Research Article

New Lax Pairs and Darboux Transformation and Its Application to a Shallow Water Wave Model of Generalized KdV Type

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New Lax pairs of a shallow water wave model of generalized KdV equation type are presented. According to this Lax pair, we constructed a new spectral problem. By using this spectral problem, we constructed Darboux transformation with the help of a gauge transformation. Applying this Darboux transformation, some new exact solutions including double-soliton solution of the shallow water wave model of generalized KdV equation type are obtained. In order to visually show dynamical behaviors of these double soliton solutions, we plot graphs of profiles of them and discuss their dynamical properties.

1. Introduction

It is well known that the Lax pair and Darboux transformation can be employed to obtain multisoliton solution of nonlinear evolution equations. Darboux transformations provide us with purely algebraic, powerful method to construct solutions for systems of nonlinear equations. In recent years, more and more researchers used the Lax pair and Darboux transformation to investigate soliton solutions of classical nonlinear wave equations and some new soliton equations which were generated by new spectral problems; see [1–30] and references cited therein. In general, a systematical theory on such Darboux transformation even for \( n \times n \) matrix spectral problem and the resulting zero curvature equation has a beautiful algebraic structure for associated evolution equations; see [31, 32] and references cited therein. Sometimes, it is found that there are many infinity symmetries from the adopted zero curvature equation.

In this paper, we will investigate the Lax pairs, Darboux transformation, and double soliton solutions of the following famous shallow water wave model of generalized KdV equation type:

\[
\frac{u_t}{3} - \frac{h_0^2}{3} u_{xxx} + c_0 u_x + \frac{3\alpha}{2} uu_x - \frac{c_0 h_0^2}{6} u_{xxx} = 0,
\]

(1)

which appeared in [33], where \( 0 < \alpha \ll 1 \) is a small-amplitude parameter. Only dropping the right-hand side of (1) gives BBM equation. Dropping the right-hand side of (1) and replacing the term \( u_{xxt} \) by the term \( -c_0 u_{xxx} \) give KdV equation [34]. Thus, (1) can be seen as a BBM equation extended by retaining higher order terms in an asymptotic expansion in terms of the small-amplitude parameter \( \alpha \). Dropping the term \(- (c_0 h_0^2 / 6) u_{xxx} \) from (1) and letting \( h_0 = \sqrt{3}, c_0 = 2\kappa, \alpha = 2, \) (1) becomes the celebrated Camassa-Holm equation [33] as follows:

\[
u_t + 2\kappa - u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx},
\]

(2)

where \( u \) is the fluid velocity in the \( x \) direction (or equivalent to the height of the water's free surface above a flat bottom), \( \kappa \) is a constant related to the critical shallow water wave speed, and subscripts denote partial derivatives. Letting \( c_0 = 1, \gamma = -(1/3)\gamma h_0, \gamma = (3/2)\alpha, \beta = -(1/6)\gamma c_0, \) (1) can be rewritten as

\[
u_t + u_x + \gamma u_{xxt} + \beta u_{xxx} + \gamma uu_x + \frac{1}{3} \nu (uu_{xxx} + 2uu_x u_{xx}) = 0,
\]

(3)
which comes from physical considerations via the methodology introduced by Fuchssteiner and Fokas in [35, 36]. The Lax pairs of (3) with \( y = 1 \) are given by [37, 38] as follows:

\[
\Psi_{xx} = \left[ k^2 (1 - v) + \frac{v}{4v} \right] \Psi = 0, \quad \sigma v = u + \gamma u_{xx},
\]

\[
\sigma = \frac{\beta - v}{2v},
\]

\[
\Psi_t + \left( \frac{1}{2} + c \right) \Psi + \left( u + \frac{\beta}{v} + \frac{2\sigma}{4\gamma k^2 - 1} \right) \Psi_x = 0,
\]  

where \( c \) is an arbitrary constant. It is a pity that the [37] is not a formal publication, and it is only preprint paper, so we cannot know its reality contents whether the authors have obtained soliton solutions of (3) by the Lax pairs (4) and (5). In fact, (3) has been studied by many authors in recent years; see the following brief introductions.

In [39], by using the bifurcation theory of dynamic system, some subsection-function and implicit function solutions such as compactons, solitary waves, smooth periodic waves, and nonsmooth periodic waves with peaks as well as the existence conditions have been presented by Bi. By using the same method, Li and Zhang [40] studied a generalization form of the modified KdV equation, which is more complex than (3). In [40], the existence of solitary wave, kink and antikink wave solutions, and uncountably many smooth and nonsmooth periodic wave solutions are discussed. By using the improved method named integral bifurcation method [41], Rui et al. [42] obtain all kinds of soliton-like or kink-like wave solutions, periodic wave solutions with loop or without loop, smooth compacton-like periodic wave solutions, and nonsmooth periodic cusp wave solutions for (3). In [43], Long and Chen discussed the existence of solitary wave, cusp wave, periodic wave, periodic cusp wave, and compactons were for (3). From the above references, (1) (i.e., (3)) is a very important water wave model.

The rest of this paper is organized as follows. In Section 2, we will derive new Lax pair and Darboux transformation of (1). In Section 3, by using this Darboux transformation, we will investigate soliton solutions of (1) and discuss the dynamic properties of these soliton solutions.

2. Lax Pair and Darboux Transformation of (1)

Through a series of tedious computation, we obtain Lax pairs of (1) as follows:

\[
\phi_{xx} = \left( -\frac{c_0}{4\lambda \alpha} + \frac{3}{4h_0^2} - \frac{u - (1/3) h_0^2 u_{xx}}{2\lambda} \right) \phi,
\]

\[
\phi_t = -\left( \frac{c_0}{2} + \frac{3\lambda \alpha}{2h_0^2} + \frac{\alpha}{2} u \right) \phi + \frac{\alpha}{4} u_x \phi.
\]

Obviously, the Lax pairs (6) and (7) are different from the Lax pairs (4) and (5) under \( c_0 = 1, v = -(1/3) h_0^2, y = (3/2)\alpha, \beta = -(1/6) h_0^2 c_0 \). They are new Lax pairs which we obtained. By using the new Lax pairs (6) and (7), we will construct a Darboux transformation for obtaining soliton solutions of (1).

First, we consider the following spectral problems:

\[
\phi_x = M \phi, \quad \phi_t = N \phi,
\]

with

\[
M = \begin{pmatrix} 0 & 1 \\ -\frac{c_0}{4\lambda \alpha} + \frac{3}{4h_0^2} & -\frac{u - (1/3) h_0^2 u_{xx}}{2\lambda} \end{pmatrix},
\]

\[
N = \begin{pmatrix} \frac{\alpha}{4} u_x \\ \frac{\alpha}{4} u_x - \left( \frac{c_0}{2} + \frac{3\lambda \alpha}{2h_0^2} + \frac{\alpha}{2} u \right) \left( -\frac{c_0}{4\lambda \alpha} + \frac{3}{4h_0^2} - \frac{u - (1/3) h_0^2 u_{xx}}{2\lambda} \right) \end{pmatrix},
\]

where \( \alpha \) is a constant, \( \lambda \) is a spectral parameter, and \( u \) is a potential function. From compatibility, condition \( \phi_{xxt} = \phi_{xxx} \) yields a zero curvature equation \( M_t - N_x + [M, N] = O \). Substituting \( M, N \) into the zero curvature equation, by a direct calculation, (1) is obtained successfully.

Next, we will construct a Darboux Transformation (DT) of the spectral problems (8). In fact, the DT is actually a gauge transformation

\[
\bar{\phi} = T \phi
\]  

of the spectral problems (8). It is required that \( \bar{\phi} \) also satisfies the same form of spectral problems

\[
\bar{\phi}_x = \bar{M} \bar{\phi}, \quad \bar{M} = (T_x + TM) T^{-1},
\]

\[
\bar{\phi}_t = \bar{N} \bar{\phi}, \quad \bar{N} = (T_t + TN) T^{-1}.
\]

It means that we have to find a matrix \( T \) such that the old potential \( \phi \) is replaced by the new one \( \bar{\phi} \).

Suppose

\[
T = T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix},
\]
where
\[ A(\lambda) = A_N \left( \lambda^N + \sum_{k=0}^{N-1} A_k \lambda^k \right), \]
(14)
\[ B(\lambda) = A_N \left( \sum_{k=0}^{N-1} B_k \lambda^k \right), \]
\[ C(\lambda) = \frac{1}{A_N} \left( \sum_{k=0}^{N-1} C_k \lambda^k \right), \]
(15)
\[ D(\lambda) = \frac{1}{A_N} \left( \lambda^N + \sum_{k=0}^{N-1} D_k \lambda^k \right), \]
and \( A_N, A_k, B_k, C_k, \) and \( D_k \) \((0 \leq k \leq N - 1)\) are functions of \( x \) and \( t \).

Let \( \phi(\lambda_j) = (\phi_1(\lambda_j), \phi_2(\lambda_j))^T, \psi(\lambda_j) = (\psi_1(\lambda_j), \psi_2(\lambda_j))^T \) be two basic solutions of (8). From (10), there exist constants \( r_j \) \((0 \leq j \leq N - 1)\), which satisfy
\[ (A(\lambda_j) \phi_1(\lambda_j) + B(\lambda_j) \phi_2(\lambda_j)) - r_j (A(\lambda_j) \psi_1(\lambda_j) + B(\lambda_j) \psi_2(\lambda_j)) = 0, \]
\[ (C(\lambda_j) \phi_1(\lambda_j) + D(\lambda_j) \phi_2(\lambda_j)) - r_j (C(\lambda_j) \psi_1(\lambda_j) + D(\lambda_j) \psi_2(\lambda_j)) = 0. \]
(16)
Further, (16) can be written as a linear algebraic system
\[ A(\lambda_j) + \delta_j B(\lambda_j) = 0, \quad C(\lambda_j) + \delta_j D(\lambda_j) = 0. \]
(17)
That is
\[ \sum_{k=0}^{N-1} (A_k + \delta_j B_k) \lambda_j^k = -\lambda_j^N, \quad \sum_{k=0}^{N-1} (C_k + \delta_j D_k) \lambda_j^k = -\delta_j A_N, \]
(18)
where
\[ \delta_j = \frac{\phi_2(\lambda_j) - r_j \psi_2(\lambda_j)}{\phi_1(\lambda_j) - r_j \psi_1(\lambda_j)}, \quad 1 \leq j \leq 2N, \]
(19)
and the constants \( \lambda_j, (\lambda_j \neq \lambda_s) \) as \( k \neq s \), \( r_j \) are suitably chosen such that determinant of coefficients for (18) is nonzero. Therefore, \( A_N, A_k, B_k, C_k, \) and \( D_k \) \((0 \leq k \leq N - 1)\) are uniquely determined by (18).

Equations (14) and (15) show that the \( \det T(\lambda) \) is a \( 2N \)-th order polynomial in \( \lambda \), and
\[ \det T(\lambda_j) = A(\lambda_j) D(\lambda_j) - B(\lambda_j) C(\lambda_j). \]
(20)
On the other hand, from (17), we have \( A(\lambda_j) = -\delta_j B(\lambda_j), C(\lambda_j) = -\delta_j D(\lambda_j) \). Thus we have
\[ \det T(\lambda) = \beta^{2N-1} \prod_{j=1}^{2N-1} (\lambda - \lambda_j), \]
(21)
where \( \beta \) is independent of \( \lambda \). Equation (21) implies that \( \lambda_j \) \((1 \leq j \leq 2N)\) are \( 2N \) roots of \( \det T(\lambda) \).

Second, we prove the following theory of Darboux transformation for special variable.

Theorem 1. Let \( A_N \) satisfy
\[ A_N^2 = 1. \]
(22)
Then the matrix \( \mathbf{M} \) determined by (11) has the same form as \( \mathbf{M} \); that is,
\[ \mathbf{M} = \left( \begin{array}{ccc} 0 & -c_0 & 1 \\ 3 \frac{1}{4 \lambda^2} & 0 - \frac{1}{2} \bar{u} & \frac{1}{2} \bar{u} \\ 0 & \frac{1}{2} \bar{u} & 0 \end{array} \right), \]
(23)
where the transformation from the old potential \( u \) into new one \( \bar{u} \) is given by
\[ \bar{u} = u + \frac{6}{h_0} A_{N-1}, \]
(24)
\[ A_{N-1} = C_{N-1} - \frac{3}{4h_0^2} B_{N-1}, \]
\[ B_{N-1} = D_{N-1} - A_{N-1}, \]
\[ C_{N-1} = \frac{3}{4h_0^2} (A_{N-1} - D_{N-1}) + \frac{c_0}{4a} + \frac{u - (1/3) h_0^2 u_{xx}}{2}, \]
\[ D_{N-1} = \frac{3}{4h_0^2} B_{N-1} - C_{N-1}. \]
(25)
Proof. Let \( T^{-1} = \mathbf{T}^* / \det T \) and
\[ (\mathbf{T}_x + \mathbf{TM}) \mathbf{T}^* = \left( \begin{array}{c} f_{11}(\lambda) \\ f_{12}(\lambda) \\ f_{22}(\lambda) \end{array} \right). \]
(26)
It is easy to see that \( f_{11}(\lambda) \) and \( f_{22}(\lambda) \) are \( 2N \)-th order polynomials in \( \lambda \), \( f_{12}(\lambda) \) and \( f_{21}(\lambda) \) are \((2N - 1)\)-th order polynomials in \( \lambda \). From (19) and (8), we find
\[ \delta_{x_j} = -\frac{c_0}{4 \lambda^2} + \frac{3}{4h_0^2} - \frac{u - (1/3) h_0^2 u_{xx}}{2 \lambda} \]
(27)
Through direct calculation, all \( \lambda_j \) \((0 \leq j \leq 2N)\) are roots of \( f_{ns} (n, s = 1, 2) \). Together with (21) and (26), we get
\[ (\mathbf{T}_x + \mathbf{TM}) \mathbf{T}^* = (\det T) P(\lambda), \]
(28)
with
\[ P(\lambda) = \left( \begin{array}{cc} p_{11}^{(0)} & p_{12}^{(0)} \\ p_{21}^{(0)} & p_{22}^{(0)} \end{array} \right), \]
(29)
where \( p_{ns}^{(0)} (n, s = 1, 2) \) are independent of spectral parameter \( \lambda \). Indeed, (28) can be written as
\[ \mathbf{T}_x + \mathbf{TM} = P(\lambda) \mathbf{T}. \]
(30)
Comparing the coefficients of $\lambda^N$ in (30), we find
\[ p_{11}^{(0)} = -p_{22}^{(0)} = \partial_1 \ln A_N, \quad p_{12}^{(0)} = A_N^2, \quad p_{21}^{(0)} = \frac{3}{A_N^2 4h_0^2}. \] \tag{31}

Substituting (22) into (31) yields
\[ p_{11}^{(0)} = -p_{22}^{(0)} = 0, \quad p_{12}^{(0)} = A_N^2 = 1. \] \tag{32}

From (22), (24), (25), and (31) and noticing $\bar{u}$ in (23), we get
\[ p_{21}^{(0)} = -\frac{c_0}{4\lambda} + \frac{3}{4h_0^2} \tau - \frac{(1/3) h_0^2 \tau_{xx}}{2\lambda}. \] \tag{33}

Thus $P(\lambda) = \overline{M}$. The proof of Theorem 1 is completed. \qed

Finally, by using same way to Theorem 1, we prove that $\overline{N}$ in (12) has the same form as $N$ under the transformation (10) and (24); see the following theory and its proof.

**Theorem 2.** The matrix $\overline{N}$ defined by (12) has the same type as $N$, in which the old potential $u$ is mapped into $\bar{u}$ via the same $DT$ (24).

**Proof.** Let $T^{-1} = T^*/\det T$ and
\[
(T_t + TN) T^* = \begin{pmatrix} g_{11}(\lambda) & g_{12}(\lambda) \\ g_{21}(\lambda) & g_{22}(\lambda) \end{pmatrix}, \tag{34}
\]

It is easy to see that $g_{11}(\lambda)$ and $g_{22}(\lambda)$ are 2Nth-order polynomials in $\lambda$, $g_{12}(\lambda)$ and $g_{21}(\lambda)$ are $(2N + 1)$th-order polynomials in $\lambda$. By using (19) and (8), we obtain
\[
\delta_{ij} = \frac{\alpha}{4u} \tau_{xx} - \left( \frac{c_0}{2} + \frac{3\lambda\alpha}{2h_0^2} + \frac{\alpha}{2} u \right) \left( \frac{c_0}{4\lambda} + \frac{3}{4h_0^2} \tau - \frac{(1/3) h_0^2 \tau_{xx}}{2\lambda} \right) \tag{35}
\]

Through direct calculation, all $\lambda_j (0 \leq j \leq 2N)$ are roots of $\xi_{ns} (n, s = 1, 2)$. Together with (21) and (34), we get
\[
(T_t + TN) T^* = (\det T) Q(\lambda), \tag{36}
\]

with
\[
Q(\lambda) = \begin{pmatrix} q_{11}^{(0)} & q_{12}^{(1)} + q_{12}^{(0)} \\ q_{21}^{(1)} + q_{21}^{(0)} & q_{22}^{(0)} \end{pmatrix}, \tag{37}
\]

where $q_{ns}^{(0)} (n, s = 1, 2, l = 0, 1)$ are independent of spectral parameter $\lambda$. Equation (37) can be written as
\[
T_t + TN = Q(\lambda) T. \tag{38}
\]

Comparing the coefficients of $\lambda^{N+1}$ and $\lambda^N$ in (38) leads to
\[
q_{12}^{(1)} = -\frac{3\alpha}{2h_0^2} A_N^2, \quad q_{21}^{(1)} = -\frac{1}{A_N^2} \frac{3\alpha}{2h_0^2} A_N^2, \tag{39}
\]
\[
q_{11}^{(0)} = \partial_1 \ln A_N + \frac{\alpha}{4} u_x - \frac{3\alpha}{2h_0^2} B_{N-1} + \frac{3\alpha}{2h_0^2} C_{N-1}, \tag{40}
\]
\[
q_{12}^{(0)} = -\left( \frac{c_0}{2} + \alpha \right) A_N^2 - \frac{3\alpha}{2h_0^2} A_N^2 A_{N-1} + \frac{3\alpha}{2h_0^2} A_N^2 D_{N-1}, \tag{41}
\]
\[
q_{21}^{(0)} = \frac{1}{A_N^2} \frac{3\alpha}{4\lambda} u_x x - \frac{3\alpha}{2h_0^2} \left( \frac{c_0}{2} + \frac{\alpha}{2} u \right) - \frac{1}{A_N^2} \frac{3\alpha}{2h_0^2} A_{N-1} \tag{42}
\]

Substituting (22), (24), and (25) into (40) to (43), we can get
\[
q_{11}^{(0)} = -\frac{3\alpha}{2h_0^2} B_{N-1} - \frac{3\alpha}{2h_0^2} C_{N-1} = -\frac{3\alpha}{4u} A_{N-1} \tag{43}
\]
\[
q_{12}^{(0)} = -\frac{3\alpha}{2h_0^2} A_{N-1} = -\frac{\alpha}{4} u_x \tag{44}
\]
\[
q_{12}^{(1)} + q_{12}^{(0)} = -\frac{3\alpha}{2h_0^2} A_{N-1} = -\frac{\alpha}{4} u_x \tag{45}
\]
\[
q_{22}^{(0)} = -\frac{3\alpha}{2h_0^2} C_{N-1} = -\frac{\alpha}{4} u_x \tag{46}
\]
\[ \varphi(\lambda) = \left( -\frac{\zeta_0}{4\lambda + 3} \right) \phi(\lambda), \]

where \( \lambda \) is a nonzero constant and \( 0 < j \leq 2N \).

Using the Cramer rule to solve the linear algebraic system (18), we obtain

\[ A_{N-1} = \frac{\Delta_{A_{N-1}}}{\Delta}, \]

where

\[ \Delta = \begin{vmatrix}
1 & \delta_1 & \lambda_1 & \delta_1 \lambda_1 & \ldots & \lambda_1^k & \delta_1 \lambda_1^k & \ldots & \lambda_1^{N-1} & \delta_1 \lambda_1^{N-1} \\
1 & \delta_2 & \lambda_2 & \delta_2 \lambda_2 & \ldots & \lambda_2^k & \delta_2 \lambda_2^k & \ldots & \lambda_2^{N-1} & \delta_2 \lambda_2^{N-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \delta_{2N} & \lambda_{2N} & \delta_{2N} \lambda_{2N} & \ldots & \lambda_{2N}^k & \delta_{2N} \lambda_{2N}^k & \ldots & \lambda_{2N}^{N-1} & \delta_{2N} \lambda_{2N}^{N-1} \\
\end{vmatrix} \]

and

\[ \Delta_{A_{N-1}} = \begin{vmatrix}
1 & \delta_1 & \lambda_1 & \delta_1 \lambda_1 & \ldots & \lambda_1^k & \delta_1 \lambda_1^k & \ldots & -\lambda_1^N & \delta_1 \lambda_1^N & \delta_1 \lambda_1^{N-1} \\
1 & \delta_2 & \lambda_2 & \delta_2 \lambda_2 & \ldots & \lambda_2^k & \delta_2 \lambda_2^k & \ldots & -\lambda_2^N & \delta_2 \lambda_2^N & \delta_2 \lambda_2^{N-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & \delta_{2N} & \lambda_{2N} & \delta_{2N} \lambda_{2N} & \ldots & \lambda_{2N}^k & \delta_{2N} \lambda_{2N}^k & \ldots & -\lambda_{2N}^N & \delta_{2N} \lambda_{2N}^N & \delta_{2N} \lambda_{2N}^{N-1} \\
\end{vmatrix} \]
As examples, we will investigate exact solutions of (1) in two simple cases $N = 1$ and $N = 2$. When $N = 1$, solving the linear algebraic system (18) leads to

$$A_0 = \frac{\delta_1 \lambda_2 - \delta_2 \lambda_1}{\delta_2 - \delta_1}. \quad (52)$$

Substituting (49) and (52) into (24), a singular double-soliton solution of (1) is obtained as follows:

$$u \left[1 \right] = \frac{6}{h_0} f \left[ \sqrt{- \frac{c_0}{4\lambda_1 \alpha} + \frac{3}{4h_0^2} - \frac{u_0}{2\lambda_1} \frac{1}{1 - r_1 \tanh \mu_1} - \lambda_2} - \sqrt{- \frac{c_0}{4\lambda_2 \alpha} + \frac{3}{4h_0^2} - \frac{u_0}{2\lambda_2} \frac{1}{1 - r_2 \tanh \mu_2} - \lambda_1} \right], \quad (53)$$

where

$$f = \sqrt{- \frac{c_0}{4\lambda_2 \alpha} + \frac{3}{4h_0^2} - \frac{u_0}{2\lambda_2} \frac{1}{1 - r_2 \tanh \mu_2} - \lambda_1} \quad \text{and} \quad \lambda = \sqrt{- \frac{c_0}{4\lambda_1 \alpha} + \frac{3}{4h_0^2} - \frac{u_0}{2\lambda_1} \frac{1}{1 - r_1 \tanh \mu_1} - \lambda_2}. \quad (54)$$

By using the program of computer, it is easy to verify that the solution (53) satisfies (1), and this shows that the Darboux transformation (24) which we obtained is correct. In order to show the properties of the above singular double-soliton solutions visually, as an example, we plot the 3-D graphs of solution (53) for some fixed parameters, which are shown in Figures 1 and 2.

**Figure 1:** The 3-D graphs of profiles of the singular double-soliton solution (53) for fixed parameters $c_0 = 1.5, \alpha = 0.2, \lambda_1 = -1, \lambda_2 = -0.2, \ r_1 = -8.0, r_2 = 2.0, \text{ and } a = 0.1$.

**Figure 2:** The 3-D graphs of profiles of the singular double-soliton solution (53) for parameters: (a) $h_0 = 0.15, c_0 = 1.5, \alpha = 0.2, \lambda_1 = -1, \lambda_2 = -0.2, \ r_1 = -8.0, r_2 = 2.0, \text{ and } a = 0.1$; (b) $h_0 = 3, c_0 = -1.5, \alpha = 0.2, \lambda_1 = 1, \lambda_2 = 0.2, \ r_1 = 8, r_2 = 2.0, \text{ and } a = -2$.

As examples, we will investigate exact solutions of (1) in two simple cases $N = 1$ and $N = 2$. When $N = 1$, solving the linear algebraic system (18) leads to

$$A_0 = \frac{\delta_1 \lambda_2 - \delta_2 \lambda_1}{\delta_2 - \delta_1}. \quad (52)$$

Substituting (49) and (52) into (24), a singular double-soliton solution of (1) is obtained as follows:

$$u \left[1 \right] = \frac{6}{h_0} f \left[ \sqrt{- \frac{c_0}{4\lambda_1 \alpha} + \frac{3}{4h_0^2} - \frac{u_0}{2\lambda_1} \frac{1}{1 - r_1 \tanh \mu_1} - \lambda_2} - \sqrt{- \frac{c_0}{4\lambda_2 \alpha} + \frac{3}{4h_0^2} - \frac{u_0}{2\lambda_2} \frac{1}{1 - r_2 \tanh \mu_2} - \lambda_1} \right], \quad (53)$$

where

$$f = \sqrt{- \frac{c_0}{4\lambda_2 \alpha} + \frac{3}{4h_0^2} - \frac{u_0}{2\lambda_2} \frac{1}{1 - r_2 \tanh \mu_2} - \lambda_1} \quad \text{and} \quad \lambda = \sqrt{- \frac{c_0}{4\lambda_1 \alpha} + \frac{3}{4h_0^2} - \frac{u_0}{2\lambda_1} \frac{1}{1 - r_1 \tanh \mu_1} - \lambda_2}. \quad (54)$$

By using the program of computer, it is easy to verify that the solution (53) satisfies (1), and this shows that the Darboux transformation (24) which we obtained is correct. In order to show the properties of the above singular double-soliton solutions visually, as an example, we plot the 3-D graphs of solution (53) for some fixed parameters, which are shown in Figures 1 and 2.
When $N = 2$, using the Cramer rule to solve the linear algebraic system (18), we obtain

$$A_1 = \frac{\Delta_{A_1}}{\Delta},$$  \hspace{1cm} (55)

with

$$\Delta = \begin{vmatrix} 1 & \delta_1 & \lambda_1 & \delta_1 \lambda_1 \\ 1 & \delta_2 & \lambda_2 & \delta_2 \lambda_2 \\ 1 & \delta_3 & \lambda_3 & \delta_3 \lambda_3 \\ 1 & \delta_4 & \lambda_4 & \delta_4 \lambda_4 \end{vmatrix},$$

$$\Delta_{A_1} = \begin{vmatrix} 1 & -\lambda_1^2 & \delta_1 & \delta_1 \lambda_1 \\ 1 & -\lambda_2^2 & \delta_2 & \delta_2 \lambda_2 \\ 1 & -\lambda_3^2 & \delta_3 & \delta_3 \lambda_3 \\ 1 & -\lambda_4^2 & \delta_4 & \delta_4 \lambda_4 \end{vmatrix},$$  \hspace{1cm} (56)

where $\delta_j$ ($j = 1, 2, 3, 4$) are given by (49). From (24), an explicit solution of (1) is obtained by the following:

$$\sigma[2] = u + \frac{6}{h_0^2} A_1,$$  \hspace{1cm} (57)

where $A_1$ is given by (55). Equation (57) is a very complex solution, and it is not soliton solution. In order to show the properties of solution (57), under the fixed parameters $\lambda_1 = -0.2, \lambda_2 = -0.3, \lambda_3 = -0.4, \lambda_4 = -0.1, a = 0, c_0 = 15, \alpha = 1/2, h_0 = 0.8, r_1 = -0.2, r_2 = -0.3, r_3 = -0.4, r_4 = -0.5, t = 0.1$, we plot its 2-D profile, which is shown in Figure 3.

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**References**


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