 .

Hence, if

$$h \leq \min \left\{ \frac{M_{x_1}^{'+}}{M_{x_2}^{'+}}, \frac{M_{x_1}^{'-}}{M_{x_2}^{'-}} \right\}, \quad (29)$$

the system dynamics will stay inside the domain of attraction; so this is a sufficient condition in the delay size for amplitude bounded oscillations to exist. $\square$

Remark 4. For a second order sliding mode system in the form (1), with a specified delay, h , and noticing that the maximum x_1 amplitude is obtained when $\text{sign}(x_1(t)) =$

$\text{sign}(x_1(t-h))$ and $x_2 = 0$, an estimate for the oscillation amplitude, A_{x_1} , can be obtained with

$$\begin{aligned} m_{x_1}' h &\leq A_{x_1} \\ &\leq \max \left\{ \sup_{t>0, \mathbf{x} \in D^+, 0 < \delta < k} \{x_1 \mid f(t, x_1, 0) - k = -2\delta\}, \right. \\ &\quad \left. \sup_{t>0, \mathbf{x} \in D^-, 0 < \delta < k} \{|x_1| \mid f(t, x_1, 0) + k = 2\delta\} \right\}, \end{aligned} \quad (30)$$

where m_{x_2}' is the lowest magnitude value of x_2 for the interval when $\text{sign}(x_1(t)) = -\text{sign}(x_1(t-h))$. Now, noticing that $\dot{x}_2 = 0$ is an extreme for x_2 in some cases, such lowest value for those cases may be obtained from

$$\begin{aligned} m_{x_2}' &= \min \left\{ \inf_{t>0, \mathbf{x} \in D^+, |x_1(t-h)| < 0, x_2 > 0} \{x_2 \mid f(t, x_1, x_2) + k = 0\}, \right. \\ &\quad \left. \inf_{t>0, \mathbf{x} \in D^-, |x_1(t-h)| > 0, x_2 < 0} \{|x_2| \mid f(t, x_1, x_2) - k = 0\} \right\}, \end{aligned} \quad (31)$$

verifying that $\ddot{x}_2 > 0$ when $\text{sign}(x_1(t)) = 1$ and $\ddot{x}_2 < 0$ when $\text{sign}(x_1(t)) = -1$.

Remark 5. If the second order sliding mode system is of the form

$$\begin{aligned} \dot{x}_1 &= x_2 \\ &\vdots \\ \dot{x}_n &= f(t, x_1, x_2) - k \text{sign}(s(x_1(t-h), \dots, x_{n-1}(t-h))), \end{aligned} \quad (32)$$

such that

$$\begin{aligned} \dot{s} &= \eta(\mathbf{x}) \\ \ddot{s} &= \varphi(t, \mathbf{x}) - m \text{sign}(s(t-h)), \end{aligned} \quad (33)$$

so that $\mathbf{x} = 0$ is stable for $h = 0$, with the domain of attraction being $\mathbf{x} \in D$ and

$$\begin{aligned} D^{s+} &= \{D \mid s > 0, \dot{s} \geq 0\}, \\ D^{s-} &= \{D \mid s < 0, \dot{s} \leq 0\}, \end{aligned} \quad (34)$$

an estimate of the maximum delay for steady oscillations in the sliding surface is

$$h \leq \min \left\{ \frac{M_s^{'+}}{M_s^{'+}}, \frac{M_s^{'-}}{M_s^{'-}} \right\}, \quad (35)$$

where

$$\begin{aligned} M_s^{'+} &= \sup_{t>0, \mathbf{x} \in D^{s+}, 0 < \delta < m} \{\dot{s} \mid \varphi(t, \mathbf{x}) + m = 2\delta\} \\ M_s^{'-} &= \inf_{t>0, \mathbf{x} \in D^{s-}, 0 < \delta < m} \{\dot{s} \mid \varphi(t, \mathbf{x}) - m = -2\delta\} \end{aligned} \quad (36)$$

$$0 < M_s^{'+} \leq M_s^+ = \sup \{s > 0 \mid \langle s \ \dot{s} = M_s^{'+} \rangle \in D\}$$

$$0 > M_s^{'-} \geq M_s^- = \inf \{s < 0 \mid \langle s \ \dot{s} = M_s^{'-} \rangle \in D\}.$$

FIGURE 1: Behavior of x_1 for different forms of $f(t, x_1, x_2)$, where it can be seen as $h < t_1 - t_0$ and maximum value for $x_2 = M_{x_2}^+$, when $0 < x_1 \leq M_{x_1}^+$.

Remark 6. If the system is a first order sliding mode system such that with the form

$$\dot{x}_1 = x_2$$

$\vdots$

$$\dot{x}_n = f(t, x_1, x_2) - k \operatorname{sign}(s(x_1(t-h), \dots, x_n(t-h))), \quad (37)$$

$$\dot{s} = \psi(t, \mathbf{x}) - p \operatorname{sign}(s(t-h)), \quad (38)$$

so that $\mathbf{x} = 0$ is stable for $h = 0$, with the domain of attraction being $\mathbf{x} \in D$ and

$$\begin{aligned} D^{s+} &= \{D \mid s > 0\} \\ D^{s-} &= \{D \mid s < 0\}, \end{aligned} \quad (39)$$

an estimate for the maximum delay where steady oscillations in the sliding surface are obtained may be gotten with

$$h \leq \min \left\{ \frac{M_s^{'+}}{M_s^{'+}}, \frac{M_s^{'-}}{M_s^{'-}} \right\}, \quad (40)$$

where

$$\begin{aligned} M_s^{'+} &= \sup_{t>0, \mathbf{x} \in D^{s+}, 0<\delta<p} \{ \dot{s} \mid \psi(t, \mathbf{x}) + p \leq 2\delta \} \\ M_s^{'-} &= \inf_{t>0, \mathbf{x} \in D^{s-}, 0<\delta<p} \{ \dot{s} \mid \psi(t, \mathbf{x}) - p \geq -2\delta \} \\ 0 < M_s^{'+} &\leq M_s^{+} = \sup \{ s > 0 \mid \dot{s} = M_s^{'+} \in D \} \\ 0 > M_s^{'-} &\geq M_s^{-} = \inf \{ s < 0 \mid \dot{s} = M_s^{'+} \in D \}. \end{aligned} \quad (41)$$

Remark 7. For a first order sliding mode system with amplitude bounded oscillations, if an upper bound for the sliding surface derivative magnitude is known (see Remark 6), together with the delay, h , an upper bound for the oscillation amplitude, A_s , of the sliding surface, s , can be obtained with

$$A_s \leq \max \{ M_s^{'+} h, |M_s^{'-}| h \}. \quad (42)$$

4. Examples

In this section, three examples are given to validate the results presented above.

4.1. Reaching Equilibrium Point in Finite Time for Second Order Sliding Modes. Let us consider the following system:

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_1 - 5x_2 - 5 \operatorname{sign}(x_1). \end{aligned} \quad (43)$$

It is easy to verify that system dynamics are unstable if $5 \operatorname{sign}(x_1)$ is not present. In order to verify if the dynamics of (43) reach in finite time the equilibrium point in zero, Proposition 1 is used. Let's consider $x_1 > 0$. Since $k = 5$, $\dot{x}_2 > -k = -5$ must be complied and also $\dot{x}_2 \leq -c_0$, such that $x_2 > -5t + x_2(0)$, and $x_1 \leq -c_0 t^2/2 + x_2(0)t + x_1(0)$.

Therefore, $\dot{x}_2 \leq -c_0 t^2/2 + (25 + x_2(0))t + x_1(0) - 5x_2(0) - 5$. From this expression, it can be observed that the time when the biggest value for $\dot{x}_2$ (less negative) is $t = 0$, since for $|f(t, x_1, x_2)| < 5$, it is required that $|x_2| < 5$, then from $t = 0$, $\dot{x}_2$ goes more negative while $x_1 > 0$. Similarly, it can be shown that if the initial condition is $x_1 < 0$, the lowest value for $\dot{x}_2$ occurs at $t = 0$, such that $k - |f(t, x_1, x_2)| \geq c_0$. Hence, a sure way to choose the initial conditions would be $|x_1(0) - 5x_2(0)| < 5$.

So that, for this example, we choose $x_1(0) = 4$, $x_2(0) = 0$. Therefore, at $t = 0$, $|f(t, x_1, x_2)| < 5$, and just afterwards, x_2 is and remains negative while $x_1 > 0$. Therefore x_1 goes down until reaching the first zero crossing. Knowing that

$$x_1(t) = -.9642e^{0.19258t} - 0.03328e^{-5.1958t} + 5, \quad (44)$$

it is easy to see that $x_1 = 0$, when $x_2 = -.93$.

Up to now, first conditions for the Proposition 1 are met in order to get the first zero crossing. Now, to see if the condition for finite time surface reaching is met, let us consider an arbitrary time of zero crossing from positive to negative t_n . In the interval $t_n \leq t \leq t_{n+1}$, the maximum value for $|f(t, x_1, x_2)|$ is at t_n , since this is the initial time in this interval. Then $f(t_n, 0, c_n) = -5c_n$ and at $t = t_{n+1}$, $f(t_{n+1}, 0, c_{n+1}) = -5c_{n+1}$. From Proposition 1, it is needed

$$\frac{|c_n|}{|c_{n+1}|} > \frac{(k - |f(t_n, 0, c_n)|)}{(k - |f(t_{n+1}, 0, c_{n+1})|)} = \frac{5 - 5|c_n|}{5 - 5|c_{n+1}|}. \quad (45)$$

Resulting in the condition $|c_n| > |c_{n+1}|$.

Let's see if the system meets such a condition. Meanwhile, $x_1 < 0$ in the interval $t_n \leq t \leq t_{n+1}$,

$$\begin{aligned} x_1(t) &= Ae^{0.19258(t-t_n)} + Be^{-5.1958(t-t_n)} - 5 \\ x_2(t) &= 0.19258Ae^{0.19258(t-t_n)} - 5.1958Be^{-5.1958(t-t_n)}, \end{aligned} \quad (46)$$

with

$$\begin{aligned} A &= \frac{5 - 0.1958c_n}{1.037}, \\ B &= \frac{5 + 5.1958c_n}{27.963}. \end{aligned} \quad (47)$$

Now, considering that in the following zero crossing, $t = t_{n+1}$,

$$\begin{aligned} 0 &= Ae^{0.19258(t_{n+1}-t_n)} + Be^{-5.1958(t_{n+1}-t_n)} - 5, \\ c_{n+1} &= 0.19258Ae^{0.19258(t_{n+1}-t_n)} - 5.1958Be^{-5.1958(t_{n+1}-t_n)}, \end{aligned} \quad (48)$$

and solving (48), using expressions for A and B in (46),

$$c_{n+1} = 0.963 - 0.963e^{-5.1958(t_{n+1}-t_n)} - e^{-5.1958(t_{n+1}-t_n)}c_n. \quad (49)$$

This is an asymptotically stable first order difference equation for c_n dynamics, since $t_{n+1} > t_n$, giving the eigenvalue magnitude less than one and guaranteeing that $|c_{n+1}| < |c_n|$, complying with the condition for reaching the surface in finite time. Eventually, when $t_{n+1} = t_n$, the equilibrium point is reached, $c_n = 0$. Therefore, system dynamics are stable and reach the surface in finite time without need of using a measure or estimate of x_2 for the control signal. In Figure 2, the behavior of x_1 and x_2 is shown using $x_1(0) = 4$, $x_2(0) = 0$. It is seen that effectively, x_1 approaches smoothly and in finite time to zero and chattering is confined to x_2 , which also approaches in finite time to zero.

4.2. Maximum Permissible Delay in the Second Order Sliding Mode System. Knowing that sliding modes are gotten for system (43), now a delay in the control is introduced as shown below:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 - 5x_2 - 5 \operatorname{sign}(x_1(t-h)). \end{aligned} \quad (50)$$

FIGURE 2: Behavior of system (43) dynamics showing that $x_1 = 0$ and $x_2 = 0$ are reached in finite time. As expected for second order sliding mode, chattering is not present in x_1 .

FIGURE 3: Estimate for biggest value of x_2 in the system with delay (50) inside the domain of attraction.

Now, using (22), $M'_{x_1} - 5M'_{x_2} + 5 = 2\delta$, which gives the maximum with $\delta = 0$

$$M'_{x_2} \leq \frac{M'_{x_1} + 5}{5}. \quad (51)$$

The domain of attraction is shown in Figure 3, together with all possible values for M'_{x_2} . It is noticed that at $x_1 = 4.5$, $M'_{x_2} = 1.9$ intersects the upper limit of the domain of attraction. Therefore

$$h < \frac{M'_{x_1}}{M'_{x_2}} = \frac{4.5}{1.9} = 2.368. \quad (52)$$

In this example, $h = 2.36$ was used along with initial conditions set to $x_1 = 2$ and $x_2 = 0$, in order to ensure the initial crossing by zero since for $t \in [0, h]$ no control is available. To estimate the oscillation amplitude, Remark 4 of

FIGURE 4: Behavior of dynamics for system (50) with delay $h = 2.36$.

Proposition 3 is used. To obtain m'_{x_2} , we have $x_1 - 5x_2 + 5 = 0$. And so $m'_{x_2} = \min_{x_1 > 0} ((5 + x_1)/5) = 1$. Therefore, $A_{x_1} \geq m'_{x_2} h = 2.36$. Now, the biggest value for x_1 is obtained from $x_1 - 5 = -2\delta$, so that $A_{x_1} \leq \max_{0 \leq \delta < 5} (5 - 2\delta) = 5$.

Hence, the oscillation amplitude for x_1 is bounded with a value between 2.36 and 5. In Figure 4, it can be verified that $h = 2.36$ is a sufficient condition for bounded oscillations to occur. Also, the amplitude is seen to be a little bigger than 3, which is inside the estimated limits. For this case the describing function approach cannot be used to estimate the amplitude, because of the instability of the associated linear system.

4.3. Amplitude Estimate for a Globally Stable System. Considering a case where the equilibrium point of the involved system is globally stable such as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -5x_2 - 5 \operatorname{sign}(x_1(t-h)). \end{aligned} \quad (53)$$

Using Remark 4 of Proposition 3, the inferior bound for amplitude A_{x_1} with $m'_{x_2} = 5/5 = 1$ is $A_{x_1} > hm'_{x_2} = h$. If $h = 5$, then $A > 5$. In Figure 5, the system response for x_1 is seen. The oscillation amplitude is a little bigger than 5, verifying the estimated value. Also, since x_2 is almost constant, the estimated amplitude is very close to the real one. If the describing function approach was used, the estimated amplitude would be around 4.44, which is lower than the minimum estimated with the proposed method. In this particular case, an estimate of the superior bound for x_1 using Remark 4 cannot be obtained, since x_1 is not explicit in $f(t, x, 0) - k = -2\delta$.

5. Conclusions

Stability analysis for a class of second order sliding mode systems, which does not need measuring or estimation of the derivative of the sliding variable for the control signal,

FIGURE 5: Behavior of globally stable system (53) with a delay $h = 5$.

was presented. Also, when delay in input is present, bounded oscillations are studied. The system can be linear or nonlinear. The class of systems treated is a more general class of the systems studied in [20], where sufficient conditions are given for stationary oscillations to exist using another approach and no means is given to estimate amplitude of oscillations. The analysis presented in this paper is based on two propositions. One of them gives sufficient conditions for sliding mode existence for a class of second order sliding mode systems. The other can be used as a method based on maximum derivative to get a delay estimate on the control signal for a class of second order sliding mode systems, giving sufficient conditions to maintain amplitude bounded oscillations on the sliding surface. Also, for a given delay in input, oscillation amplitude for the sliding surface dynamics can be estimated if the system state is in the attraction region. On the other hand, when oscillation amplitude is known, such as hysteresis happening in the discontinuous input, frequency of oscillation can be estimated. Numerical examples for second order sliding modes were presented to illustrate the proposed approach, first two examples have unstable dynamics when no input is applied. Third example has global stability for the considered input. It is clear that very conservative results using this method would arise for systems where the expected sliding surface derivative is much lower than the maximum derivative. For linear stable processes, the describing function method has been used in the literature to estimate oscillations amplitude. In the second example such approach cannot be used since the linear process is unstable. In the third example, the simulation shows the amplitude just a little bigger than the minimum estimated by the approach presented here. For this example, the describing function method gives a lower estimate than the minimum given by the proposed approach.

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